数理解析研究所講究録 第2121巻 2019年 15-28

Singular limit problem for the Navier–Stokes equations in a curved thin domain

Tatsu-Hiko Miura*
Graduate School of Mathematical Sciences, the University of Tokyo

1 Introduction

Let Γ be a closed (i.e. compact and without boundary), connected, oriented, and smooth surface in $\mathbb{R}^3$ with unit outward normal vector field n. Also, let g_0 and g_1 be smooth functions on Γ satisfying $g := g_1 - g_0 \geq c$ on Γ with some constant $c > 0$. For a sufficiently small $\varepsilon \in (0,1)$ we define a curved thin domain Ω_ε in $\mathbb{R}^3$ by

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma,\ r \in (\varepsilon g_0(y), \varepsilon g_1(y))\} \tag{1.1}$$

and consider the Navier–Stokes equations with Navier's slip boundary conditions

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon - \nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ \operatorname{div} u^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ u^\varepsilon \cdot n_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ 2\nu P_\varepsilon D(u^\varepsilon) n_\varepsilon + \gamma_\varepsilon u^\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon. \end{cases} \tag{1.2}$$

Here Γ_ε is the boundary of Ω_ε that is the union of the inner and outer boundaries Γ_ε^0 and Γ_ε^1 given by $\Gamma_\varepsilon^i := \{y + \varepsilon g_i(y) n(y) \mid y \in \Gamma\}$ for $i = 0,1$, n_ε the unit outward normal vector field of Γ_ε, $\nu > 0$ the viscosity coefficient independent of ε, and $\gamma_\varepsilon \geq 0$ the friction coefficient given by $\gamma_\varepsilon := \gamma_\varepsilon^i$ on Γ_ε^i for $i = 0,1$ with γ_ε^0 and γ_ε^1 nonnegative constants. Also, $D(u^\varepsilon)$ and P_ε are the strain rate tensor and the orthogonal projection onto the tangent plane of Γ_ε given by

$$D(u^\varepsilon) := \frac{\nabla u^\varepsilon + (\nabla u^\varepsilon)^T}{2}, \quad P_\varepsilon := I_3 - n_\varepsilon \otimes n_\varepsilon,$$

where I_3 is the 3×3 identity matrix and $n_\varepsilon \otimes n_\varepsilon$ the tensor product of n_ε with itself.

PDEs in thin domains have been studied for a long time since the pioneering works [4, 5] by Hale and Raugel on reaction-diffusion and damped wave equations. In the study of the Navier–Stokes equations in a three-dimensional thin domain we naturally expect to get the global-in-time existence of a strong solution for large data since a three-dimensional thin domain with small width can be seen as almost two-dimensional. Raugel and Sell [19] first established a global existence of a strong

*The work of the author was supported by Grant-in-Aid for JSPS Fellows No. 16J02664 and the Program for Leading Graduate Schools, MEXT, Japan.

solution in the case of a flat product thin domain $\Omega_\varepsilon = Q_2 \times (0,\varepsilon)$ with a rectangle Q_2 and a sufficiently small $\varepsilon \in (0,1)$ under the purely periodic or mixed Dirichlet-periodic boundary conditions. Temam and Ziane [22] generalized the results of [19] to the case of a flat product thin domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$ with a bounded domain ω in $\mathbb{R}^2$ and boundary conditions which are combinations of the Dirichlet, periodic, and Hodge boundary conditions. They also studied in [23] the Navier–Stokes equations with Hodge boundary conditions in a thin spherical shell

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < a + \varepsilon a\}, \quad a > 0$$

to give a mathematical justification of derivation of the primitive equations for the atmosphere and ocean dynamics [11, 12]. Later, Iftimie, Raugel, and Sell [8] considered the Navier–Stokes equations in a flat thin domain with a nonflat top boundary

$$\Omega_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in (0,1)^2,\ x_3 \in (0, \varepsilon g(x'))\}, \quad g\colon (0,1)^2 \to \mathbb{R}$$

under the periodic boundary conditions on the lateral boundaries and the slip boundary conditions on the top and bottom boundaries. Hoang [6] and Hoang and Sell [7] generalized the results in [8] to the case of nonflat top and bottom boundaries.

In the resent work [14] we considered the curved thin domain Ω_ε of the form (1.1) as a new type of thin domain in the study of the Navier–Stokes equations (see [17, 18] for the study of a reaction-diffusion equation in a curved thin domain around a lower dimensional manifold). Our thin domain has a nonconstant width in the thin direction as in the case of flat thin domains in [6, 7, 8]. Moreover, its limit set Γ is a general closed surface with nonconstant curvatures. Such complicated shapes of Ω_ε and Γ make the analysis of the equations very difficult. In particular, we need to analyze carefully the behavior of vector fields on the boundary of Ω_ε that satisfy the slip boundary conditions to find out the dependence on ε of boundary integrals of such vector fields. We provided in [14] mathematical tools for analysis of vector fields in the curved thin domain and established the global existence of a strong solution to (1.2) for a large data when ε is sufficiently small.

In the study of PDEs in a thin domain we are also concerned with the behavior of a solution as the width of the thin domain tends to zero. When the thin domain shrinks to a lower dimensional set, it is important to derive limit equations on the limit set and compare solutions to the bulk and limit equations in order to study the effects of the limit set and the thin direction on the bulk equations in the thin domain. Such a problem for the Navier–Stokes equations was first studied by Temam and Ziane [22, 23] in the cases of a flat product thin domain and a thin spherical shell. They proved the convergence of the average in the thin direction of a solution to the bulk equations and derived limit equations by characterizing the limit as a solution to the limit equations. Iftimie, Raugel, and Sell [8] also compared a solution of the Navier–Stokes equations in a flat thin domain with a nonflat top boundary and that of limit equations which contains a function describing the top boundary of the thin domain. In [15] the present author formally derived limit equations of the Navier–Stokes equations in a tubular neighborhood of an evolving surface. They are basically the same as incompressible viscous fluid equations on an evolving surface derived by Jankuhn, Olshanskii, and Reusken [9] and Koba, Liu, and Giga [10]. In this paper we present the result in [14] that gives a rigorous derivation of the surface Navier–Stokes type equations by the thin width limit of the bulk Navier–Stokes equations (1.2) in the stationary curved thin domain of the form (1.1).

2 Notations

To state our result we fix notations on a closed surface and a curved thin domain. Unless otherwise stated we assume that all functions given here are sufficiently smooth. Also, we denote by I_3 the identity matrix of size three and by

$$a \otimes b := \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}, \quad \nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{pmatrix}$$

the tensor product of $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ and a vector field $u = (u_1, u_2, u_3)$ on an open set in $\mathbb{R}^3$.

2.1 Closed surface

Let Γ be a two-dimensional closed (i.e. compact and without boundary), connected, oriented, and smooth surface in $\mathbb{R}^3$ with unit outward normal vector field n. We define the tangential gradient and tangential derivatives of a function η on Γ by

$$\nabla_\Gamma \eta := P\nabla\tilde{\eta}, \quad \underline{D}_i \eta := \sum_{j=1}^{3} P_{ij}\partial_j\tilde{\eta} \quad \text{on} \quad \Gamma,\ i = 1, 2, 3$$

so that $\nabla_\Gamma \eta = (\underline{D}_1\eta, \underline{D}_2\eta, \underline{D}_3\eta)$. Here $\tilde{\eta}$ is an extension of η to an open neighborhood of Γ and $P = (P_{ij})_{i,j} := I_3 - n \otimes n$ the orthogonal projection onto the tangent plane of Γ. Note that the values of $\nabla_\Gamma \eta$ are independent of a choice of $\tilde{\eta}$ (see e.g. [3, Section 16.1]). For a (not necessarily tangential) vector field $v = (v_1, v_2, v_3)$ on Γ we define the tangential gradient matrix of v and the surface strain rate tensor by

$$\nabla_\Gamma v := \begin{pmatrix} \underline{D}_1 v_1 & \underline{D}_1 v_2 & \underline{D}_1 v_3 \\ \underline{D}_2 v_1 & \underline{D}_2 v_2 & \underline{D}_2 v_3 \\ \underline{D}_3 v_1 & \underline{D}_3 v_2 & \underline{D}_3 v_3 \end{pmatrix}, \quad D_\Gamma(v) := P\left(\frac{\nabla_\Gamma v + (\nabla_\Gamma v)^T}{2}\right)P$$

on Γ and the surface divergence of v by $\mathrm{div}_\Gamma v := \mathrm{tr}[\nabla_\Gamma v]$ on Γ. Moreover, for a matrix-valued function $A\colon \Gamma \to \mathbb{R}^{3\times 3}$ and $j = 1, 2, 3$ we set

$$[\mathrm{div}_\Gamma A]_j := \sum_{i=1}^{3} \underline{D}_i A_{ij} \quad \text{on} \quad \Gamma, \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

and define $\mathrm{div}_\Gamma A := ([\mathrm{div}_\Gamma A]_1, [\mathrm{div}_\Gamma A]_2, [\mathrm{div}_\Gamma A]_3)$ on Γ.

Next we define function spaces on Γ. For $\eta, \xi \in C^1(\Gamma)$ we have an integration by parts formula (see e.g. [3, Lemma 16.1])

$$\int_\Gamma (\eta \underline{D}_i \xi + \xi \underline{D}_i \eta)\, d\mathcal{H}^2 = -\int_\Gamma \eta\xi H n_i\, d\mathcal{H}^2, \quad i = 1, 2, 3,$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure and $H := -\mathrm{div}_\Gamma n$ is (twice) the mean curvature of Γ. Based on this formula, for $i = 1, 2, 3$ we say that $\eta \in L^2(\Gamma)$ has the weak tangential derivative η_i in $L^2(\Gamma)$ if

$$(\eta_i, \xi)_{L^2(\Gamma)} = -(\eta, \underline{D}_i \xi + \xi H n_i)_{L^2(\Gamma)} \quad \text{for all} \quad \xi \in C^1(\Gamma).$$

In this case we write $\underline{D}_i\eta = \eta_i$ and define the Sobolev space

$$H^1(\Gamma) := \{\eta \in L^2(\Gamma) \mid \underline{D}_i\eta \in L^2(\Gamma) \text{ for all } i = 1, 2, 3\}.$$

Also, for $\mathcal{X} = L^2, H^1$ and the function g on Γ given in (2.1) below we set

$$\begin{aligned}\mathcal{X}(\Gamma, T\Gamma) &:= \{v \in \mathcal{X}(\Gamma)^3 \mid v \cdot n = 0 \text{ on } \Gamma\},\\ \mathcal{X}_{g\sigma}(\Gamma, T\Gamma) &:= \{v \in \mathcal{X}(\Gamma, T\Gamma) \mid \mathrm{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}\end{aligned}$$

and denote by $H^{-1}(\Gamma, T\Gamma)$ the dual of $H^1(\Gamma, T\Gamma)$ (via the $L^2(\Gamma)$-inner product).

2.2 Curved thin domain

Let g_0 and g_1 be functions on Γ such that

$$g := g_1 - g_0 \geq c \quad \text{on} \quad \Gamma \tag{2.1}$$

with some constant $c > 0$. For a sufficiently small $\varepsilon \in (0, 1)$ we define a curved thin domain Ω_ε and its inner and outer boundaries Γ_ε^0 and Γ_ε^1 by

$$\begin{aligned}\Omega_\varepsilon &:= \{y + rn(y) \mid y \in \Gamma,\ r \in (\varepsilon g_0(y), \varepsilon g_1(y))\} \subset \mathbb{R}^3,\\ \Gamma_\varepsilon^i &:= \{y + \varepsilon g_i(y)n(y) \mid y \in \Gamma\}, \quad i = 0, 1\end{aligned}$$

and denote by $\Gamma_\varepsilon := \Gamma_\varepsilon^0 \cup \Gamma_\varepsilon^1$ the boundary of Ω_ε with unit outward normal n_ε. Let

$$L^2_\sigma(\Omega_\varepsilon) = \{u \in L^2(\Omega_\varepsilon)^3 \mid \mathrm{div}\, u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$$

be the standard L^2-solenoidal space on Ω_ε and $\mathcal{V}_\varepsilon := L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$. We denote by A_ε the Stokes operator on $L^2_\sigma(\Omega_\varepsilon)$ associated with slip boundary conditions and by $D(A_\varepsilon)$ its domain. They are of the form

$$\begin{aligned}A_\varepsilon u &= -\nu \mathbb{P}_\varepsilon \Delta u, \quad u \in D(A_\varepsilon),\\ D(A_\varepsilon) &= \{u \in L^2_\sigma(\Omega_\varepsilon) \cap H^2(\Omega_\varepsilon)^3 \mid 2\nu P_\varepsilon D(u)n_\varepsilon + \gamma_\varepsilon u = 0 \text{ on } \Gamma_\varepsilon\}\end{aligned}$$

with Helmholtz–Leray projection $\mathbb{P}_\varepsilon$ from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$.

By the definition of Ω_ε we have a change of variables formula

$$\int_{\Omega_\varepsilon} \varphi(x)\, dx = \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y)) J(y, r)\, dr\, d\mathcal{H}^2(y) \tag{2.2}$$

for a function φ on Ω_ε. Here $J(y, r)$ is the Jacobian that satisfies

$$|J(y, r) - 1| \leq c\varepsilon, \quad y \in \Gamma,\ r \in (\varepsilon g_0(y), \varepsilon g_1(y))$$

with a constant $c > 0$ independent of ε (see [14, Section 2.2] for details). Based on (2.2) we define the average in the thin direction of a function φ on Ω_ε by

$$M\varphi(y) := \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y))\, dr, \quad y \in \Gamma \tag{2.3}$$

and write $M_\tau u := PMu$ for the averaged tangential component of $u \colon \Omega_\varepsilon \to \mathbb{R}^3$.

3 Main result

We make the following assumptions on the friction coefficient γ_ε appearing in the slip boundary conditions. Recall that γ_ε can take different values γ_ε^0 and γ_ε^1 on the inner and outer boundaries Γ_ε^0 and Γ_ε^1, where γ_ε^0 and γ_ε^1 are nonnegative constants.

Assumption 3.1. There exists a constant $c > 0$ such that

$$\gamma_\varepsilon^0 \leq c\varepsilon, \quad \gamma_\varepsilon^1 \leq c\varepsilon \quad \text{for all} \quad \varepsilon \in (0,1).$$

Assumption 3.2. There exists a constant $c > 0$ such that

$$\gamma_\varepsilon^0 \geq c\varepsilon \quad \text{for all} \quad \varepsilon \in (0,1) \quad \text{or} \quad \gamma_\varepsilon^1 \geq c\varepsilon \quad \text{for all} \quad \varepsilon \in (0,1).$$

There assumptions are used to show the uniform equivalence of the norms

$$c^{-1}\|u\|_{H^k(\Omega_\varepsilon)} \leq \|A_\varepsilon^{k/2}u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^k(\Omega_\varepsilon)} \quad u \in D(A_\varepsilon^{k/2}),\ k = 1,2$$

for the Stokes operator A_ε with a constant $c > 0$ independent of ε (see [14]).

Remark 3.3. By Assumption 3.2 we exclude the perfect slip boundary conditions

$$u \cdot n_\varepsilon = 0, \quad P_\varepsilon D(u)n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon.$$

In [14] we also consider these boundary conditions under other assumptions on Γ.

Now we present our main result of [14] in a slightly modified form.

Theorem 3.4 ([14, Theorem 1.6])**.** *Under Assumptions 3.1 and 3.2, let*

$$u_0^\varepsilon \in \mathcal{V}_\varepsilon, \quad f^\varepsilon \in L^\infty(0,\infty; L^2_\sigma(\Omega_\varepsilon)), \quad \varepsilon \in (0,1).$$

Suppose that the following conditions are satisfied:

(a) There exist constants $c > 0$, $\varepsilon_1 \in (0,1)$, and $\alpha \in (0,1)$ such that

$$\|u_0^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} + \|f^\varepsilon\|^2_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))} \leq c\varepsilon^{-1+\alpha} \quad \textit{for all} \quad \varepsilon \in (0,\varepsilon_1).$$

(b) There exist $v_0 \in L^2(\Gamma, T\Gamma)$ and $f \in L^\infty(0,\infty; H^{-1}(\Gamma, T\Gamma))$ such that

$$\begin{aligned} &\lim_{\varepsilon\to 0} M_\tau u_0^\varepsilon = v_0 \quad \textit{weakly in} \quad L^2(\Gamma, T\Gamma), \\ &\lim_{\varepsilon\to 0} M_\tau f^\varepsilon = f \quad \textit{weakly-}\star \textit{ in} \quad L^\infty(0,\infty; H^{-1}(\Gamma, T\Gamma)). \end{aligned}$$

(c) For $i = 0,1$ there exists $\gamma^i \geq 0$ such that $\lim_{\varepsilon\to 0} \varepsilon^{-1}\gamma_\varepsilon^i = \gamma^i$.

Then there exists a constant $\varepsilon_2 \in (0,1)$ such that the problem (1.2) *admits a global-in-time strong solution*

$$u^\varepsilon \in C([0,\infty); \mathcal{V}_\varepsilon) \cap L^2_{loc}([0,\infty); D(A_\varepsilon)) \cap H^1_{loc}([0,\infty); L^2_\sigma(\Omega_\varepsilon))$$

for each $\varepsilon \in (0,\varepsilon_2)$ and

$$\lim_{\varepsilon\to 0} Mu^\varepsilon \cdot n = 0 \quad \textit{strongly in} \quad C([0,\infty); L^2(\Gamma)).$$

Moreover, there exists a vector field

$$v \in C([0,\infty); L^2_{g\sigma}(\Gamma, T\Gamma)) \cap L^2_{loc}([0,\infty); \mathcal{V}_g) \cap H^1_{loc}([0,\infty); H^{-1}(\Gamma, T\Gamma))$$

such that

$$\begin{aligned} \lim_{\varepsilon\to 0} M_\tau u^\varepsilon &= v \quad \text{weakly in} \quad L^2(0,T; H^1(\Gamma, T\Gamma)), \\ \lim_{\varepsilon\to 0} \partial_t M_\tau u^\varepsilon &= \partial_t v \quad \text{weakly in} \quad L^2(0,T; H^{-1}(\Gamma, T\Gamma)) \end{aligned}$$

for each $T > 0$ and v is a unique weak solution to

$$\begin{cases} g\Big(\partial_t v + \overline{\nabla}_v v\Big) - 2\nu\left\{P\mathrm{div}_\Gamma[gD_\Gamma(v)] - \dfrac{1}{g}(\nabla_\Gamma g \otimes \nabla_\Gamma g)v\right\} \\ \qquad\qquad\qquad\qquad + (\gamma^0 + \gamma^1)v + g\nabla_\Gamma q = gf \quad \text{on} \quad \Gamma \times (0,\infty), \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{div}_\Gamma(gv) = 0 \quad \text{on} \quad \Gamma \times (0,\infty), \\ \qquad\qquad\qquad\qquad\qquad\qquad v|_{t=0} = v_0 \quad \text{on} \quad \Gamma \end{cases} \tag{3.1}$$

with an associated pressure q.

Here $\mathcal{V}_g := H^1_{g\sigma}(\Gamma, T\Gamma)$ and $\overline{\nabla}_v v := P(v \cdot \nabla_\Gamma)v$ is the covariant derivative of the tangential vector field v on Γ along itself. We also define a weak solution to (3.1) as follows: for $v_1, v_2, v_3 \in H^1(\Gamma, T\Gamma)$ let

$$\begin{aligned} a_g(v_1, v_2) := 2\nu\left\{\big(gD_\Gamma(v_1), D_\Gamma(v_2)\big)_{L^2(\Gamma)} + (g^{-1}(v_1 \cdot \nabla_\Gamma g), v_2 \cdot \nabla_\Gamma g)_{L^2(\Gamma)}\right\} \\ + (\gamma^0 + \gamma^1)(v_1, v_2)_{L^2(\Gamma)} \end{aligned}$$

be a bilinear form corresponding to the viscous and friction terms and

$$b_g(v_1, v_2, v_3) := -(g(v_1 \otimes v_2), \nabla_\Gamma v_3)_{L^2(\Gamma)}$$

a trilinear form corresponding to the convection term. For each $T > 0$ we say that a vector field $v \in L^\infty(0,T; L^2_{g\sigma}(\Gamma, T\Gamma)) \cap L^2(0,T; \mathcal{V}_g)$ with $\partial_t v \in L^2(0,T; H^{-1}(\Gamma, T\Gamma))$ is a weak solution to (3.1) on $[0,T)$ if it satisfies $v|_{t=0} = v_0$ in $H^{-1}(\Gamma, T\Gamma)$ and

$$\int_0^T \{[g\partial_t v, \eta]_{T\Gamma} + a_g(v, \eta) + b_g(v, v, \eta)\}\, dt = \int_0^T [gf, \eta]_{T\Gamma}\, dt \tag{3.2}$$

for all $\eta \in L^2(0,T; \mathcal{V}_g)$, where $[\cdot,\cdot]_{T\Gamma}$ denotes the duality product between $H^{-1}(\Gamma, T\Gamma)$ and $H^1(\Gamma, T\Gamma)$. Moreover, we call v a weak solution to (3.1) on $[0,\infty)$ if it is a weak solution to (3.1) on $[0,T)$ for all $T > 0$.

Theorem 3.4 provides only a weak convergence result, but in [14] we also derived estimates for the difference between $M_\tau u^\varepsilon$ and v and established a strong convergence result. See [14, Section 10.6] for details.

Remark 3.5. Formally, if $g \equiv 1$ and $\gamma^0 = \gamma^1 = 0$ in (3.1), then we have

$$\partial_t v + \overline{\nabla}_v v - 2\nu P\mathrm{div}_\Gamma[D_\Gamma(v)] + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma \times (0,\infty). \tag{3.3}$$

It is shown in [15, Lemma 2.5] that $2P\mathrm{div}_\Gamma[D_\Gamma(v)] = \Delta_B v + Kv$ on Γ for a tangential and surface divergence-free vector field v on Γ, where Δ_B and K are the Bochner Laplacian on Γ and the Gaussian curvature of Γ. Also, since Γ is two-dimensional, K agrees with the Ricci curvature Ric of Γ, i.e. $Kw = \mathrm{Ric}(w)$ for a tangential vector field w on Γ (see e.g. [21, Appendix C]). Hence the equations (3.3) read

$$\partial_t v + \overline{\nabla}_v v - \nu\{\Delta_B v + \mathrm{Ric}(v)\} + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma \times (0,\infty). \tag{3.4}$$

Note that the equations (3.4) are described only in terms of the intrinsic quantities of the Riemannian manifold Γ. They were called the "correct" Navier–Stokes equations on a manifold in [2, 20] and studied by Mitrea and Taylor [13], Nagasawa [16], and Taylor [20]. Hence our limit equations (3.1) can be seen as the damped and weighted Navier–Stokes equations on a manifold.

4 Outline of the proof

In this section we explain the outline of the proof of Theorem 3.4. For details of the proof and construction of an associated pressure in (3.1), see [14, Section 10].

4.1 Average of the weak formulation

First we take the average in the thin direction of the weak formulation for the bulk equations (1.2) satisfied by a strong solution. Under the assumptions in Theorem 3.4 we can show the global-in-time existence of a strong solution

$$u^\varepsilon \in C([0,\infty);\mathcal{V}_\varepsilon) \cap L^2_{loc}([0,\infty);D(A_\varepsilon)) \cap H^1_{loc}([0,\infty);\mathcal{H}_\varepsilon)$$

to (1.2) for a sufficiently small $\varepsilon \in (0,1)$ and uniform estimates

$$\begin{gathered} \|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \leq c\varepsilon, \quad \int_0^t \|u^\varepsilon(s)\|^2_{H^1(\Omega_\varepsilon)}\,ds \leq c\varepsilon(1+t), \\ \|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{-1+\alpha}, \quad \int_0^t \|u^\varepsilon(s)\|^2_{H^2(\Omega_\varepsilon)}\,ds \leq c\varepsilon^{-1+\alpha}(1+t), \\ \int_0^t \|\partial_t u^\varepsilon(s)\|^2_{L^2(\Omega_\varepsilon)}\,ds \leq c\varepsilon^{-1+\alpha}(1+t) \end{gathered} \tag{4.1}$$

for all $t \geq 0$ with a constant $c > 0$ independent of ε and t (see [14, Theorem 8.4]). The strong solution u^ε to (1.2) satisfies

$$\int_0^T \{(\partial_t u^\varepsilon, \varphi)_{L^2(\Omega_\varepsilon)} + a_\varepsilon(u^\varepsilon,\varphi) + b_\varepsilon(u^\varepsilon,u^\varepsilon,\varphi)\}\,dt = \int_0^T (f^\varepsilon,\varphi)_{L^2(\Omega_\varepsilon)}\,dt \tag{4.2}$$

for all $T > 0$ and $\varphi \in L^2(0,T;\mathcal{V}_\varepsilon)$, and $u^\varepsilon|_{t=0} = u_0^\varepsilon$ in $\mathcal{V}_\varepsilon$, where

$$a_\varepsilon(u_1,u_2) := 2\nu\big(D(u_1),D(u_2)\big)_{L^2(\Omega_\varepsilon)} + \gamma_\varepsilon^0(u_1,u_2)_{L^2(\Gamma_\varepsilon^0)} + \gamma_\varepsilon^1(u_1,u_2)_{L^2(\Gamma_\varepsilon^1)}$$

is a bilinear form for $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$ corresponding to the Stokes problem in Ω_ε with slip boundary conditions and

$$b_\varepsilon(u_1,u_2,u_3) := -(u_1 \otimes u_2, \nabla u_3)_{L^2(\Omega_\varepsilon)}$$

is a trilinear form for $u_1, u_2, u_3 \in H^1(\Omega_\varepsilon)^3$.

Let $M_\tau u^\varepsilon$ be the averaged tangential component of the strong solution u^ε. By the space-time regularity of u^ε we have

$$M_\tau u^\varepsilon \in C([0,\infty); H^1(\Gamma, T\Gamma)) \cap H^1_{loc}([0,\infty); L^2(\Gamma, T\Gamma)).$$

We transform (4.2) into a weak formulation for $M_\tau u^\varepsilon$. To this end, we construct an appropriate test function for (4.2) from a test function $\eta \in L^2(0,T;\mathcal{V}_g)$ for the weak formulation (3.2) of the limit equations. We extend η to a vector field on Ω_ε that satisfies the impermeable boundary condition, i.e. the first boundary condition of (1.2) and then apply the Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$. Then we get a test function $\eta_\varepsilon \in L^2(0,T;\mathcal{V}_\varepsilon)$ for (4.2) that satisfies (we suppress t)

$$\begin{gathered} \|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla\eta_\varepsilon - \overline{F(\eta)}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|\eta\|_{H^1(\Gamma)}, \\ \|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Gamma_\varepsilon)} \leq c\varepsilon\|\eta\|_{H^1(\Gamma)} \end{gathered} \tag{4.3}$$

with a constant $c > 0$ independent of ε, where $\bar{\eta}$ is the constant extension of η in the normal direction of Γ and $F(\eta) := \nabla_\Gamma\eta + g^{-1}(\eta\cdot\nabla_\Gamma g)n\otimes n$ on Γ. Substituting η_ε for φ in (4.2) and using the estimates (4.1) and (4.3), the change of variables formula (2.2), and the average operator (2.3) we derive a weak formulation for $M_\tau u^\varepsilon$:

$$\begin{aligned} &\int_0^T \{(g\partial_t M_\tau u^\varepsilon, \eta)_{L^2(\Gamma)} + a_g(M_\tau u^\varepsilon, \eta) + b_g(M_\tau u^\varepsilon, M_\tau u^\varepsilon, \eta)\}\,dt \\ &\qquad\qquad = \int_0^T (gM_\tau f^\varepsilon, \eta)_{L^2(\Gamma)}\,dt + R^1_\varepsilon(\eta) \end{aligned} \tag{4.4}$$

for all $\eta \in L^2(0,T;\mathcal{V}_g)$ with a residual term $R^1_\varepsilon(\eta)$ satisfying

$$|R^1_\varepsilon(\eta)| \leq c\left(\varepsilon^{\alpha/4} + \sum_{i=0,1}|\varepsilon^{-1}\gamma^i_\varepsilon - \gamma^i|\right)(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}, \tag{4.5}$$

where $c > 0$ is a constant independent of ε. Note that to prove (4.5) we require the uniform estimates (4.1) for the strong solution u^ε to (1.2), especially its H^2-estimate. This is due to the fact that the limit equations (3.1) are essentially described only in terms of the intrinsic quantities of Γ, while the bulk equations (1.2) contain the extrinsic quantities of Γ. In other words, the H^2-regularity of the strong solution to (1.2) supplements a lack of the extrinsic information of Γ in (3.1).

4.2 Energy estimate for the average of a solution

Next we derive the energy estimate for $M_\tau u^\varepsilon$. In derivation of the energy estimate for an approximate solution to the Navier–Stokes equations we usually substitute the approximate solution itself for its weak formulation. However, we cannot take $M_\tau u^\varepsilon$ as a test function for its weak formulation (4.4) since it is not in $\mathcal{V}_g$, i.e. $\mathrm{div}_\Gamma(gM_\tau u^\varepsilon)$ does not vanish on Γ in general. To overcome this difficulty we establish the weighted Helmholtz–Leray decomposition of a surface tangential vector field

$$v = v_g + g\nabla_\Gamma q \quad \text{in} \quad L^2(\Gamma, T\Gamma), \quad v_g \in L^2_{g\sigma}(\Gamma, T\Gamma),\ g\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma, T\Gamma)^\perp.$$

Using this we get the weighted solenoidal part

$$v^\varepsilon \in C([0,\infty);\mathcal{V}_g) \cap H^1_{loc}([0,\infty);L^2_{g\sigma}(\Gamma,T\Gamma))$$

of $M_\tau u^\varepsilon$ satisfying

$$\begin{aligned} \max_{t\in[0,T]} \|M_\tau u^\varepsilon(t) - v^\varepsilon(t)\|^2_{L^2(\Gamma)} &\leq c\varepsilon^2, \\ \int_0^T \|M_\tau u^\varepsilon(t) - v^\varepsilon(t)\|^2_{H^1(\Gamma)}\,dt &\leq c\varepsilon^2(1+T), \\ \int_0^T \|\partial_t M_\tau u^\varepsilon(t) - \partial_t v^\varepsilon(t)\|^2_{L^2(\Gamma)}\,dt &\leq c\varepsilon^\alpha(1+T) \end{aligned} \tag{4.6}$$

for all $T>0$ and transform (4.4) into a weak formulation for v^ε of the form

$$\begin{aligned} &\int_0^T \{(g\partial_t v^\varepsilon,\eta)_{L^2(\Gamma)} + a_g(v^\varepsilon,\eta) + b_g(v^\varepsilon,v^\varepsilon,\eta)\}\,dt \\ &\qquad = \int_0^T (gM_\tau f^\varepsilon,\eta)_{L^2(\Gamma)}\,dt + R^1_\varepsilon(\eta) + R^2_\varepsilon(\eta) \end{aligned} \tag{4.7}$$

for all $T>0$ and $\eta \in L^2(0,T;\mathcal{V}_g)$, where $R^2_\varepsilon(\eta)$ satisfies

$$|R^2_\varepsilon(\eta)| \leq c\varepsilon^{\alpha/2}(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}$$

with a constant $c>0$ independent of ε. Since $v^\varepsilon \in L^2(0,T;\mathcal{V}_g)$ we can substitute it for (4.7) to derive the energy estimate

$$\max_{t\in[0,T]} \|v^\varepsilon(t)\|^2_{L^2(\Gamma)} + \int_0^T \|\nabla_\Gamma v^\varepsilon(t)\|^2_{L^2(\Gamma)}\,dt \leq c_T \tag{4.8}$$

for all $T>0$ with a constant $c_T>0$ depending only on T. Then we combine (4.6) and (4.8) to obtain the energy estimate

$$\max_{t\in[0,T]} \|M_\tau u^\varepsilon(t)\|^2_{L^2(\Gamma)} + \int_0^T \|\nabla_\Gamma M_\tau u^\varepsilon(t)\|^2_{L^2(\Gamma)}\,dt \leq c_T \tag{4.9}$$

for the original averaged tangential component $M_\tau u^\varepsilon$.

4.3 Estimate for the time derivative of the average

By the energy estimate (4.9) we see that (a subsequence of) $M_\tau u^\varepsilon$ converges weakly in appropriate function spaces on Γ as $\varepsilon \to 0$. However, we also require the strong convergence of $M_\tau u^\varepsilon$ to show the convergence of the trilinear term in (4.4). We use the Lions–Aubin lemma to get the strong convergence. For this purpose, we derive a uniform estimate for the time derivative of $M_\tau u^\varepsilon$.

First we estimate the time derivative of the weighted solenoidal part v^ε of $M_\tau u^\varepsilon$ in $H^{-1}(\Gamma,T\Gamma)$. To this end, we take $w \in L^2(0,T;H^1(\Gamma,T\Gamma))$ and construct a test function $\eta \in L^2(0,T;\mathcal{V}_g)$ for (4.7) and $q \in L^2(0,T;H^2(\Gamma))$ such that

$$w = g\eta + g\nabla_\Gamma q \quad \text{on} \quad \Gamma, \quad \|\eta\|_{H^1(\Gamma)} \leq c\|w\|_{H^1(\Gamma)}.$$

Then we substitute η for (4.7) and use the above relations and

$$\int_0^T (g\partial_t v^\varepsilon, \eta)_{L^2(\Gamma)}\,dt = \int_0^T (\partial_t v^\varepsilon, g\eta)_{L^2(\Gamma)}\,dt = \int_0^T (\partial_t v^\varepsilon, w)_{L^2(\Gamma)}\,dt$$

by $\partial_t v^\varepsilon \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and $g\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma, T\Gamma)^\perp$ to obtain

$$\left|\int_0^T (\partial_t v^\varepsilon, w)_{L^2(\Gamma)}\,dt\right| \leq c_T \|w\|_{L^2(0,T;H^1(\Gamma))}$$

for all $w \in L^2(0,T;H^1(\Gamma, T\Gamma))$, which yields

$$\|\partial_t v^\varepsilon\|_{L^2(0,T;H^{-1}(\Gamma,T\Gamma))} \leq c_T$$

with a constant $c_T > 0$ depending only on T. By this estimate and the last inequality of (4.6) with $\|v\|_{H^{-1}(\Gamma,T\Gamma)} \leq \|v\|_{L^2(\Gamma)}$ for $v \in L^2(\Gamma, T\Gamma)$ we obtain

$$\|\partial_t M_\tau u^\varepsilon\|_{L^2(0,T;H^{-1}(\Gamma,T\Gamma))} \leq c_T. \tag{4.10}$$

Remark 4.1. In construction of a weak solution to the Navier–Stokes equations we usually estimate the time derivative of an approximate solution in the dual of a solenoidal space, but here we estimate $\partial_t M_\tau u^\varepsilon$ in the dual $H^{-1}(\Gamma, T\Gamma)$ of $H^1(\Gamma, T\Gamma)$, not in the dual $\mathcal{V}_g'$ of $\mathcal{V}_g = H^1_{g\sigma}(\Gamma, T\Gamma)$. This is because we multiply $\partial_t M_\tau u^\varepsilon$ by g in (4.4). For $f \in \mathcal{V}_g'$ we cannot define gf as an element of $\mathcal{V}_g'$ by $gf\colon v \mapsto {}_{\mathcal{V}_g'}\langle f, gv\rangle_{\mathcal{V}_g}$ for $v \in \mathcal{V}_g$ since gv does not belong to $\mathcal{V}_g$ in general (here ${}_{\mathcal{V}_g'}\langle\cdot,\cdot\rangle_{\mathcal{V}_g}$ is the duality product between $\mathcal{V}_g'$ and $\mathcal{V}_g$). On the other hand, for $f \in H^{-1}(\Gamma, T\Gamma)$ we can define $gf \in H^{-1}(\Gamma, T\Gamma)$ by $[gf, v]_{T\Gamma} := [f, gv]_{T\Gamma}$ for $v \in H^1(\Gamma, T\Gamma)$ since $gv \in H^1(\Gamma, T\Gamma)$ by the smoothness of g on Γ. We consider $\partial_t M_\tau u^\varepsilon$ in $H^{-1}(\Gamma, T\Gamma)$ to avoid a problem with multiplication of a function in dual spaces.

4.4 Convergence of the average and characterization of the limit

Now let us prove the convergence of the average of the strong solution u^ε to (1.2) and characterize the limit as a unique weak solution to (3.1). First note that, since u^ε satisfies $u^\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε and (4.1), we can show

$$\sup_{t\in[0,\infty)} \|Mu^\varepsilon(t)\cdot n\|_{L^2(\Gamma)} \leq c\varepsilon^{1/2} \sup_{t\in[0,\infty)} \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{\alpha/2} \to 0$$

as $\varepsilon \to 0$ (see [14, Lemma 6.4] for the first inequality). Hence $\{Mu^\varepsilon \cdot n\}_\varepsilon$ converges strongly to zero in $C([0,\infty); L^2(\Gamma))$.

Next we consider the averaged tangential component $M_\tau u^\varepsilon$. For each fixed $T > 0$ we observe by (4.9) and (4.10) that

- $\{M_\tau u^\varepsilon\}_\varepsilon$ is bounded in $L^\infty(0,T;L^2(\Gamma,T\Gamma)) \cap L^2(0,T;H^1(\Gamma,T\Gamma))$,
- $\{\partial_t M_\tau u^\varepsilon\}_\varepsilon$ is bounded in $L^2(0,T;H^{-1}(\Gamma,T\Gamma))$.

Thus there exist a vector field

$$v \in L^\infty(0,T;L^2(\Gamma,T\Gamma)) \cap L^2(0,T;H^1(\Gamma,T\Gamma))$$
$$\text{with} \quad \partial_t v \in L^2(0,T;H^{-1}(\Gamma,T\Gamma))$$

and a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of positive numbers convergent to zero such that

$$\begin{aligned}
\lim_{k\to\infty} M_\tau u^{\varepsilon_k} &= v && \text{weakly-}\star \text{ in} && L^\infty(0,T;L^2(\Gamma,T\Gamma)),\\
\lim_{k\to\infty} M_\tau u^{\varepsilon_k} &= v && \text{weakly in} && L^2(0,T;H^1(\Gamma,T\Gamma)),\\
\lim_{k\to\infty} \partial_t M_\tau u^{\varepsilon_k} &= \partial_t v && \text{weakly in} && L^2(0,T;H^{-1}(\Gamma,T\Gamma)).
\end{aligned} \tag{4.11}$$

By the Lions–Aubin lemma (see e.g. [1, Theorem II.5.16]) we also have

$$\lim_{k\to\infty} M_\tau u^{\varepsilon_k} = v \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma,T\Gamma)). \tag{4.12}$$

Note that we do not a priori know that v is a weighted solenoidal vector field on Γ. However, by $u^\varepsilon \in L^2_\sigma(\Omega_\varepsilon)$, the first inequality of (4.1), and (4.12) we can prove

$$v \in L^\infty(0,T;L^2_{g\sigma}(\Gamma,T\Gamma)) \cap L^2(0,T;\mathcal{V}_g).$$

For details, we refer to [14, Lemma 10.24].

Let us show that v satisfies the weak formulation (3.2) for the limit equations. First we take $\eta \in C_c(0,T;\mathcal{V}_g)$ and consider the weak formulation (4.4) for $M_\tau u^{\varepsilon_k}$:

$$\begin{aligned}
\int_0^T \{[g\partial_t M_\tau u^{\varepsilon_k},\eta]_{T\Gamma} + a_g(M_\tau u^{\varepsilon_k},\eta) + b_g(M_\tau u^{\varepsilon_k}, M_\tau u^{\varepsilon_k},\eta)\}\,dt \\
= \int_0^T [gM_\tau f^{\varepsilon_k},\eta]_{T\Gamma}\,dt + R^1_{\varepsilon_k}(\eta).
\end{aligned} \tag{4.13}$$

We send $k\to\infty$ in (4.13). Then, by the assumption (b) of Theorem 3.4 and (4.11),

$$\begin{aligned}
\lim_{k\to\infty}\int_0^T [g\partial_t M_\tau u^{\varepsilon_k},\eta]_{T\Gamma}\,dt &= \int_0^T [g\partial_t v,\eta]_{T\Gamma}\,dt,\\
\lim_{k\to\infty}\int_0^T a_g(M_\tau u^{\varepsilon_k},\eta)\,dt &= \int_0^T a_g(v,\eta)\,dt,\\
\lim_{k\to\infty}\int_0^T [gM_\tau f^{\varepsilon_k},\eta]_{T\Gamma}\,dt &= \int_0^T [gf,\eta]_{T\Gamma}\,dt.
\end{aligned} \tag{4.14}$$

Also, by (4.5), the assumption (c), and $\alpha>0$ we have

$$|R^1_{\varepsilon_k}(\eta)| \le c\left(\varepsilon_k^{\alpha/4} + \sum_{i=0,1}|\varepsilon_k^{-1}\gamma^i_{\varepsilon_k} - \gamma^i|\right)(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \to 0 \tag{4.15}$$

as $k\to\infty$. To prove the convergence of the trilinear term we set

$$\begin{aligned}
J_1^k &:= \int_0^T b_g(M_\tau u^{\varepsilon_k}, M_\tau u^{\varepsilon_k},\eta)\,dt - \int_0^T b_g(v, M_\tau u^{\varepsilon_k},\eta)\,dt,\\
J_2^k &:= \int_0^T b_g(v, M_\tau u^{\varepsilon_k},\eta)\,dt - \int_0^T b_g(v,v,\eta)\,dt.
\end{aligned}$$

By Ladyzhenskaya's inequality $\|\xi\|_{L^4(\Gamma)} \le c\|\xi\|^{1/2}_{L^2(\Gamma)}\|\nabla_\Gamma\xi\|^{1/2}_{L^2(\Gamma)}$ for $\xi\in H^1(\Gamma)$ (see [14, Lemma 3.1]) and Hölder's inequality we have

$$|J_1^k| \le c\int_0^T \|M_\tau u^{\varepsilon_k} - v\|^{1/2}_{L^2(\Gamma)}\|M_\tau u^{\varepsilon_k} - v\|^{1/2}_{H^1(\Gamma)}\|M_\tau u^{\varepsilon_k}\|_{H^1(\Gamma)}\|\eta\|_{H^1(\Gamma)}\,dt.$$

Moreover, since $\{M_\tau u^\varepsilon\}_\varepsilon$ is bounded in $L^2(0,T;H^1(\Gamma,T\Gamma))$ and satisfies (4.12), and since $\|\eta(t)\|_{H^1(\Gamma)}$ is bounded on $[0,T]$ by $\eta\in C_c(0,T;\mathcal{V}_g)$,

$$|J_1^k|\leq c\|M_\tau u^{\varepsilon_k}-v\|_{L^2(0,T;L^2(\Gamma))}^{1/2}\to 0 \quad \text{as} \quad k\to 0. \tag{4.16}$$

Also, since the linear functional

$$\Phi(\xi):=\int_0^T b_g(v,\xi,\eta)\,dt, \quad \xi\in L^2(0,T;H^1(\Gamma,T\Gamma))$$

is bounded on $L^2(0,T;H^1(\Gamma,T\Gamma))$ by $v\in L^2(0,T;\mathcal{V}_g)$ and $\eta\in C_c(0,T;\mathcal{V}_g)$, we get

$$\lim_{k\to\infty}J_2^k=\lim_{k\to\infty}\{\Phi(M_\tau u^{\varepsilon_k})-\Phi(v)\}=0 \tag{4.17}$$

by (4.11). Hence it follows from (4.16) and (4.17) that

$$\lim_{k\to\infty}\int_0^T b_g(M_\tau u^{\varepsilon_k},M_\tau u^{\varepsilon_k},\eta)\,dt=\int_0^T b_g(v,v,\eta)\,dt \tag{4.18}$$

and we see by (4.13)–(4.15) and (4.18) that v satisfies (3.2) for all $\eta\in C_c(0,T;\mathcal{V}_g)$. By the space-time regularity of v and the density of $C_c(0,T;\mathcal{V}_g)$ in $L^2(0,T;\mathcal{V}_g)$ we can also show that $v\in C([0,T];L^2_{g\sigma}(\Gamma,T\Gamma))$ and (3.2) is valid for all $\eta\in L^2(0,T;\mathcal{V}_g)$.

To show that v is a weak solution to (3.1) it is also necessary to verify the initial condition. Let $\xi\in\mathcal{V}_g$ and $\varphi\in C^\infty([0,T])$ such that $\varphi(0)=1$ and $\varphi(T)=0$. We substitute $\eta:=\varphi\xi\in L^2(0,T;\mathcal{V}_g)$ for (3.2) and (4.13), carry out integration by parts for the time derivatives, and send $k\to\infty$. Then by $\varphi(0)=1$ and $\varphi(T)=0$, the assumption (b) of Theorem 3.4, (4.12), (4.14), (4.15), and (4.18) we obtain

$$(gv(0),\xi)_{L^2(\Gamma)}=(gv_0,\xi)_{L^2(\Gamma)} \quad \text{for all} \quad \xi\in\mathcal{V}_g.$$

Since $\mathcal{V}_g$ is dense in $L^2_{g\sigma}(\Gamma,T\Gamma)$, the above equality is also valid for all $\xi\in L^2_{g\sigma}(\Gamma,T\Gamma)$ and we can set $\xi:=v(0)-v_0$ to get

$$(g\{v(0)-v_0\},v(0)-v_0)_{L^2(\Gamma)}=\|g^{1/2}\{v(0)-v_0\}\|^2_{L^2(\Gamma)}=0,$$

which combined with (2.1) implies $v|_{t=0}=v_0$ on Γ. Therefore, v is a weak solution to (3.1) on $[0,T)$. Moreover, we can show that v is a unique weak solution to (3.1) as in the case of the two-dimensional Navier–Stokes equations (see e.g. [1]). By the boundedness of $\{M_\tau u^\varepsilon\}_\varepsilon$ and $\{\partial_t M_\tau u^\varepsilon\}_\varepsilon$ and the uniqueness of a weak solution to (3.1) we also have the convergence of the full sequence

$$\begin{aligned}\lim_{\varepsilon\to 0}M_\tau u^\varepsilon&=v &&\text{weakly in} && L^2(0,T;H^1(\Gamma,T\Gamma)),\\ \lim_{\varepsilon\to 0}\partial_t M_\tau u^\varepsilon&=\partial_t v &&\text{weakly in} && L^2(0,T;H^{-1}(\Gamma,T\Gamma)).\end{aligned} \tag{4.19}$$

Since the strong solution u^ε to (1.2) exists globally in time, by the above arguments we obtain a unique weak solution

$$v_T\in C([0,T];L^2_{g\sigma}(\Gamma,T\Gamma))\cap L^2(0,T;\mathcal{V}_g)\cap H^1(0,T;H^{-1}(\Gamma,T\Gamma))$$

to (3.1) on $[0,T)$ satisfying (4.19) for all $T>0$. Moreover, if $T<T'$ then $v_T=v_{T'}$ on $[0,T]$ by the uniqueness of a weak solution. Therefore, setting $v:=v_T$ on $[0,T]$ for each $T>0$ we can define a vector field

$$v\in C([0,\infty);L^2_{g\sigma}(\Gamma,T\Gamma))\cap L^2_{loc}([0,\infty);\mathcal{V}_g)\cap H^1_{loc}([0,\infty);H^{-1}(\Gamma,T\Gamma)),$$

which is a unique weak solution to (3.1) on $[0,\infty)$ and satisfies (4.19) for all $T>0$.

References

[1] F. Boyer and P. Fabrie, *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, vol. 183 of Applied Mathematical Sciences, Springer, New York, 2013.

[2] D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2), 92 (1970), pp. 102–163.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[4] J. K. Hale and G. Raugel, *A damped hyperbolic equation on thin domains*, Trans. Amer. Math. Soc., 329 (1992), pp. 185–219.

[5] ——, *Reaction-diffusion equation on thin domains*, J. Math. Pures Appl. (9), 71 (1992), pp. 33–95.

[6] L. T. Hoang, *Incompressible fluids in thin domains with Navier friction boundary conditions (I)*, J. Math. Fluid Mech., 12 (2010), pp. 435–472.

[7] L. T. Hoang and G. R. Sell, *Navier-Stokes equations with Navier boundary conditions for an oceanic model*, J. Dynam. Differential Equations, 22 (2010), pp. 563–616.

[8] D. Iftimie, G. Raugel, and G. R. Sell, *Navier-Stokes equations in thin 3D domains with Navier boundary conditions*, Indiana Univ. Math. J., 56 (2007), pp. 1083–1156.

[9] T. Jankuhn, M. A. Olshanskii, and A. Reusken, *Incompressible fluid problems on embedded surfaces: modeling and variational formulations*, Interfaces Free Bound., 20 (2018), pp. 353–377.

[10] H. Koba, C. Liu, and Y. Giga, *Energetic variational approaches for incompressible fluid systems on an evolving surface*, Quart. Appl. Math., 75 (2017), pp. 359–389.

[11] J.-L. Lions, R. Temam, and S. H. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity, 5 (1992), pp. 237–288.

[12] ——, *On the equations of the large-scale ocean*, Nonlinearity, 5 (1992), pp. 1007–1053.

[13] M. Mitrea and M. Taylor, *Navier-Stokes equations on Lipschitz domains in Riemannian manifolds*, Math. Ann., 321 (2001), pp. 955–987.

[14] T.-H. Miura, *Navier–stokes equations in a curved thin domain*, preprint, arXiv:1811.09816 [math.AP].

[15] ——, *On singular limit equations for incompressible fluids in moving thin domains*, Quart. Appl. Math., 76 (2018), pp. 215–251.

[16] T. Nagasawa, *Construction of weak solutions of the Navier-Stokes equations on Riemannian manifold by minimizing variational functionals*, Adv. Math. Sci. Appl., 9 (1999), pp. 51–71.

[17] M. Prizzi, M. Rinaldi, and K. P. Rybakowski, *Curved thin domains and parabolic equations*, Studia Math., 151 (2002), pp. 109–140.

[18] M. Prizzi and K. P. Rybakowski, *On inertial manifolds for reaction-diffusion equations on genuinely high-dimensional thin domains*, Studia Math., 154 (2003), pp. 253–275.

[19] G. Raugel and G. R. Sell, *Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc., 6 (1993), pp. 503–568.

[20] M. E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations, 17 (1992), pp. 1407–1456.

[21] ——, *Partial differential equations II. Qualitative studies of linear equations*, vol. 116 of Applied Mathematical Sciences, Springer, New York, second ed., 2011.

[22] R. Temam and M. Ziane, *Navier-Stokes equations in three-dimensional thin domains with various boundary conditions*, Adv. Differential Equations, 1 (1996), pp. 499–546.

[23] ——, *Navier-Stokes equations in thin spherical domains*, in Optimization methods in partial differential equations (South Hadley, MA, 1996), vol. 209 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1997, pp. 281–314.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo, 153-8914
Japan
E-mail address: `thmiura@ms.u-tokyo.ac.jp`

東京大学・大学院数理科学研究科　三浦達彦