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Remarks on test function methods for blowup of solutions to semilinear evolution equations in sectorial domain

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1. Introduction

This paper is a joint work with Dr. Masahiro Ikeda (Keio University/RIKEN) and a part of joint works [**6**] and [**5**]. In this paper we consider the following initial-boundary value problem

$$(1.1)\qquad \begin{cases} \tau\partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = |u(x,t)|^p, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0 & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=\varepsilon f(x),\ \tau\partial_t u(x,0)=\tau\varepsilon g(x), & x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ ($N\in\mathbb{N}$) is an unbounded domain, for instance,

$$\Omega_1=\mathbb{R}^N\setminus\overline{B(0,1)},\quad \text{or}\quad \Omega(\Sigma)=\{\rho\omega\in\mathbb{R}^N\ ;\ \rho>0,\omega\in\Sigma\}$$

with $\Sigma\subset S^{N-1}$ having smooth boundary. The region $\Omega(\Sigma)$ is so-called sectorial domain. The parameters $p\in(1,\frac{N}{(N-2)_+})$ and $\varepsilon>0$ describe the the effect of nonlinearity and the smallness of initial data, respectively. Finally, the constant $\tau\in\{0,1\}$ switches the parabolicity and hyperbolicity of the problem (1.1).

The interest of the present paper is the lifespan of blowup solutions to (1.1) for small initial data. Therefore we first fix the pair $(f,\tau g)$, then we discuss blowup of solutions to the problem (1.1) with sufficiently small $\varepsilon>0$.

The study of global existence and blowup of solutions to (1.1) has a long history. In the case $\tau=0$, the problem

$$(1.2)\qquad \begin{cases} \partial_t u(x,t)-\Delta u(x,t)=u(x,t)^p, & (x,t)\in\mathbb{R}^N\times(0,T),\\ u(x,0)=u_0(x)\ge 0, & x\in\mathbb{R}^N \end{cases}$$

is initially studied in [**2**] to understand the effects of dimension and nonlinearity. In [**2**], it is proved that

(i) if $1<p<1+\frac{2}{N}$, then (1.2) does not have non-trivial global-in-time solutions.

(ii) if $p>1+\frac{2}{N}$, then (1.2) possesses nontrivial global-in-time solutions for small initial data.

The exponent $p_F = 1 + \frac{2}{N}$ is called Fujita exponent. The same consequence as (i) for the critical case $p = p_F$ is given in Hayakawa [**3**], Sugitani [**25**] and Kobayashi–Sirao–Tanaka [**11**]. The lifespan estimate of solutions to (1.2) is also studied in Lee-Ni [**13**] by using the heat kernel and the maximum principle as

$$\text{(Lifespan of } u \text{ with } u_0 = \varepsilon f) \sim \begin{cases} C\varepsilon^{-(\frac{1}{p-1}-\frac{N}{2})^{-1}} & \text{if } 1 < p < p_F, \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_F. \end{cases}$$

In the case $\tau = 1$, the problem forms a Cauchy problem of the usual damped wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = |u(x,t)|^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = \varepsilon f(x), \ \partial_t u(x,0) = \varepsilon g(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

The blowup phenomena and estimates of the lifespan of solutions to (1.3) has been studied from the work of Li–Zhou [**16**]. They proved the blowup of solutions with upper lifespan estimates for $1 < p \le p_F$ when $N = 1, 2$. The three dimensional problem with sharp upper lifespan estimates is proved by Nishihara [**22**]. For general case, Todorova–Yordanov [**26**] showed blowup of small solutions for $1 < p < p_F$. Zhang [**29**] derived blowup of small solutions in the critical case $p = p_F$. For the lifespan estimates for the critical case $p = p_F$, recently, Lai–Zhou [**12**] succeeded in proving the sharp upper estimates by applying the consideration in [**13**]. The precise lifespan estimates for semilinear damped wave equation (1.3) is the same as that of semilinear heat equation (1.2).

Similar study of respective problems for halved spaces $\mathbb{R}^k \times \mathbb{R}^{N-k}$, for exterior domains and for sectorial domains has been separately done in the literature (see e.g., Meier [**20, 21**], Levine–Meier [**14, 15**], Ikehata [**7, 8, 9, 10**], Ogawa–Takeda [**23**], Pinsky [**24**] and Wakasugi [**28**]).

We point out that most of the blowup solutions in various equations (like (1.2) and (1.3)) can be treated in the framework of Mitidieri–Pokhozhaev [**17**] and also their lifespan estimates can be given by this argument with small modification (see Mitidieri–Pokhozhaev [**19, 18**]). However, in the critical case $p = p_F$ in (1.2) and (1.3), their argument does not give sharp lifespan estimates (see also, Ikeda–Ogawa [**4**]).

The purpose of the present paper is to propose a new test function method from the viewpoint of ordinary differential inequalities with respect to the parameter.

Here we introduce the definition of weak solutions to (1.1) which is used in the present paper.

Definition 1. For $(f, \tau g) \in H_0^1(\Omega) \times L^2(\Omega)$, the function $u : \Omega \times [0,T) \to \mathbb{R}$ is called the weak solution of (1.1) with initial data (f, g) in $(0,T)$ if u belongs the following class

$$S_T = \begin{cases} C([0,T); H_0^1(\Omega)) \cap C((0,T); L^{2p}(\Omega)) & \text{if } \tau = 0, \\ C^1([0,T); L^2(\Omega)) \cap C([0,T); H_0^1(\Omega)) \cap C([0,T); L^{2p}(\Omega)) & \text{if } \tau = 1 \end{cases}$$

and satisfies $u(0) = f$, $\tau\partial_t u(0) = \tau g$ and for every $\varphi \in C^1(\overline{\Omega} \times [0,T))$ with $\varphi = 0$ on $\partial\Omega \times [0,T)$,

$$\int_\Omega (\tau g + f)\varphi\,dx + \int_0^T\int_\Omega |u|^p\varphi\,dx\,dt = \int_0^T\int_\Omega \nabla u\cdot\nabla\varphi\,dx\,dt - \int_0^T\int_\Omega (\tau\partial_t u + u)\partial_t\varphi\,dx\,dt.$$

Remark 1.1. Existence of local-in-time weak solutions to (1.1) in the sense verifies by the standard argument of mild solutions

$$\begin{cases} U(t) = e^{t\Delta_D}f + \int_0^t e^{(t-s)\Delta_D}|u(s)|^p\,ds \text{ in } L^2(\Omega) & \text{if } \tau = 0,\\ u(t) = e^{-t\mathcal{A}}U_0 + \int_0^t e^{-(t-s)\mathcal{A}}\mathcal{N}(U(s))\,ds \text{ in } H_0^1(\Omega)\times L^2(\Omega) & \text{if } \tau = 1,\end{cases}$$

where $\mathcal{A}(u,v) = (-v, -\Delta_D u + v)$ and $\mathcal{N}(u,v) = (0, |u|^p)$. The domains of Δ_D and $\mathcal{A}$ are given by $D(\Delta_D) = H^2(\Omega)\cap H_0^1(\Omega)$ and $D(\mathcal{A}) = D(\Delta_D)\times H_0^1(\Omega)$. In this argument we require $p \le \frac{N}{N-2}$ (see e.g., Cazenave–Haraux [**1**]).

The main result of this paper could be Proposition 2.1, which provides a sufficient condition on the shape of domain and the parameter p for the blowup phenomena of small solutions to (1.1). In Section 2, a positive harmonic function satisfying Dirichlet boundary condition plays a crucial role. The profile of this function can be regarded as the one of the shape of Ω. In Section 3, we give a result of lifespan estimates of blowup solutions in specified domains (Propositions 3.1, 3.2 and 3.3) as corollaries of Proposition 2.1.

2. Analysis of blowup via a test function method

Here we assume that there exists a positive harmonic function Φ satisfying Dirichlet boundary condition, that is, $\Phi \in C(\overline{\Omega})\cap C^\infty(\Omega)$ satisfies

$$\begin{cases} \Delta\Phi(x) = 0 & x\in\Omega,\\ \Phi(x) > 0 & x\in\Omega,\\ \Phi(x) = 0 & x\in\partial\Omega.\end{cases} \tag{2.1}$$

Moreover, existence of a nonnegative auxiliary function $w \in C^2(\overline{\Omega})$ satisfying that there exists $k > 0$ such that

$$|\nabla w(x)|^2 \le k w(x), \quad |\Delta w| \le k. \tag{2.2}$$

is required. Note that unboundedness of Ω is required to ensure the existence of Φ.

Remark 2.1. Here we give examples of the choices of the pair (Φ, w) as follows:

$$(\Phi, w) = \begin{cases} (1, |x|^2) & \text{if } \Omega = \mathbb{R}^N\ (N\in\mathbb{N}),\\ (x_N, |x|^2) & \text{if } \Omega = \{x = (x_1,\dots,x_N)\in\mathbb{R}^N\ ;\ x_N > 0\},\\ (1-|x|^{2-N}, (|x|-1)^2) & \text{if } \Omega = \{x\in\mathbb{R}^N\ ;\ |x| > 1\}\ (N\ge 3),\\ (\log|x|, (|x|-1)^2) & \text{if } \Omega = \{x\in\mathbb{R}^2\ ;\ |x|>1\}.\end{cases}$$

The following assertion is the essential tool of the present work, which is derived via a test function method with a harmonic function Φ satisfying Dirichlet boundary condition.

Proposition 2.1. *Suppose that there exists a pair* (Φ, w) *such that* Φ *and* w *satisfy* (2.1) *and* (2.2)*, respectively. Assume that* $(f, \tau g) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $(\tau g + f)\Phi \in L^1(\Omega)$ *with*

$$c_0 = \int_\Omega (\tau g + f)\Phi\, dx > 0. \tag{2.3}$$

If the function

$$h(T) = \int_1^T \left(\int_{\{x\in\Omega\,;\, w(x)<R\}} \left(1 + \frac{|\nabla w \cdot \nabla \Phi|}{\Phi}\right)^{p'} \Phi\, dx \right)^{1-p} dR \tag{2.4}$$

diverges as $T \to \infty$*, then the solution of* (1.1) *with* $\varepsilon \ll 1$ *blows up until* $T = h^{-1}(C\varepsilon^{-(p-1)})$*, where* C *is a positive constant depending only on* N*,* p*,* f*,* τg*.*

Remark 2.2. Proposition 2.1 asserts that blowup phenomena of solutions to (1.1) with small initial data are governed by the relation between the structure of the positive harmonic function Φ satisfying Dirichlet boundary condition and the exponent of nonlinearity p.

Proof of Proposition 2.1. Here we fix two kinds of functions $\eta \in C^\infty([0,\infty))$ and $\eta^* \in L^\infty((0,\infty))$ as follows, which will be used in the cut-off functions:

$$\eta(s) \begin{cases} = 1 & \text{if } s \in [0,1/2] \\ \text{is decreasing} & \text{if } s \in (1/2,1) \\ = 0 & \text{if } s \in [1,\infty), \end{cases} \qquad \eta^*(s) = \begin{cases} 0 & \text{if } s \in [0,1/2), \\ \eta(s) & \text{if } s \in [1/2,\infty). \end{cases}$$

For $p > 1$, we define for $R > 0$,

$$\psi_R(x,t) = [\eta(s_R(x,t))]^{2p'}, \quad (x,t) \in \mathbb{R}^N \times [0,\infty),$$
$$\psi_R^*(x,t) = [\eta^*(s_R(x,t))]^{2p'}, \quad (x,t) \in \mathbb{R}^N \times [0,\infty)$$

with

$$s_R(x,t) = \frac{w(x)^2 + t}{R}.$$

We also set $P(R) = \operatorname{supp}\psi_R \cap (\Omega \times [0,R])$. The second function is useful in the sense of the equality $\frac{d}{ds}(\eta^q(s)) = q[\eta^*(s)]^{q-1}\eta'(s)$. This kind of test functions with $w(x) = |x|^\alpha$ is introduced in Mitidieri–Pokhozhaev [**17**].

Let u be a weak solution of (1.1) with initial data $(\varepsilon f, \varepsilon g)$ in $(0, T_\varepsilon)$. Without loss of generality, we assume that $T_\varepsilon > R_*$ with $R_* \geq 1$ satisfying for every $R \geq R_*$

$$I_R = \int_\Omega (\tau g + f)\Phi\psi_R(\cdot,0)\, dx \geq \frac{c_0}{2} > 0.$$

This is possible by a consequence of the dominated convergence theorem. For $R \in (R_*, T_\varepsilon)$, take $\varphi = \Phi\psi_R$ as a test function which satisfies Dirichlet boundary condition in view of (2.1). Noting that by (2.2),

$$\begin{aligned}
&|\tau\Phi\partial_t^2\psi_R - \Phi\partial_t\psi_R - \Delta(\Phi\psi_R)| \\
&\le \frac{2p'\tau}{R^2}\Big(|\eta''(s_R)|\eta(s_R) + (2p'-1)(\eta'(s_R))^2\Big)\Phi[\psi_R^*]^{\frac{1}{p}} + \frac{2p'}{R}|\eta'(s_R)|\eta(s_R)\Phi[\psi_R^*]^{\frac{1}{p}} \\
&\quad + \frac{2p'}{R}\Big(|\eta''(s_R)|\eta(s_R)|\Delta w| + (2p'-1)(\eta'(s_R))^2\frac{|\nabla w|^2}{R}\Big)\Phi[\psi_R^*]^{\frac{1}{p}} + \frac{2p'}{R}|\nabla w\cdot\nabla\Phi|[\psi_R^*]^{\frac{1}{p}} \\
&\le \frac{C'}{R}\left(1 + \frac{|\nabla w\cdot\nabla\Phi|}{\Phi}\right)\Phi[\psi_R^*]^{\frac{1}{p}},
\end{aligned}$$

where C is a positive constant depending only on $N, p, \|\eta\|_{W^{2,\infty}((0,\infty))}$ and k, we see from the definition of weak solutions and Hölder's inequality that

$$\begin{aligned}
\text{(2.5)}\qquad &\varepsilon I_R + \int_0^R\int_\Omega |u|^p\Phi\psi_R\,dx\,dt \\
&= \int_0^R\int_\Omega u\Big(\tau\Phi\partial_t^2\psi_R - \Phi\partial_t\psi_R - \Delta(\Phi\psi_R)\Big)\,dx\,dt \\
&\le C\left(R^{-p'}\iint_{P(R)}\Theta^{p'}\Phi\,dx\,dt\right)^{\frac{1}{p'}}\left(\int_0^R\int_\Omega |u|^p\Phi\psi_R^*\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned}$$

where

$$\Theta(x) = 1 + \frac{|\nabla w(x)\cdot\nabla\Phi(x)|}{\Phi(x)}.$$

Remark that the previous computation requires $\Delta\Phi = 0$. Now we define

$$Y(R) = \int_0^R\left(\int_0^\rho\int_\Omega |u|^p\Phi\psi_\rho^*\,dx\,dt\right)\frac{d\rho}{\rho}, \quad R_* \le R < T_\varepsilon.$$

Then as in [**6**, Lemma 3.9], we have $Y \in C^1([R_*, T_\varepsilon))$ and

$$Y'(R) = \frac{1}{R}\int_0^R\int_\Omega |u|^p\Phi\psi_R^*\,dx\,dt, \quad Y(R) \le \int_0^R\int_\Omega |u|^p\Phi\psi_R\,dx\,dt.$$

On the other hand, by the definition of $P(R)$, we have

$$\iint_{P(R)}\Theta^{p'}\Phi\,dx\,dt \le \int_0^R\int_{\{x\in\Omega\,;\,w(x)<R\}}\Theta^{p'}\Phi\,dx\,dt = R\int_{\{x\in\Omega\,;\,w(x)<R\}}\Theta^{p'}\Phi\,dx.$$

Therefore (2.5) is reduced to the following ordinary differential inequality with respect to the parameter R:

$$\text{(2.6)}\qquad \left(\frac{c_0\varepsilon}{2} + Y(R)\right)^p \le C^p\left(\int_{\{x\in\Omega\,;\,w(x)<R\}}\Theta^{p'}\Phi\,dx\right)^{p-1} Y'(R).$$

Solving the above equation, we deduce

$$\begin{aligned}
0 &\le \left(\frac{c_0\varepsilon}{2} + Y(R)\right)^{1-p} \\
&\le \left(\frac{c_0\varepsilon}{2} + Y(R_*)\right)^{1-p} - (p-1)C^{-p}\int_{R_*}^{R}\left(\int_{\{x\in\Omega\ ;\ w(x)<\rho\}} \Theta^{p'}\Phi\,dx\right)^{1-p} d\rho \\
&\le \left(\frac{c_0\varepsilon}{2}\right)^{1-p} - (p-1)C^{-p}\int_{R_*}^{R}\left(\int_{\{x\in\Omega\ ;\ w(x)<\rho\}} \Theta^{p'}\Phi\,dx\right)^{1-p} d\rho.
\end{aligned}$$

If (2.4) is satisfied, then we have $R \le R_\varepsilon$ such that

$$\left(\frac{c_0\varepsilon}{2}\right)^{1-p} = (p-1)C^{-p}\int_{R_*}^{R_\varepsilon}\left(\int_{\{x\in\Omega\ ;\ w(x)<\rho\}} \Theta^{p'}\Phi\,dx\right)^{1-p} d\rho.$$

Since $R \in [R_*, T_\varepsilon)$ is arbitrary, we obtain $T_\varepsilon \le R_\varepsilon$. □

3. The result of the problem in particular cases

In this section we give blowup results for the following cases:

- the whole space case $\Omega = \mathbb{R}^N$
- the case $\Omega = \Omega_1 = \{x \in \mathbb{R}^N\ ;\ |x| > 1\}$ $(N \ge 2)$
- the case $\Omega = \Omega(\Sigma) = \{\rho\omega \in \mathbb{R}^N\ ;\ \rho > 0, \omega \in \Sigma\}$ with $\Sigma \subset S^{N-1}$ $(N \ge 2)$

as application of Proposition 2.1.

3.1 The case $\Omega = \mathbb{R}^N$

The following assertion is well-known as mentioned in Introduction.

Proposition 3.1 (The case of $\Omega = \mathbb{R}^N$)**.** *Let $(f, \tau g) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with*

$$\tau g + f \in L^1(\mathbb{R}^N), \ \int_{\mathbb{R}^N} (\tau g + f)\,dx > 0.$$

If $1 < p \le p_F$, then the solution u of (1.1) blows up until

$$T = \begin{cases} C\varepsilon^{-(\frac{1}{p-1}-\frac{N}{2})^{-1}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = 1 + \frac{2}{N}. \end{cases}$$

Proof. As in Remark 2.1, we choose

$$\Phi(x) \equiv 1, \quad w(x) = |x|^2$$

which satisfy (2.1) and (2.2), respectively. Then the function h in (2.4) is given by

$$\begin{aligned} h(T) &= \int_1^T \left(\int_{\{x\in\mathbb{R}^N\ ;\ w(x)<R\}} \left(1 + \frac{|\nabla w\cdot\nabla\Phi|}{\Phi}\right)^{p'} \Phi\,dx \right)^{1-p} dR \\ &= |B(0,1)|^{1-p} \int_1^T R^{-\frac{N}{2}(p-1)}\,dR \\ &= \begin{cases} \frac{|B(0,1)|^{1-p}}{1-\frac{N}{2}(p-1)}(T^{1-\frac{N}{2}(p-1)} - 1) & \text{if } 1 < p < 1 + \frac{2}{N} \\ |B(0,1)|^{1-p}\log T & \text{if } p = 1 + \frac{2}{N}, \end{cases} \end{aligned}$$

where $|B(0,1)| = \int_{B(0,1)} dx$. Therefore Proposition 2.1 implies the desired upper bound for the lifespan of solution u. □

3.2 Exterior problem $\Omega = \Omega_1$ $(N \geq 2)$

Next assertion is a blowup result of small solutions to (1.1) in the exterior domain Ω_1.

Proposition 3.2 (The case of $\Omega = \Omega_1$)**.** *Let $(f, \tau g) \in H_0^1(\Omega_1) \times L^2(\Omega_1)$ with*

$$(\tau g + f)\log|x| \in L^1(\Omega_1), \quad \begin{cases} \int_{\Omega_1}(\tau g + f)(1 - |x|^{2-N})\,dx > 0, & \text{if } N \geq 3, \\ \int_{\Omega_1}(\tau g + f)\log|x|\,dx > 0, & \text{if } N = 2. \end{cases}$$

If $1 < p \leq p_F$, then the solution u of (1.1) blows up until

$$T = \begin{cases} C\varepsilon^{-(\frac{1}{p-1}-\frac{N}{2})^{-1}} & \text{if } N \geq 3, 1 < p < 1 + \frac{2}{N}, \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } N \geq 3,\ p = 1 + \frac{2}{N}, \\ C(\varepsilon^{-1}\log(\varepsilon^{-1}))^{\frac{p-1}{2-p}} & \text{if } N = 2, 1 < p < 2, \\ \exp(\exp(C\varepsilon^{-(p-1)})) & \text{if } N = 2, p = 2. \end{cases}$$

Remark 3.1. The cases $N \geq 3$ and $N = 2$, $1 < p < 2$ are known (See Pinsky [**24**]). The main contribution of the paper [**5**] is the two-dimensional critical case $N = p = 2$.

Proof of Proposition 3.2. First we treat the case $N \geq 3$. We choose

$$\Phi(x) = 1 - |x|^{2-N}, \quad w(x) = (|x| - 1)^2.$$

Then we easily see that Φ satisfies $\Delta\Phi = 0$ and

$$|\nabla w|^2 = 4w, \quad |\Delta w| \leq 2N + 4, \quad |\nabla w \cdot \nabla\Phi| \leq 2(N-2)\Phi.$$

By direct computation the function h in (2.4) can be estimated as follows:

$$h(T) \geq C\int_1^T \left(\int_{\{x\in\mathbb{R}^N\ ;\ |x|<2R^{\frac{1}{2}}\}} dx \right)^{1-p} dR.$$

By Proposition 2.1, we obtain the same conclusion as the case of $\Omega = \mathbb{R}^N$.

If $N=2$, then we change the choice of Φ as

$$\Phi = \log|x|.$$

Then noting that

$$|\nabla w \cdot \nabla \Phi| = 2\frac{|x|-1}{|x|} \leq 2\log|x| \leq 2\Phi,$$

we deduce that the function h in (2.4) is estimated as follows:

$$\begin{aligned} h(T) &\geq C\int_1^T (R\log R)^{1-p}\, dR \\ &\geq \begin{cases} CT^{2-p}(\log T)^{1-p} & \text{if } 1<p<2, \\ C\log\log T & \text{if } p=2. \end{cases} \end{aligned}$$

Proposition 2.1 implies the desired upper bound for the lifespan of solution u. $\square$

3.3 Problems in $\Omega(\Sigma) = \{\rho\omega \in \mathbb{R}^N\ ;\ \rho > 0, \omega \in \Sigma\}$ $(N \geq 2)$

Finally, we treat the case of sectorial domain $\Omega = \Omega(\Sigma)$. By using Friedrichs extension, we can find local-in-time weak solutions to (1.1) for every $(f, \tau g) \in H^1_0(\Omega(\Sigma)) \times L^2(\Omega(\Sigma))$. The condition on essential selfadjointness of Laplacian Δ endowed with domain

$$D = \{u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})\ ;\ u = 0 \text{ on } \partial\Omega(\Sigma)\ (= \Omega(\partial\Sigma))\}$$

is written in [**6**].

To state the result for sectorial domain, First we state the assertion for the first eigenvalue and eigenfunction of the Laplace–Beltrami operator in Σ endowed with Dirichlet boundary condition (see [**27**, Chapter IX] for detail).

Lemma 1. *The Laplace–Beltrami operator $-\Delta_\Sigma$ in $L^2(\Sigma)$ endowed with domain $H^2(\Sigma) \cap H^1_0(\Sigma)$ is selfadjoint and all resolvent operator of $-\Delta_\Sigma$ are compact. The first eigenvalue λ_Σ is nonnegative and simple, and the corresponding eigenfunction φ_Σ is positive in Σ. Moreover, λ_Σ is positive if and only if $\Sigma \neq S^{N-1}$.*

Here we define γ as a smallest root of the quadratic equation $\gamma^2 + (N-2)\gamma - \lambda_\Sigma = 0$. Then the positive harmonic function on $\Omega(\Sigma)$ satisfying Dirichlet boundary condition is given as follows.

Lemma 2. *Set*

$$\Phi_\Sigma(x) = |x|^\gamma \varphi_\Sigma\left(\frac{x}{|x|}\right), \quad x \in \Omega(\Sigma).$$

Then Φ_Σ satisfies

$$\begin{cases} \Delta\Phi_\Sigma(x) = 0 & x \in \Omega(\Sigma), \\ \Phi_\Sigma(x) > 0 & x \in \Omega(\Sigma), \\ \Phi_\Sigma(x) = 0 & x \in \partial\Omega(\Sigma), \\ x \cdot \nabla\Phi_\Sigma(x) = \gamma\Phi_\Sigma(x) & x \in \Omega(\Sigma). \end{cases}$$

According to the blowup result with upper lifespan estimates for the case of $\Omega = \Omega(\Sigma)$ is the following. The critical exponent for (1.1) in $\Omega(\Sigma)$ seems to depend on γ which comes from the first eigenvalue of Laplace–Beltrami operator $-\Delta_\Sigma$.

Proposition 3.3 (The case of $\Omega = \Omega(\Sigma)$)**.** *Let* $(f, \tau g) \in H_0^1(\Omega(\Sigma)) \times L^2(\Omega(\Sigma))$ *with*

$$(\tau g + f)\Phi_\Sigma \in L^1(\Omega(\Sigma)), \quad \int_{\Omega(\Sigma)} (\tau g + f)\Phi_\Sigma \, dx > 0.$$

If $1 < p \le p_F$*, then the solution* u *of* (1.1) *blows up until*

$$T = \begin{cases} C\varepsilon^{-(\frac{1}{p-1} - \frac{N+\gamma}{2})^{-1}} & \text{if } 1 < p < 1 + \frac{2}{N+\gamma}, \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = 1 + \frac{2}{N+\gamma}. \end{cases}$$

Proof. It is verified by choosing $\Phi = \Phi_\Sigma$ and $w(x) = |x|^2$. $\square$

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