Equations and dynamic boundary conditions of Allen–Cahn type and their approximation with Robin boundary conditions

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Abstract

In this note we summarize the main findings of some recent works [4, 9] for a class of Allen–Cahn equation with dynamic boundary conditions and affine linear transmission conditions, and their approximation by Robin boundary conditions.

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1 Introduction

The Allen–Cahn equation [1] is an important phenomenological model used to describe the motion of antiphase boundaries in binary alloys. For a bounded domain $\Omega \subset \mathbb{R}^d$, the equation can be obtained as the L^2-gradient flow of the energy functional

$$E(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\,dx$$

where $\varepsilon > 0$ is a small parameter related to the interfacial thickness of the thin regions separating the alloys, and F is a double well potential. The typical example is a smooth potential $F(s) = (s^2-1)^2$, but non-smooth (or singular) potentials such as

$$F(s) = \frac{1}{2}[(1+s)\log(1+s) + (1-s)\log(1-s)] - \frac{\theta}{2}s^2,$$

$$F(s) = \begin{cases} \frac{1}{2}(1-s^2), & s \in [-1,1], \\ +\infty, & s \notin [-1,1]. \end{cases}$$

have also been studied. The above energy E is commonly known as the Ginzburg–Landau energy functional or the Modica–Mortola energy functional. The latter name is attributed to the well-known result by Modica and Mortola [11] that $E(u)$ converges to a scalar multiple of the perimeter functional in the sense of Γ-convergence as $\varepsilon \to 0$, which allows for the interpretation that the Allen–Cahn equation

$$u_t = \Delta u - \frac{1}{\varepsilon^2} F'(u)$$

is an approximation of the L^2-gradient flow of the perimeter functional, i.e., the geometric motion by mean curvature, as $\varepsilon \to 0$.

For classical mathematical analysis of the Allen–Cahn equation, many authors often consider homogeneous Neumann or Dirichlet boundary conditions:

$$\partial_n u = 0 \text{ (Neumann) or } u = 0 \text{ (Dirichlet) on } \partial\Omega.$$

One then obtains the following energy identity for solutions to the Allen–Cahn equation:

$$\frac{d}{dt} E(u(t)) + \int_\Omega |u_t|^2 \, dx = 0 \text{ for } t > 0.$$

Recently, physicists [6, 8] have proposed to include effective short-range interactions between the bulk domain Ω and its boundary $\partial\Omega$ by introducing an additional energy functional defined on the boundary. We use the notation $\Gamma := \partial\Omega$, and define the surface energy

$$E_s(\phi) := \int_\Gamma \frac{\kappa}{2} |\nabla_\Gamma \phi|^2 + G(\phi) \, dS$$

where ∇_Γ denotes the surface gradient operator defined on Γ, and G is a surface potential function that may account for potential phase separation on the boundary. When the coefficient κ is positive, we allow for the possibility of lateral diffusion on the boundary.

In the sequel, we set $\varepsilon = \kappa = 1$ as their values have no consequence with the analytical results we report below. By combining the bulk energy E and the surface energy E_s, taking the surface variable ϕ as the trace of the bulk variable u, and then computing the associated L^2-gradient flow, one obtains the following Allen–Cahn system:

$$\begin{aligned} u_t &= \Delta u - F'(u) \text{ in } \Omega, \\ \phi_t &= \Delta_\Gamma \phi - G'(\phi) - \partial_n u \text{ on } \Gamma, \\ u &= \phi \text{ on } \Gamma \end{aligned} \tag{1.1}$$

where Δ_Γ is the Laplace–Beltrami operator on Γ. The second and third equations can be combined to read as

$$u_t = \Delta_\Gamma u - G'(u) - \partial_n u \text{ on } \Gamma,$$

which in the literature is called a dynamic boundary condition for the bulk variable u. Furthermore, in light of the dynamic boundary condition, we obtain the energy identity

$$\frac{d}{dt}\Big(E(u(t)) + E_s(u(t))\Big) + \int_\Omega |u_t|^2 \, dx + \int_\Gamma |u_t|^2 \, dS = 0 \text{ for } t > 0.$$

Since their introduction, many authors have studied the Allen–Cahn equation with dynamic boundary conditions, see for instance [2, 3, 5, 14], and the references cited in [4]. For the rest of this note, we report on some recent mathematical investigations on a modification of (1.1), in which the third equation is replaced by

$$u = h(\phi) \text{ on } \Gamma$$

for some continuous function $h : \mathbb{R} \to \mathbb{R}$. For the special case $h(s) = s$, we recover the original system (1.1), and so our investigation aims to generalise the current results in the literature [2, 3, 14] for equations and dynamic boundary conditions of Allen–Cahn type. By setting g as the inverse of h, i.e., $\phi = g(u)$, using the relation $h'(\phi) = (g'(u))^{-1}$ we can reformulate (1.1) as

$$\begin{aligned} u_t - \Delta u + F'(u) &= 0 \text{ in } \Omega, \\ g'(u)((g(u))_t - \Delta_\Gamma(g(u)) + G'(g(u))) + \partial_n u &= 0 \text{ on } \Gamma. \end{aligned} \tag{1.2}$$

We immediately observe that there will be some difficult in passing to the limit for some approximation scheme (such as Faedo–Galerkin) to recover the Laplace–Beltrami term if g is a nonlinear function. This is more evident in the weak formulation of (1.2):

$$\begin{aligned} 0 = &\int_\Omega (u_t + F'(u))\zeta + \nabla u \cdot \nabla \zeta \, dx \\ &+ \int_\Gamma g'(u)\Big((g(u))_t + G'(g(u))\Big)\zeta + \nabla_\Gamma g(u) \cdot \nabla_\Gamma \Big(g'(u)\zeta\Big) \, dS \end{aligned}$$

for all $\zeta \in H^1(\Omega)$ such that $\zeta|_\Gamma \in H^1(\Gamma)$. In particular, for an approximation scheme such as Faedo–Galerkin approximation, the most difficult part in passing to the limit is the term with the highly nonlinear surface gradient. However, in the case where g (and also h) is an affine linear function, i.e., $g(s) = \alpha^{-1}(s - \beta)$ for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$, so that $h(s) = \alpha s + \beta$, we are able to establish strong well-posedness to the system

$$\begin{aligned} u_t - \Delta u + F'(u) &= 0 \text{ in } \Omega, \\ \phi_t - \Delta_\Gamma \phi + G'(\phi) + \alpha \partial_n u &= 0 \text{ on } \Gamma, \\ u &= \alpha\phi + \beta \text{ on } \Gamma, \end{aligned} \tag{1.3}$$

where by strong solutions we mean that the above equations are satisfied a.e. in Ω and a.e. on Γ. Indeed, substituting the relation $u = \alpha\phi + \beta$ into the surface equation, we can express (1.3) equivalently as

$$\begin{aligned} u_t - \Delta u + F'(u) &= 0 \text{ in } \Omega, \\ u_t - \Delta_\Gamma u + \alpha G'(\alpha^{-1}(u - \beta)) + \alpha^2 \partial_n u &= 0 \text{ on } \Gamma, \end{aligned}$$

which differs only slightly from the standard case $\alpha = 1$, $\beta = 0$ studied previously in the literature.

To tackle the more general problem (1.2), we turn to a well-known technique in numerical analysis, in which we replace the Dirichlet-like condition $u = h(\phi)$ by a Robin boundary condition and study the system

$$\begin{aligned} u_t^K - \Delta u^K + F'(u^K) &= 0 \text{ in } \Omega, \\ \phi_t^K - \Delta_\Gamma \phi^K + G'(\phi^K) + h'(\phi^K)\partial_n u^K &= 0 \text{ on } \Gamma, \\ K\partial_n u^K &= h(\phi^K) - u^K \text{ on } \Gamma. \end{aligned} \tag{1.4}$$

In the formal limit $K \to 0$, if $u^K \to u$ and $\phi^K \to \phi$ for some limit functions (u, ϕ), then we recover the condition $u = h(\phi)$ on Γ, which serves to motivate the parallel study of (1.4) in addition to (1.3).

2 Main Results

We work with the following assumptions:

(1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary Γ.

(2) The function $h \in C^2(\mathbb{R})$ with $h', h'' \in L^\infty(\mathbb{R})$.

(3) F and G are $C^3(\mathbb{R})$ functions satisfying

$$\begin{aligned} |F'''(s)| \leq c_0(1 + |s|^p), \quad |G'''(s)| &\leq c_0(1 + |s|^q) \text{ for all } s \in \mathbb{R}, \\ F(s) \geq c_1|s| - c_2, \quad G(s) &\geq c_1|s| - c_2 \text{ for } |s| > c_3, \\ F''(s) \geq -c_4, \quad G''(s) &\geq -c_4 \text{ for all } s \in \mathbb{R} \end{aligned}$$

for positive constants $c_0, \ldots, c_4$ with exponents $p \in [0, 3)$ and $q \in [0, \infty)$.

(4) The initial data $(u_0, \phi_0) \in H^2(\Omega) \times H^2(\Gamma)$ such that $K\partial_n u_0 + u_0 = h(\phi_0)$ holds a.e. on Γ.

While in the introduction we mentioned non-smooth potentials, which are excluded by these assumptions, we will come back to them at the last section of this note. Unless stated otherwise, we assume that assumptions (1)–(4) hold throughout this note.

The first result is a strong well-posedness theorem for the Robin system (1.4).

Theorem 2.1 ([9, Theorems 2.1, 2.2]). *For any $\delta > 0$, there exist a unique pair of functions (u, ϕ) with*

$$\begin{aligned} &u \in L^\infty(0, \infty; H^2(\Omega)) \cap L^\infty(\delta, \infty; H^3(\Omega)), \\ &u_t \in L^\infty(\delta, \infty; H^1(\Omega)) \cap L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)), \\ &\phi \in L^\infty(0, \infty; H^2(\Gamma)) \cap L^\infty(\delta, \infty; H^3(\Gamma)), \end{aligned}$$

$$\phi_t \in L^\infty(\delta,\infty;H^1(\Gamma)\cap L^\infty(0,\infty;L^2(\Gamma))\cap L^2(0,\infty;H^1(\Gamma)),$$

that satisfy the equations in (1.4) *a.e. in* Ω *and a.e. on* Γ *with* $u(0) = u_0$, $\phi(0) = \phi_0$. *Furthermore, if* $(u_i, \phi_i)_{i=1,2}$ *denote two solutions to* (1.4) *corresponding to initial data* $(u_{0,i}, \phi_{0,i})_{i=1,2}$, *then, there exist positive constants* c_1 *and* c_2 *depending on the initial data,* Ω *and* Γ *such that*

$$\begin{aligned}&\Big(\|u_1(t)-u_2(t)\|^2_{L^2(\Omega)} + \|\phi_1(t)-\phi_2(t)\|^2_{L^2(\Gamma)}\Big)\\&\quad+\int_0^t \Big(\|u_1(s)-u_2(s)\|^2_{H^1(\Omega)} + \|\phi_1(s)-\phi_2(s)\|^2_{H^1(\Gamma)}\Big)\,ds\\&\quad\le c_1 e^{c_2 t}\Big(\|u_{0,1}-u_{0,2}\|^2_{L^2(\Omega)} + \|\phi_{0,1}-\phi_{0,2}\|^2_{L^2(\Gamma)}\Big).\end{aligned} \tag{2.5}$$

The above theorem yields the global well-posedness of the Robin problem (1.4), and in light of the solution existing for all positive times, the next series of results address the long-time behaviour of the solution.

We begin first with the stationary problem of (1.4), which reads as

$$\begin{aligned}-\Delta u_* + F'(u_*) &= 0 \text{ in } \Omega,\\ K\partial_n u_* + u_* &= h(\phi_*) \text{ on } \Gamma,\\ -\Delta_\Gamma \phi_* + G'(\phi_*) + K^{-1}h'(\phi_*)\partial_n u_* &= 0 \text{ on } \Gamma.\end{aligned} \tag{2.6}$$

It turns out that solutions to (2.6) have a characterisation, as the following theorem shows.

Theorem 2.2 ([9, Theorem 3.1]). *A pair* $(u_*, \phi_*) \in H^2(\Omega)\times H^2(\Gamma)$ *is a strong solution to the stationary problem* (2.6) *if and only if* (u_*, ϕ_*) *is a critical point to the functional*

$$\mathcal{E}(u,\phi) := \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx + \int_\Gamma \frac{1}{2}|\nabla_\Gamma \phi|^2 + G(\phi) + \frac{1}{2K}|u-h(\phi)|^2\,dS.$$

We now strengthening the regularity assumptions of h, F and G to analytical regularities. This is essential as we aim to establish an extended Łojasiewicz–Simon inequality for critical points $(u_*, \phi_*) \in H^2(\Omega)\times H^2(\Gamma)$ of the energy $\mathcal{E}$. Let $\mathcal{M} : (H^1(\Omega)\times H^1(\Gamma)) \to (H^1(\Omega)\times H^1(\Gamma))^*$ be defined as

$$\begin{aligned}&\langle \mathcal{M}(u,\phi),(w,\xi)\rangle_{H^1(\Omega)\times H^1(\Gamma)}\\&\quad:= \int_\Omega \nabla u\cdot\nabla w + F'(u)w\,dx + \int_\Gamma \nabla_\Gamma\phi\cdot\nabla_\Gamma\xi + G'(\phi)\xi\,dS\\&\quad+\int_\Gamma \frac{1}{K}(u-h(\phi))(w-h'(\phi)\xi)\,dS,\end{aligned} \tag{2.7}$$

that is, $\mathcal{M}(u,\phi)$ can be seen as the first variation of $\mathcal{E}$ at (u,ϕ). Then, we have the following:

Theorem 2.3 ([9, Theorem 4.1]). *Let (u_*, ϕ_*) be any critical point of the energy $\mathcal{E}$. There exist $\theta \in (0, \frac{1}{2})$ and $\gamma > 0$ depending on (u_*, ϕ_*) such that for any $(u, \phi) \in H^1(\Omega) \times H^1(\Gamma)$ satisfying*

$$\|(u,\phi) - (u_*, \phi_*)\|_{H^1(\Omega)\times H^1(\Gamma)} = \sqrt{\|u - u_*\|^2_{H^1(\Omega)} + \|\phi - \phi_*\|^2_{H^1(\Gamma)}} < \gamma,$$

it holds that

$$|\mathcal{E}(u,\phi) - \mathcal{E}(u_*, \phi_*)|^{1-\theta} \leq \|\mathcal{M}(u,\phi)\|_{(H^1(\Omega)\times H^1(\Gamma))^*}. \tag{2.8}$$

One important point to note is the above theorem asserts that the Łojasiewicz–Simon inequality (2.8) is valid for any pair of function (u, ϕ), even if they are not solutions to (1.4), as long as they are sufficiently close to the equilibrium point (u_*, ϕ_*). In particular, the Łojasiewicz–Simon inequality is only a statement about an energy functional $\mathcal{E}$, its first variation $\mathcal{M}$, and its critical points (u_*, ϕ_*).

However, if (u, ϕ) is a solution to (1.4), we can refine the above result, namely:

Theorem 2.4 ([9, Theorem 4.2]). *Let (u_*, ϕ_*) be any critical point of the energy $\mathcal{E}$. There exist $\theta \in (0, \frac{1}{2})$ and $\gamma > 0$ depending on (u_*, ϕ_*) such that for any strong solution (u, ϕ) to (1.4) satisfying*

$$\|(u,\phi) - (u_*, \phi_*)\|_{H^2(\Omega)\times H^2(\Gamma)} < \gamma,$$

it holds that

$$\begin{aligned}|\mathcal{E}(u,\phi) - \mathcal{E}(u_*, \phi_*)|^{1-\theta} &\leq \|F'(u) - \Delta u\|_{L^2(\Omega)} + \|G'(\phi) - \Delta_\Gamma \phi + K^{-1} h'(\phi)\partial_n u\|_{L^2(\Gamma)} \\ &= \|u_t\|_{L^2(\Omega)} + \|\phi_t\|_{L^2(\Gamma)}.\end{aligned}$$

Notice that the norms have been modified from $H^1(\Omega) \times H^1(\Gamma)$ to $H^2(\Omega) \times H^2(\Gamma)$ for the closeness to the critical point, and the right-hand side of the Łojasiewicz–Simon inequality has been modified from $\|\mathcal{M}(u,\phi)\|_{(H^1(\Omega)\times H^1(\Gamma))^*}$ to $\|\mathcal{M}(u,\phi)\|_{L^2(\Omega)\times L^2(\Gamma)}$, which according to (2.7) is

$$\begin{aligned}\|\mathcal{M}(u,\phi)\|_{L^2(\Omega)\times L^2(\Gamma)} &= \|F'(u) - \Delta u\|_{L^2(\Omega)} + \|G'(\phi) - \Delta_\Gamma \phi + K^{-1} h'(\phi)\partial_n u\|_{L^2(\Gamma)} \\ &= \|u_t\|_{L^2(\Omega)} + \|\phi_t\|_{L^2(\Gamma)}\end{aligned}$$

if (u, ϕ) is a strong solution to (1.4).

Thanks to the extended Łojasiewicz–Simon inequality we can establish the long-time behaviour of solutions to (1.4). This is formulated as follows.

Theorem 2.5 ([9, Theorem 5.4]). *For any initial condition (u_0, ϕ_0) satisfying condition (4), the unique global strong solution to (1.4) converges to an equilibrium $(u_*, \phi_*) \in H^2(\Omega) \times H^2(\Gamma)$ which is a strong solution to the stationary problem. Moreover, there exist a positive constant C and $\theta \in (0, \frac{1}{2})$ depending on (u_*, ϕ_*) such that*

$$\|(u_*, \phi_*) - (u(t), \phi(t))\|_{H^2(\Omega)\times H^2(\Gamma)} \leq C(1+t)^{-\theta/(1-2\theta)}.$$

We now turn to the limit problem of (1.4) as $K \to 0$, which was the original motivation to study the Robin problem. Due to the highly nonlinear nature of (1.2) we are only able to investigate the well-posedness and convergence in the limit $K \to 0$ for the case of affine linear relations, i.e., the problem (1.3). The first result concerns the strong well-posedness of (1.3).

Theorem 2.6 ([4, Theorem 3.6]). *Let $u_0 \in H^2(\Omega)$ with $F(u_0) \in L^1(\Omega)$ and $G(\alpha^{-1}(u_0 - \beta)) \in L^1(\Gamma)$. Then, for any $T > 0$, there exists a unique strong solution (u, ϕ) satisfying*

$$u \in L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)),$$
$$\phi = \alpha^{-1}(u|_\Gamma - \beta) \in L^\infty(0,T;H^2(\Gamma)) \cap H^1(0,T;H^1(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma))$$

to (1.3) with $u(0) = u_0$. Furthermore, if $(u_i, \phi_i)_{i=1,2}$ denote two solutions to (1.3) corresponding to initial data $(u_{0,i}, \phi_{0,i})_{i=1,2}$, then, there exist positive constants c_1 and c_2 such that (2.5) holds.

The next result states the weak convergence of solutions (u_K, ϕ_K) of (1.4) to the solution (u, ϕ) of (1.3) as $K \to 0$.

Theorem 2.7 ([4, Theorem 3.4]). *For $K > 0$, let (u_K, ϕ_K) be the unique strong solution to (1.4) with the specific case $h(s) = \alpha s + \beta$ for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$, with corresponding initial condition $(u_{0,K}, \phi_{0,K})$. Suppose there exists functions $u_0 \in H^1(\Omega)$ and $\phi_0 \in H^1(\Gamma)$ such that*

$$u_{0,K} \to u_0 \text{ in } H^1(\Omega), \quad \phi_{0,K} \to \phi_0 \text{ in } H^1(\Gamma),$$

with

$$u_0 = h(\phi_0) = \alpha\phi_0 + \beta \text{ on } \Gamma, \quad \|u_{0,K} - h(\phi_{0,K})\|^2_{L^2(\Gamma)} \leq CK$$

for some positive constant C independent of K. Then, it holds that

$$u_K \to u \text{ weakly-* in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$
$$\phi_K \to \phi \text{ weakly-* in } L^\infty(0,T;H^1(\Gamma)) \cap H^1(0,T;L^2(\Gamma)),$$
$$u_K - h(\phi_K) \to 0 \text{ strongly in } L^2(0,T;L^2(\Gamma)),$$

where the pair of function (u, ϕ) is a weak solution to (1.3) in the following sense: for all $\zeta \in H^1(\Omega)$ such that $\zeta|_\Gamma \in H^1(\Gamma)$ and for a.e. $t \in (0,T)$ it holds that

$$0 = \int_\Omega u_t\zeta + \nabla u \cdot \nabla\zeta + F'(u)\zeta \, dx + \int_\Gamma \frac{1}{\alpha}\Big(\phi_t\zeta|_\Gamma + G'(\phi)\zeta|_\Gamma + \nabla_\Gamma\phi \cdot \nabla_\Gamma\zeta|_\Gamma\Big) \, dS$$

and $u(0) = u_0$, $\phi(0) = \phi_0$.

Thanks to the strong well-posedness of (1.3) and of (1.4), it is possible to establish strong convergence of (u_K, ϕ_K) to (u, ϕ) and also obtain a rate of convergence. This is given in the following theorem.

Theorem 2.8 ([4, Theorem 3.7]). *For $K > 0$, let (u_K, ϕ_K) denote the unique strong solution to* (1.4) *corresponding to initial condition $(u_{0,K}, \phi_{0,K})$, and let (u, ϕ) denote the unique strong solution to* (1.3) *corresponding to initial condition (u_0, ϕ_0), where $\phi = \alpha^{-1}(u|_\Gamma - \beta)$ and $\phi_0 = \alpha^{-1}(u_0|_\Gamma - \beta)$. Suppose further that F and G have the following decomposition:*

$$F(s) = \hat{\beta}(s) + \hat{\pi}(s), \quad G(s) = \hat{\beta}_\Gamma(s) + \hat{\pi}_\Gamma(s) \quad \forall s \in \mathbb{R},$$

where $\hat{\beta}, \hat{\beta}_\Gamma \in C^2(\mathbb{R})$ are convex, while $\hat{\pi}, \hat{\pi}_\Gamma \in C^2(\mathbb{R})$ have globally Lipschitz derivatives. Then, there exists a positive constant C independent of K such that

$$\begin{aligned}&\|u_K - u\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \|\phi_K - \phi\|^2_{L^\infty(0,T;L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma))}\\ &\quad + K^{-1}\|\alpha\phi_k + \beta - u_K\|^2_{L^2(0,T;L^2(\Gamma))}\\ &\le C\Big(\|u_{0,K} - u_0\|^2_{L^2(\Omega)} + \|\phi_{0,K} - \phi_0\|^2_{L^2(\Gamma)} + K\|\partial_n u\|^2_{L^2(0,T;L^2(\Gamma))}\Big).\end{aligned}$$

Let us mention that the assumption on the splitting of F and G into a convex part and a non-convex part is a natural assumption, since many of the potentials (such as those discussed in the Introduction) used in the literature exhibit this kind of decomposition. Moreover, we note that in order for the above error estimate to be valid, the minimum regularity for the solution (u, ϕ) of the limit problem (1.3) is that $u \in L^2(0, T; H^2(\Omega))$. Hence, the strong existence of solutions to (1.3) is essential for strong convergences as $K \to 0$.

3 Non-smooth potentials

The well-posedness results for (1.3) and (1.4) in the previous section can also be shown for the case where F and G are non-smooth, i.e., the classical derivative of F and G need not exist. For the rest of this section we assume that

(5) There exist splitting $F = \hat{\beta} + \hat{\pi}$ and $G = \hat{\beta}_\Gamma + \hat{\pi}_\Gamma$ for the potentials, where

  (i) $\hat{\beta}, \hat{\beta}_\Gamma : \mathbb{R} \to [0, \infty]$ are proper, convex, lower semicontinuous functions with $\hat{\beta}(0) = 0$ and $\hat{\beta}_\Gamma(0) = 0$.

  (ii) $\hat{\pi}, \hat{\pi}_\Gamma \in C^2(\mathbb{R})$ are nonnegative functions whose first derivative $\pi = \hat{\pi}'$ and $\pi_\Gamma = \hat{\pi}'_\Gamma$ are Lipschitz continuous.

(iii) The subdifferentials $\beta := \partial\hat{\beta}$ and $\beta_\Gamma := \partial\hat{\beta}_\Gamma$ are maximal monotone graphs on $\mathbb{R} \times \mathbb{R}$ with effective domains $D(\beta)$ and $D(\beta_\Gamma)$, respectively, and $D(\beta), D(\beta_\Gamma)$ need not be equal to the whole real line $\mathbb{R}$. Furthermore, for the problem (1.3) we assume that for some $p < 5$ and $q < \infty$, there is a positive constant $C > 0$ such that

$$|\xi| \leq C(1 + |u|^p), \quad |\xi_\Gamma| \leq C(1 + |\phi|^q) \text{ for all } \xi \in \beta(u), \, \xi_\Gamma \in \beta_\Gamma(\phi).$$

While for (1.4) we assume that for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\xi| \leq \delta u\xi + C_\delta \text{ for all } \xi \in \beta(u).$$

(iv) For (1.4) the initial data (u_0, ϕ_0) satisfy $u_0 \in H^2(\Omega)$ with $\beta^0(u_0) \in L^2(\Omega)$, $\phi_0 \in H^2(\Gamma)$ with $\beta_\Gamma^0(\phi_0) \in L^2(\Gamma)$, and $K\partial_n u_0 + u_0 = h(\phi_0)$ holds a.e. on Γ, where $\beta^0(u_0)$ is the element in the set $\beta(u_0)$ with the minimal $L^2(\Omega)$-norm, and vice versa for $\beta_\Gamma^0(\phi_0)$. While for (1.3) the initial data u_0 satisfies $u_0 \in H^2(\Omega)$ with $\beta^0(u_0) \in L^2(\Omega)$ and trace $u_0|_\Gamma \in H^2(\Gamma)$ with $\beta_\Gamma^0(\alpha^{-1}(u_0|_\Gamma - \beta)) \in L^2(\Gamma)$.

We mention that in comparison with earlier works [2,3] for the problem (1.3) with $\alpha = 1$ and $\beta = 0$, we do not impose a dominating assumption between the subdifferentials β and β_Γ such as [2, (2.22)-(2.23)]. In fact, for some $\alpha \neq 1$, $\alpha \neq 0$ and $\beta \neq 0$ there is a simple counterexample in which the dominating assumption of [2] does not hold when we have the affine linear transmission condition $u = \alpha\phi + \beta$ in (1.3), see for example [4, Remark 7.1] for more details. This motivates the growth assumptions in (iii) to obtain sufficient uniform estimates in an approximation scheme to deduce the strong existence of solutions.

Under these assumptions we have the following strong well-posedness for (1.4) with non-smooth potentials.

Theorem 3.1 ([4, Theorems 3.1 and 3.2])**.** *For any $T > 0$, there exists a unique quadruple $(u, \phi, \xi, \xi_\Gamma)$ with*

$$\begin{aligned}
&u \in L^\infty(0,T; H^2(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega)) \cap H^1(0,T; H^1(\Gamma)), \quad \partial_n u \in H^1(0,T; L^2(\Gamma)),\\
&\xi \in L^\infty(0,T; L^2(\Omega)), \quad \xi \in \beta(u) \text{ a.e. in } \Omega,\\
&\phi \in L^\infty(0,T; H^2(\Gamma)) \cap W^{1,\infty}(0,T; L^2(\Gamma)) \cap H^1(0,T; H^1(\Gamma)),\\
&\xi_\Gamma \in L^\infty(0,T; L^2(\Gamma)), \quad \xi_\Gamma \in \beta_\Gamma(\phi) \text{ a.e. on } \Gamma,
\end{aligned}$$

satisfying $u(0) = u_0$, $\phi(0) = \phi_0$ and

$$\begin{aligned}
u_t &= \Delta u - \xi - \pi(u) \text{ a.e. in } \Omega,\\
\phi_t &= \Delta_\Gamma \phi - \xi_\Gamma - \pi_\Gamma(\phi) - h'(\phi)\partial_n u \text{ a.e. on } \Gamma,\\
K\partial_n u + u &= h(\phi) \text{ a.e. on } \Gamma.
\end{aligned}$$

Meanwhile for (1.3) with non-smooth potentials we have the following.

Theorem 3.2 ([4, Theorem 3.6]). *For any $T > 0$, there exists a unique triplet (u, ξ, ξ_Γ) with*

$$\begin{aligned} u &\in L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Gamma)), \\ \xi &\in L^\infty(0,T;L^2(\Omega)), \quad \xi \in \beta(u) \ \text{a.e. in } \Omega, \\ u|_\Gamma &\in L^\infty(0,T;H^2(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma)) \cap H^1(0,T;H^1(\Gamma)), \\ \xi_\Gamma &\in L^\infty(0,T;L^2(\Gamma)), \quad \xi_\Gamma \in \beta_\Gamma(\alpha^{-1}((u|_\Gamma) - \beta)) \ \text{a.e. on } \Gamma, \end{aligned}$$

satisfying $u(0) = u_0$ and

$$\begin{aligned} u_t &= \Delta u - \xi - \pi(u) \ \text{a.e. in } \Omega, \\ (u|_\Gamma)_t &= \Delta_\Gamma(u|_\Gamma) - \alpha\Big(\xi_\Gamma - \pi_\Gamma(\alpha^{-1}((u|_\Gamma) - \beta))\Big) - \alpha^2 \partial_n u \ \text{a.e. on } \Gamma. \end{aligned}$$

We mention that analogues of the weak and strong convergences as $K \to 0$ stated in Theorems 2.7 and 2.8 for non-smooth potentials can also be derived, and we refer the reader to [4] for more details.

4 Outlook

In this section we present some interesting open problems and suggest some methodologies to tackle said problems. The first concerns the existence of solutions (weak or strong) to the original problem (1.2) for a general function g (or h). As alluded in the Introduction, the main difficulty lies in the Laplace–Beltrami term. One promising method is to employ the well-developed machinery of maximal L^p regularity (see for example [12]) to deduce a local-in-time strong well-posedness result. Then, perhaps one can extend the local-in-time solution to a global-in-time solution by making use of the fact that (1.2) is a gradient flow.

Another approach will be to use Gamma-convergence. In particular, the Robin energy functional

$$E_K(u, \phi) := \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx + \int_\Gamma \frac{1}{2}|\nabla_\Gamma \phi|^2 + G(\phi) + \frac{1}{2K}|u - h(\phi)|^2\,dS$$

should converge to the limiting functional

$$E_0(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx + \int_\Gamma \frac{1}{2}|\nabla_\Gamma g(u)|^2 + G(g(u))\,dS$$

as $K \to 0$ in the sense of Gamma-convergence. Then, using the framework of Sandier and Serfaty [13], it is also interesting to address the Gamma-convergence of the L^2-gradient flow of E_K (which is (1.4)) to the L^2-gradient flow of E_0.

A second open problem concerns whether a similar modification can be made to equations and dynamic boundary conditions of Cahn–Hilliard type [7, 10]. In comparison, the Cahn–Hilliard equation is fourth order, and so some of the techniques used for the second order Allen–Cahn equation may not work for the Cahn–Hilliard equation. However, the approximation of the affine linear transmission condition with the Robin boundary condition is interesting in its own right, and to the best of the author's knowledge, Cahn–Hilliard systems with Robin boundary conditions have not received much attention in the literature.

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