数理解析研究所講究録 第2121巻 2019年 129-144

Mathematical analysis for a model system of complex fluids

Shuichi Kawashima[1] and Shogo Taniue[2]

[1] Faculty of Science and Engineering, Waseda University, Tokyo 169-8555, Japan

[2] Financial Information Systems 2nd Division, Hitachi Ltd., Kanagawa 214-0006, Japan

Abstract

We study a model system of complex fluids in one-dimensional case. We observe that our model system is transformed into a hyperbolic system of balance laws. Moreover we show that the system has a mathematical entropy and satisfies the stability condition. As the result, by applying the general theory for hyperbolic balance laws, we can prove the global existence and asymptotic decay of solutions to our model system.

Keywords. Complex fluids, Hyperbolic balance laws, Mathematical entropy, Global solution, Stability.

1 Introduction

Complex fluids are interesting materials and may include many examples such as shampoo, toothpaste, blood, liquid crystals, $\cdots$. Some of them are considered as viscoelastic fluids. In this paper we consider the following model system of a compressible viscoelastic fluid:

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho u) = 0, \\
&(\rho u)_t + \operatorname{div}(\rho u \otimes u + pI) + \operatorname{div} \Pi = 0, \qquad (1.1)\\
&\Gamma_t + (u \cdot \nabla)\Gamma + (\partial_x u)\Gamma + \Gamma(\partial_x u)^T - \big((\partial_x u) + (\partial_x u)^T\big) = -\Gamma.
\end{aligned}$$

Here $\rho > 0$ is the fluid density, $u = (u^1, u^2, u^3) \in \mathbb{R}^3$ is the velocity, and $\Gamma = (\Gamma_{ij})$ is the configuration tensor (3×3 real symmetric matrix), which are the unknown functions of $t > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Also, $p = p(\rho)$ is the pressure satisfying $p'(\rho) > 0$ for $\rho > 0$, Π is the stress tensor which is given by the constitutive relation of the form:

$$\Pi = -2(\Gamma - \Gamma^2) - \frac{1}{2}\operatorname{tr}(\Gamma^2)\, I,$$

and $(\partial_x u) = (u^i_{x_j})$ denotes the deformation tensor; I is the identity matrix and the superscript T denotes the transpose. In our system (1.1), the first two equations are the standard conservation laws (of mass and momentum) and the third one is the time evolution of the configuration tensor Γ. This model system was proposed by Ötinger [15] and was studied by Huo and Yong [7].

The one-dimensional version of the system (1.1) is written in the form

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + (\rho u^2 + p(\rho))_x - \Big(2\gamma - \frac{3}{2}\gamma^2\Big)_x = 0, \\ &\gamma_t + u\gamma_x - 2(1-\gamma)u_x = -\gamma. \end{aligned} \tag{1.2}$$

Here $\rho > 0$, $u \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ are the unknown function of $t > 0$ and $x \in \mathbb{R}$.

In [13, 14] we developed the general mathematical theory for hyperbolic systems of balance laws:

$$w_t + \sum_{j=1}^{n} f^j(w)_{x_j} = g(w), \tag{1.3}$$

where $w \in \mathbb{R}^m$ is the unknown function of $t > 0$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We imposed two structural conditions on the system (1.3). The one is the existence of a mathematical entropy for (1.3) which is equivalent to the symmetrization of the system (1.3) (see [13]). The other is the stability condition formulated in [16] which is equivalent to the craftsmanship condition in [17]. Under these two structural conditions we proved in [14] that the initial value problem for (1.3) has a unique global solution $w(t, x)$ which decays in L^2 at the rate $t^{-n/4}$ as $t \to \infty$, provided that the initial perturbation is small in $H^s \cap L^1$.

The main purpose of this paper is to check that the above general theory in [13, 14] is applicable to our one-dimensional system (1.2). More precisely, we first show that the system (1.2) is written in the form of (1.3). Then we verify that the system (1.2), in the form of (1.3), has a mathematical entropy and satisfies the stability condition. Consequently, we can apply the general theory in [13, 14] to our system (1.2) and obtain the results concerning the global existence and asymptotic decay of solutions to the system (1.2).

Notations. Let $1 \le p \le \infty$. Then $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space over $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p}$. For a nonnegative integer s, H^s denotes the s-th order Sobolev space over $\mathbb{R}^n$ in the L^2 sense, equipped with the norm $\|\cdot\|_{H^s}$. We note that $L^2 = H^0$. Let I be an interval in $[0, \infty)$ and X be a Banach space over $\mathbb{R}^n$. Then, for a nonnegative integer k, $C^k(I; X)$ denotes the space of k-times continuously differentiable functions on I with values in X.

Finally, in this paper, we use C and c to denote generic positive constants, which may change from line to line, when the exact value of the constant is not essential.

2 Hyperbolic balance laws

The aim of this section is to review the general theory for hyperbolic balance laws, which were developed in [13, 14].

2.1 Mathematical entropy and symmetrization

Following to [13], we give the definition of the mathematical entropy for hyperbolic balance laws (1.3). We assume that the state variable w takes values in a convex open set $\mathcal{O}_w$ in $\mathbb{R}^m$. Put

$$\mathcal{M} := \{\psi \in \mathbb{R}^m; \quad \langle \psi, g(w) \rangle = 0 \ \text{ for any } w \in \mathcal{O}_w\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^m$. Then $\mathcal{M}$ is a subspace of $\mathbb{R}^m$ such that $g(w) \in \mathcal{M}^\perp$ for any $w \in \mathcal{O}_w$, where $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$. Also we introduce the set $\mathcal{E}$ of equilibrium states for hyperbolic balance laws (1.3):

$$\mathcal{E} := \{w \in \mathcal{O}_w; \quad g(w) = 0\}.$$

In discrete kinetic theory [8], $\mathcal{M}$ is called the space of collision invariants and $\mathcal{E}$ is the set of Maxwellians. Then the mathematical entropy is defined as follows.

Definition 2.1 (Mathematical entropy [13]). *Let $\eta = \eta(w)$ be a smooth function defined in a convex open set $\mathcal{O}_w$. Then $\eta(w)$ is called a mathematical entropy for hyperbolic balance laws* (1.3) *if the following four statements hold true:*

(a) *$\eta(w)$ is strictly convex in $\mathcal{O}_w$ in the sense that the Hessian $D_w^2\eta(w)$ is positive definite for $w \in \mathcal{O}_w$.*

(b) *$D_w f^j(w)(D_w^2\eta(w))^{-1}$ is symmetric for $w \in \mathcal{O}_w$ and $j = 1, \cdots, n$.*

(c) *Let $w \in \mathcal{O}_w$. Then $w \in \mathcal{E}$ holds if and only if $u := ((D_w\eta(w))^T \in \mathcal{M}$.*

(d) *For $w \in \mathcal{E}$, the matrix $-D_w g(w)(D_w^2\eta(w))^{-1}$ is symmetric and nonnegative definite such that its kernel space coincides with $\mathcal{M}$.*

The notion of mathematical entropy was first introduced by Godunov [5] in 1961 (cf. [4]) for hyperbolic systems of conservation laws (including the compressible Euler equation as an example). Then in 1988, this notion was extended by Kawashima and Shizuta [11] (cf. [9, 10]) to hyperbolic-parabolic systems of conservation laws (including the compressible Navier-Stokes equation as an example). The above definition of mathematical entropy for hyperbolic balance laws was formulated by Kawashima and Yong [13] in 2004 (cf. [3, 21]).

Next we introduce the notion of symmetrization for the system (1.3). Let $w = w(u)$ be a diffeomorphism from an open set $\mathcal{O}_u$ onto $\mathcal{O}_w$. By using this diffeomorphism $w = w(u)$, we can rewrite (1.3) in the form

$$A^0(u)u_t + \sum_{j=1}^{n} A^j(u)u_{x_j} = h(u), \tag{2.1}$$

where

$$
\begin{aligned}
A^0(u) &= D_u w(u), \\
A^j(u) &= D_u f^j(w(u)) = D_w f^j(w(u)) D_u w(u), \\
h(u) &= g(w(u)).
\end{aligned} \tag{2.2}
$$

Also we put

$$L(u) := -D_u h(u) = -D_w g(w(u)) D_u w(u). \tag{2.3}$$

This matrix $L(u)$ is called the relaxation matrix.

Definition 2.2 (Symmetric dissipative system [13]). *The system* (2.1) *is called symmetric dissipative if the following four statements hold true:*

(a) $A^0(u)$ *is symmetric and positive definite for* $u \in \mathcal{O}_u$.

(b) $A^j(u)$ *is symmetric for* $u \in \mathcal{O}_u$ *and* $j = 1, \cdots, n$.

(c) *For* $u \in \mathcal{O}_u$, $h(u) = 0$ *holds if and only if* $u \in \mathcal{M}$.

(d) *For* $u \in \mathcal{M} \cap \mathcal{O}_u$, *the relaxation matrix* $L(u)$ *is symmetric and nonnegative definite such that its kernel space coincides with* $\mathcal{M}$.

As in [5, 4, 11] for hyperbolic (hyperbolic-parabolic) systems of conservation laws, we know that the symmetrization of hyperbolic balance laws is characterized by the existence of a mathematical entropy.

Theorem 2.3 (Mathematical entropy and symmetrization [13]). *The following two statements are equivalent.*

(i) *The system* (1.3) *has a mathematical entropy.*

(ii) *There is a diffeomorphism by which* (1.3) *is transformed to a symmetric dissipative system* (2.1).

We give a short outline of the proof. Suppose that the system (1.3) has a mathematical entropy $\eta = \eta(w)$. We define the mapping $u = u(w)$ by

$$u = u(w) := (D_w \eta(w))^T. \tag{2.4}$$

This mapping $u = u(w)$ is a diffeomorphism from the convex open set $\mathcal{O}_w$ onto an open set $\mathcal{O}_u$ and satisfies $D_w u(w) = D_w^2 \eta(w)$. Let $w = w(u)$ be the corresponding inverse mapping. Then this $w = w(u)$ is a diffeomorphism from $\mathcal{O}_u$ onto $\mathcal{O}_w$ satisfying $D_u w(u) = (D_w u(w))^{-1} = (D_w^2 \eta(w))^{-1}$. By this diffeomorphism $w = w(u)$, the system (1.3) can be transformed into a symmetric dissipative system (2.1), where

$$
\begin{aligned}
A^0(u) &= D_u w(u) = (D_w^2 \eta(w))^{-1}, \\
A^j(u) &= D_w f^j(w(u)) D_u w(u) = D_w f^j(w) (D_w^2 \eta(w))^{-1}, \\
h(u) &= g(w(u)), \\
L(u) &= -D_w g(w(u)) D_u w(u) = -D_w g(w) (D_w^2 \eta(w))^{-1}.
\end{aligned} \tag{2.5}
$$

This shows that (i) implies (ii).

The converse assertion is shown as follows. Suppose that (1.3) is transformed to a symmetric dissipative system (2.1) by a diffeomorphism $w = w(u)$. Since $D_u w(u) = A^0(u)$ is symmetric, we know from the Poincaré lemma that there is a smooth function $\tilde{\eta} = \tilde{\eta}(u)$ satisfying $(D_u\tilde{\eta}(u))^T = w(u)$. By using this $\tilde{\eta}(u)$, we define $\eta(w)$ by

$$\eta(w) = \langle u(w), w\rangle - \tilde{\eta}(u(w)),$$

where $u = u(w)$ denotes the inverse mapping of $w = w(u)$. This $\eta(w)$ is the desired mathematical entropy of the system (1.3), which satisfies $D_w\eta(w) = u(w)^T$ and $D_w^2\eta(w) = (D_u w(u))^{-1} = A^0(u)^{-1}$. Thus we have verified that (ii) implies (i). We omit the details and refer the reader to [13, 11].

Next, as in [13, 11], we derive the equation satisfied by our mathematical entropy $\eta(w)$. Since $D_u f^j(w(u)) = A^j(u)$ are symmetric due to (b) of Definition 2.2, we again deduce from the Poincaré lemma that there exist smooth functions $\tilde{q}^j = \tilde{q}^j(u)$ satisfying $(D_u\tilde{q}^j(u))^T = f^j(w(u))$, $j = 1, \cdots, n$. We put

$$q^j(w) = \langle u(w), f^j(w)\rangle - \tilde{q}^j(u(w)), \qquad j = 1, \cdots, n.$$

Then this $q^j(w)$ becomes the corresponding entropy flux. In fact, a simple computation using $u(w)^T = D_w\eta(w)$ shows that $D_w q^j(w) = D_w\eta(w)D_w f^j(w)$, $j = 1, \cdots, n$. Therefore, taking the inner product of (1.3) with $u(w) = (D_w\eta(w))^T$, we arrive at the equation

$$\eta(w)_t + \sum_{j=1}^{n} q^j(w)_{x_j} = \langle u(w), g(w)\rangle, \tag{2.6}$$

which is the equation of our mathematical entropy $\eta(w)$.

Similarly, we derive the equation of the energy form associated with the mathematical entropy. Let $\bar{w} \in \mathcal{E}$ be a fixed constant equilibrium state, namely, $g(\bar{w}) = 0$, and define the energy form $\mathcal{H}(w)$ by

$$\mathcal{H}(w) = \eta(w) - \eta(\bar{w}) - \langle \bar{u}, w - \bar{w}\rangle, \tag{2.7}$$

where $\bar{u} = u(\bar{w}) = (D_w\eta(\bar{w}))^T \in \mathcal{M}$. Since the entropy $\eta(w)$ is a strictly convex function of $w \in \mathcal{O}_w$, our energy form $\mathcal{H}(w)$ is equivalent to the quadratic function $|w - \bar{w}|^2$ for small $|w - \bar{w}|$. We easily see that the energy form $\mathcal{H}(w)$ satisfies

$$\mathcal{H}(w)_t + \sum_{j=1}^{n} \mathcal{Q}^j(w)_{x_j} = \langle u(w), g(w)\rangle, \tag{2.8}$$

where $\mathcal{Q}^j(w) = q^j(w) - q^j(\bar{w}) - \langle \bar{u}, f^j(w) - f^j(\bar{w})\rangle$ is the corresponding flux function.

Finally in this subsection, we introduce another symmetrization of the system (1.3) which has a mathematical entropy. This symmetrization is useful to treat concrete physical examples. Let us consider a new state variable $v \in \mathbb{R}^m$ and assume that $w = w(v)$ is a diffeomorphism from an open set $\mathcal{O}_v$ onto the convex open set $\mathcal{O}_w$. We put $w = w(v)$ in (1.3) and then multiply the resulting system by $(D_v w)^T D_w^2 \eta$ from the left, where $\eta = \eta(w)$ is a mathematical entropy of (1.3). This yields

$$\tilde{A}^0(v) v_t + \sum_{j=1}^{3} \tilde{A}^j(v) v_{x_j} = \tilde{h}(v), \tag{2.9}$$

where

$$\begin{aligned} \tilde{A}^0(v) &= (D_v w)^T D_w^2 \eta \, D_v w = (D_v u)^T D_v w, \\ \tilde{A}^j(v) &= (D_v w)^T D_w^2 \eta \, D_v f^j = (D_v u)^T D_v f^j, \\ \tilde{h}(v) &= (D_v w)^T D_w^2 \eta \, g(w) = (D_v u)^T g(w), \end{aligned} \tag{2.10}$$

with $u := (D_w \eta)^T$. Here in the second equalities in (2.10) we used the elementary fact that $(D_v w)^T D_w^2 \eta = (D_w^2 \eta \, D_v w)^T = (D_w u \, D_v w)^T = (D_v u)^T$. The corresponding relaxation matrix is defined by

$$\tilde{L}(v) := -D_v \tilde{h}(v) = -D_v((D_v u)^T g(w)). \tag{2.11}$$

On the other hand, by using the diffeomorphism $w = w(u)$ with $u = (D_w \eta)^T$ directly in (1.3), we already have the symmetric system (2.1) with (2.5). We have the following relations between (2.10) and (2.5).

$$\begin{aligned} \tilde{A}^0(v) &= (D_v u)^T A^0(u) D_v u, \\ \tilde{A}^j(v) &= (D_v u)^T A^j(u) D_v u, \\ \tilde{h}(v) &= (D_v u)^T h(u), \end{aligned} \tag{2.12}$$

Concerning the relation between two relaxation matrices $\tilde{L}(v)$ and $L(u)$, we have the following result: When v and u are corresponding to the equilibrium state $w \in \mathcal{E}$ (*i.e.*, $g(w) = 0$), we have

$$\tilde{L}(v) = -(D_v u)^T D_v g = (D_v u)^T L(u) D_v u, \tag{2.13}$$

where we used the equality $-D_v g = -D_v h = -D_u h D_v u = L(u) D_v u$.

2.2 Stability condition and global existence

In this subsection we review the general theory on the dissipativity structure for symmetric dissipative system (2.1), which was developed in [16, 17]. Also we summarize the corresponding results on the global existence and decay of solutions to the hyperbolic balance laws (1.3), which were obtained in [14] (cf. [2, 6, 21]).

In order to formulate the stability condition for the symmetric dissipative system (2.1) obtained from (1.3), we consider the linearized system of (2.1) at $u = \bar{u}$, where $\bar{u} \in \mathcal{M} \cap \mathcal{O}_u$ is a constant state.

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{2.14}$$

where we write $A^0 = A^0(\bar{u})$, $A^j = A^j(\bar{u})$ and $L = L(\bar{u})$ for simplicity. Taking the Fourier transform of (2.14) with respect to $x \in \mathbb{R}^n$, we obtain

$$A^0 \hat{u}_t + i|\xi|A(\omega)\hat{u} + L\hat{u} = 0, \tag{2.15}$$

where $A(\omega) := \sum_{j=1}^n A^j \omega_j$ with $\omega = \xi/|\xi| \in S^{n-1}$ (the unit sphere). We denote by $\lambda = \lambda(i\xi)$ the eigenvalues of (2.15), which are the solutions to the corresponding characteristic equation

$$\det(\lambda A^0 + i|\xi|A(\omega) + L) = 0.$$

The stability condition for (2.14) is then formulated as follows.

Definition 2.4 (Stability condition [16])**.** *The system* (2.14) *satisfies the stability condition if the following holds true: Let $\phi \in \mathbb{R}^m$ satisfy $L\phi = 0$ and $\mu A^0 \phi + A(\omega)\phi = 0$ for some $\mu \in \mathbb{R}$ and $\omega \in S^{n-1}$. Then $\phi = 0$.*

This stability condition was first formulated in [16] for a general class of linear symmetric hyperbolic-parabolic systems including our linear symmetric hyperbolic systems (2.14). On the other hand, another condition was introduced in [17] to derive the decay estimate of solutions for linearized symmetric hyperbolic-parabolic systems. This condition is now called "craftsmanship condition" and is formulated as follows.

Definition 2.5 (Craftsmanship condition [17])**.** *The system* (2.14) *satisfies the craftsmanship condition if there is an $m \times m$ matrix $K(\omega)$ depending smoothly on $\omega \in S^{n-1}$ with the following properties:*

(i) *$K(-\omega) = K(\omega)$ for $\omega \in S^{n-1}$.*

(ii) *$K(\omega)A^0$ is skew-symmetric for $\omega \in S^{n-1}$.*

(iii) *$(K(\omega)A(\omega))^{sy} + L$ is positive definite for $\omega \in S^{n-1}$, where X^{sy} denotes the symmetric part of the matrix X, i.e., $X^{sy} = (X + X^T)/2$.*

The following characterization of the dissipative structure was also given in [16].

Theorem 2.6 (Dissipative structure [16])**.** *The following four conditions are equivalent to each other.*

(i) *The system* (2.14) *satisfies the stability condition.*

(ii) *The system* (2.14) *satisfies the craftsmanship condition.*

(iii) *The system* (2.14) *is uniformly dissipative in the sense that* $\mathrm{Re}\,\lambda(i\xi) \le -c\rho(\xi)$ *for $\xi \in \mathbb{R}^n$, where $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$.*

(iv) *The system* (2.14) *is strictly dissipative in the sense that* $\mathrm{Re}\,\lambda(i\xi) < 0$ *for $\xi \neq 0$.*

Remark 1. The above two structural conditions (stability condition and craftsmanship condition) formulated for the system (2.14) are equivalent to the corresponding structural conditions for the linearized system of (2.9), respectively, because we have the relations (2.12) and (2.13) between these two systems.

Remark 2. The Kalman rank condition was shown to be equivalent to the above stability condition. For the details we refer to [1].

The craftsmanship condition in Definition 2.5 is the key to show the decay estimate of solutions to the system (2.14).

Theorem 2.7 (Linear decay [17]). *Suppose that the system* (2.14) *satisfies the craftsmanship condition. Then the solution u of* (2.14) *with the initial data u_0 satisfies the following pointwise estimate in the Fourier space:*

$$|\hat{u}(t,\xi)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \tag{2.16}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. Moreover the solution u satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \tag{2.17}$$

for $k \ge 0$.

Finally, we review the global existence and decay results for hyperbolic balance laws (1.3). We assume that the system (1.3) has a mathematical entropy and the corresponding symmetric dissipative system (2.1) satisfies the stability condition at a given constant state $\bar{u} \in \mathcal{M} \cap \mathcal{O}_u$. We prescribe the initial data

$$u(0,x) = u_0(x). \tag{2.18}$$

The global existence result obtained in [14] is then stated as follows.

Theorem 2.8 (Global existence [14]). *Assume the above structural conditions for* (1.3) *and* (2.1). *Let $n \ge 1$ and $s \ge s_0 + 1$, where $s_0 = [n/2] + 1$. For a given fixed constant state $\bar{u} \in \mathcal{M} \cap \mathcal{O}_u$, we suppose that $u_0 - \bar{u} \in H^s$ and put $E_0 = \|u_0 - \bar{u}\|_{H^s}$. Then there is a positive constant δ_0 such that if $E_0 \le \delta_0$, the initial value problem* (2.1), (2.18) *has a unique global solution u with $u - \bar{u} \in C^0([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$, which satisfies the following uniform estimate*

$$\|(u-\bar{u})(t)\|_{H^s}^2 + \int_0^t \|(I-P)u(\tau)\|_{H^s}^2 + \|\partial_x u(\tau)\|_{H^{s-1}}^2 \, d\tau \le CE_0^2$$

for $t \ge 0$. Here P is the orthogonal projection onto $\mathcal{M}$. Moreover, the solution u converges to the constant state $\bar{u}$ as $t \to \infty$, namely, we have

$$\|\partial_x^l (u-\bar{u})(t)\|_{L^\infty} \longrightarrow 0$$

for $t \to \infty$, where $0 \le l \le s - s_0$.

When the initial data are also in L^1, we have the sharp decay estimates of solutions.

Theorem 2.9 (Decay estimate [14]). *Assume the same structural conditions as in Theorem 2.8. Let $n \ge 1$ and let $s \ge 3$ for $n = 1$ and $s \ge s_0 + 1$ for $n \ge 2$. Suppose that $u_0 - \bar{u} \in H^s \cap L^1$. If the norm $E_1 = \|u_0 - \bar{u}\|_{H^s \cap L^1}$ is sufficiently small, then the global solution u obtained in Theorem 2.8 satisfies the following decay estimates:*

$$\|\partial_x^k (u-\bar{u})(t)\|_{L^2} \le CE_1(1+t)^{-n/4-k/2}, \tag{2.19}$$

$$\|(I-P)\partial_x^k u(t)\|_{L^2} \le CE_1(1+t)^{-n/4-(k+1)/2}, \tag{2.20}$$

where $0 \le k \le s-1$ in (2.19) *and $0 \le k \le s-2$ in* (2.20).

Remark 3. In the above results we assumed the H^s regularity with $s = [n/2] + 2$ on the initial data. This regularity requirement can be relaxed to $s = n/2 + 1$ if we use the Besov spaces $B^s_{2,1}$. For the details, we refer the readers to [18, 19, 20].

3 One-dimensional complex fluids

In this section we treat the one-dimensional model system (1.2) of complex fluids. This model system (1.2) was studied by Hua and Yong [7] and global existence of solutions was proved there by direct computations. Our aim in this section is to verify that the general theory reviewed in the previous section is applicable to the system (1.2). As the consequence, we prove the global existence and asymptotic decay of solutions to the system (1.2) by applying Theorems 2.8 and 2.9.

Our first step is to show that the system (1.2) is rewritten in the form of hyperbolic balance laws.

Claim 1. *The system* (1.2) *is written in the form of hyperbolic balance laws for* $\gamma < 1$.

Proof. To verify this claim, we need to rewrite the third equation in (1.2). Let $\gamma < 1$ and we divide the third equation in (1.2) by $(1-\gamma)^{1/2}$. This yields

$$\frac{\gamma_t}{(1-\gamma)^{1/2}} + u\frac{\gamma_x}{(1-\gamma)^{1/2}} - 2(1-\gamma)^{1/2}u_x = -\frac{\gamma}{(1-\gamma)^{1/2}}.$$

Since $\frac{d}{d\gamma}\{-2(1-\gamma)^{1/2}\} = \frac{1}{(1-\gamma)^{1/2}}$, we can rewrite the above equation in the form

$$2\{1-(1-\gamma)^{1/2}\}_t - 2\{(1-\gamma)^{1/2}u\}_x = -\frac{\gamma}{(1-\gamma)^{1/2}}.$$

Thus we find that the system (1.2) is rewritten as

$$W_t + F(W)_x = G(W), \tag{3.1}$$

where

$$W = \begin{pmatrix} \rho \\ \rho u \\ \beta(\gamma) \end{pmatrix}, \quad F(W) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) - (2\gamma - \frac{3}{2}\gamma^2) \\ -2(1-\gamma)^{1/2}u \end{pmatrix}, \quad G(W) = \begin{pmatrix} 0 \\ 0 \\ -d(\gamma) \end{pmatrix}. \tag{3.2}$$

Here we put

$$\beta(\gamma) = 2\{1-(1-\gamma)^{1/2}\}, \qquad d(\gamma) = \frac{\gamma}{(1-\gamma)^{1/2}}.$$

This completes the proof of Claim 1. □

We choose the set of state variables W for our system (3.1) as

$$\mathcal{O}_W = \{W = (\rho, \rho u, \beta(\gamma))^T;\ \rho > 0,\ u \in \mathbb{R},\ \gamma < 2/3\} \subset \mathbb{R}^3. \tag{3.3}$$

The restriction $\gamma < 2/3$ will be used in Claim 2 below. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of $\mathbb{R}^3$. Then the subspace $\mathcal{M}$ and the set $\mathcal{E}$ of equilibrium states for our system (3.1) are given by

$$\mathcal{M} = \mathrm{span}\,\{e_1, e_2\}, \qquad \mathcal{E} = \{W = (\rho, \rho u, 0)^T;\ \rho > 0,\ u \in \mathbb{R}\}, \tag{3.4}$$

respectively; we see that $\dim \mathcal{M} = 2$ and $\mathcal{E} \subset \mathcal{M}$.

Next we consider a mathematical entropy for the system (3.1) corresponding to the original system (1.2).

Claim 2. *The system* (3.1) *corresponding to* (1.2) *has a mathematical entropy in* $\mathcal{O}_W$ *in the sense of Definition 2.1.*

Proof. We consider the following total energy for the original system (1.2).

$$\eta(W) = \rho\Big(e(\rho) + \frac{1}{2}u^2\Big) + \frac{1}{2}\gamma^2, \qquad e(\rho) = \int^\rho \frac{p(s)}{s^2}\,ds. \tag{3.5}$$

Here $e(\rho)$ denotes the internal energy. We show that this $\eta(W)$ becomes a mathematical entropy of the system (3.1) in $\mathcal{O}_W$. We need to check all the conditions in Definition 2.1. The computations below are similar to those used in [12].

Condition (a): It is useful to use the physical state variable $V = (\rho, u, \gamma)^T$ for actual computations. We first calculate $U = (D_W\eta)^T$ by using $D_W\eta = D_V\eta(D_VW)^{-1}$. A simple computation gives $D_V\eta = \big(a(\rho) + \frac{1}{2}u^2, \rho u, \gamma\big)$, where $a(\rho) := e(\rho) + \frac{p(\rho)}{\rho}$. Also we have

$$D_VW = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ 0 & 0 & \beta'(\gamma) \end{pmatrix}, \qquad (D_VW)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 \\ 0 & 0 & \frac{1}{\beta'(\gamma)} \end{pmatrix},$$

where $\beta'(\gamma) = \frac{1}{(1-\gamma)^{1/2}}$. Consequently we obtain

$$U = (D_W\eta)^T = \Big(a(\rho) - \frac{1}{2}u^2, u, b(\gamma)\Big)^T, \tag{3.6}$$

where we put $b(\gamma) := \frac{\gamma}{\beta'(\gamma)} = (1-\gamma)^{1/2}\gamma$.

Next we show the strict convexity of η in (3.5) with respect to W. We need to verify that $D_W^2\eta$ is positive definite. For this purpose it suffices to show the same for the matrix $\tilde{A}^0(V)$ in (2.10) because we have the relation $\tilde{A}^0(V) = (D_VW)^T D_W^2\eta\, D_VW$. We compute $\tilde{A}^0(V)$ by the expression $\tilde{A}^0(V) = (D_VU)^T D_VW$ in (2.10) with $U = (D_W\eta)^T$. By direct computation, using $a'(\rho) = \frac{p'(\rho)}{\rho}$, we have

$$D_VU = \begin{pmatrix} \frac{p'(\rho)}{\rho} & -u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b'(\gamma) \end{pmatrix}, \qquad (D_VU)^T = \begin{pmatrix} \frac{p'(\rho)}{\rho} & 0 & 0 \\ -u & 1 & 0 \\ 0 & 0 & b'(\gamma) \end{pmatrix},$$

where $b'(\gamma) = \frac{2-3\gamma}{2(1-\gamma)^{1/2}}$. Therefore we arrive at the expression

$$\tilde{A}^0(V) = \begin{pmatrix} \frac{p'(\rho)}{\rho} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \frac{2-3\gamma}{2(1-\gamma)} \end{pmatrix}. \tag{3.7}$$

Here we used the fact that $b'(\gamma)\beta'(\gamma) = \frac{2-3\gamma}{2(1-\gamma)}$. Since we restrict to $\gamma < 2/3$, we see that $\tilde{A}^0(V)$ in (3.7) is (diagonal and) positive definite. Thus the condition (a) of Definition 2.1 has been verified.

Condition (b): We need to show the matrix $D_W F(D_W^2\eta)^{-1} = A(U)$ in (2.5) is symmetric. Since we have the relation $\tilde{A}(V) = (D_V U)^T A(U) D_V U$ in (2.12), it suffices to show that $\tilde{A}(V)$ is symmetric. We compute $\tilde{A}(V)$ by using the expression $\tilde{A}(V) = (D_V U)^T D_V F$ in (2.10). Technical computation gives

$$D_V F = \begin{pmatrix} u & \rho & 0 \\ u^2 + p'(\rho) & 2\rho u & -(2-3\gamma) \\ 0 & -2(1-\gamma)^{1/2} & \frac{u}{(1-\gamma)^{1/2}} \end{pmatrix}.$$

Therefore we obtain

$$\tilde{A}(V) = \begin{pmatrix} \frac{p'(\rho)}{\rho}u & p'(\rho) & 0 \\ p'(\rho) & \rho u & -(2-3\gamma) \\ 0 & -(2-3\gamma) & \frac{2-3\gamma}{2(1-\gamma)}u \end{pmatrix}, \tag{3.8}$$

where we used the fact that $-2(1-\gamma)^{1/2}b'(\gamma) = -(2-3\gamma)$ and $b'(\gamma)\frac{u}{(1-\gamma)^{1/2}} = \frac{2-3\gamma}{2(1-\gamma)}u$. This implies that $\tilde{A}(V)$ is symmetric. Thus we have verified (b) of Definition 2.1.

Condition (c): Let $W = (\rho, \rho u, \beta(\gamma))^T \in \mathcal{O}_W$ (namely, $\rho > 0$ and $\gamma < 2/3$ by (3.3)) and assume that $W \in \mathcal{E}$. Then we have $\beta(\gamma) = 0$ which implies $\gamma = 0$. Therefore we have $W = (\rho, \rho u, 0)^T$. In this case the associated U in (3.6) becomes $U = \big(a(\rho) - \frac{1}{2}u^2, u, 0\big)^T$ because $b(0) = 0$. Therefore we see that $U \in \mathcal{M} = \operatorname{span}\{e_1, e_2\}$.

Conversely, we assume that $U = \big(a(\rho) - \frac{1}{2}u^2, u, b(\gamma)\big)^T \in \mathcal{M}$. Then we have $b(\gamma) = 0$ which implies $\gamma = 0$ since $\gamma < 2/3$. Therefore the corresponding W becomes $W = (\rho, \rho u, 0)^T \in \mathcal{E}$, where we used $\beta(0) = 0$. Thus we have verified (c) of Definition 2.1.

Condition (d): To check the condition (d) of Definition 2.1, we have to calculate the matrix $-D_W G(D_W^2\eta)^{-1} = L(U)$ in (2.5) for $W \in \mathcal{E}$. Using $D_W G = D_V G(D_V W)^{-1}$ and $D_W^2\eta = D_V U(D_V W)^{-1}$ with $U = (D_W\eta)^T$, we can express $L(U)$ in the form

$$L(U) = -D_V G(D_V U)^{-1}. \tag{3.9}$$

We see that

$$(D_V U)^{-1} = \begin{pmatrix} \frac{\rho}{p'(\rho)} & \frac{\rho}{p'(\rho)}u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{b'(\gamma)} \end{pmatrix}.$$

Also a direct computation yields

$$D_V G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d'(\gamma) \end{pmatrix}$$

where $d(\gamma) = \frac{\gamma}{(1-\gamma)^{1/2}}$ so that $d'(\gamma) = \frac{2-\gamma}{2(1-\gamma)^{3/2}}$. Therefore we obtain

$$L(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2-\gamma}{(1-\gamma)(2-3\gamma)} \end{pmatrix}, \tag{3.10}$$

where we used $\frac{d'(\gamma)}{b'(\gamma)} = \frac{2-\gamma}{(1-\gamma)(2-3\gamma)}$. In particular, we have $L(U) = \mathrm{diag}(0,0,1)$ for $W \in \mathcal{E}$ (namely, $\gamma = 0$). This relaxation matrix $L(U)$ is symmetric and nonnegative definite such that $\ker L(U) = \mathcal{M} = \mathrm{span}\{e_1, e_2\}$. Thus we have verified (d) of Definition 2.1 and the proof of Claim 2 is complete. $\square$

Let us derive the equation satisfied by the mathematical entropy (total energy) $\eta(W)$ in (3.5). By direct computations, using (1.2), we have

$$\begin{aligned} &\Big\{\rho\big(e(\rho) + \frac{1}{2}u^2\big) + \frac{1}{2}\gamma^2\Big\}_t \\ &\quad + \Big\{\rho u\big(e(\rho) + \frac{1}{2}u^2\big) + \frac{1}{2}\gamma^2 u + p(\rho)u - \big(2\gamma - \frac{3}{2}\gamma^2\big)u\Big\}_x + \gamma^2 = 0, \end{aligned} \tag{3.11}$$

where $e(\rho) = \int^\rho \frac{p(s)}{s^2}\,ds$. Also we consider the energy form associated with our mathematical entropy. Let us fix a constant equilibrium state $\bar{W} = (\bar\rho, \bar u, 0)^T \in \mathcal{E}$ (see (3.4)). The corresponding constant state $\bar U$ for $U = (D_W \eta)^T$ in (3.6) is given by $\bar U = \big(e(\bar\rho) + \frac{p(\bar\rho)}{\bar\rho} - \frac{1}{2}\bar u^2, \bar u, 0\big)^T$. Therefore the associated energy form is given by

$$\begin{aligned} \mathcal{H}(W) &= \eta(W) - \eta(\bar W) - \langle \bar U, W - \bar W\rangle \\ &= \rho\big(e_*(\rho) + \frac{1}{2}(u-\bar u)^2\big) + \frac{1}{2}\gamma^2, \end{aligned} \tag{3.12}$$

where

$$\epsilon_*(\rho) := e(\rho) - e(\bar\rho) + p(\bar\rho)\big(\frac{1}{\rho} - \frac{1}{\bar\rho}\big) = \int_{\bar\rho}^{\rho} \frac{p(s) - p(\bar\rho)}{s^2}\,ds.$$

We note that $e(\rho) = \int^\rho \frac{p(s)}{s^2}\,ds$ is a strictly convex function of $v = \frac{1}{\rho}$ with $\frac{d}{dv}e(\rho) = -p(\rho)$. By direct computations, using (3.11) and (1.2), we find that the equation satisfied by the energy form is

$$\begin{aligned} &\Big\{\rho\big(e_*(\rho) + \frac{1}{2}(u-\bar u)^2\big) + \frac{1}{2}\gamma^2\Big\}_t + \Big\{\rho u\big(e_*(\rho) + \frac{1}{2}(u-\bar u)^2\big) + \frac{1}{2}\gamma^2 u \\ &\quad + (p(\rho) - p(\bar\rho))(u - \bar u) - \big(2\gamma - \frac{3}{2}\gamma^2\big)(u-\bar u)\Big\} + \gamma^2 = 0. \end{aligned} \tag{3.13}$$

This equality plays an important role in deriving uniform a priori estimates of solutions.

Next we investigate the stability condition in Definition 2.4 for our system (3.1). For this purpose we derive the corresponding symmetric system in the form of (2.9) for the physical state variable $V = (\rho, u, \gamma)^T$:

$$\tilde A^0(V)V_t + \tilde A(V)V_x = \tilde H(V). \tag{3.14}$$

Here we already derived the coefficient matrices $\tilde A^0(V)$ and $\tilde A(V)$ in (3.7) and (3.8), respectively. We compute the term $\tilde H(V)$ in (3.14) by the expression $\tilde H(V) = (D_V U)^T G$ (see (2.10)) and find that

$$\tilde H(V) = \begin{pmatrix} 0 \\ 0 \\ -\frac{2-3\gamma}{2(1-\gamma)}\gamma \end{pmatrix}, \tag{3.15}$$

where we used $b'(\gamma)d(\gamma) = \frac{2-3\gamma}{2(1-\gamma)}\gamma$. Consequently, we have

$$\tilde{H}(V) = -\tilde{L}_0(V)V, \qquad \tilde{L}_0(V) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2-3\gamma}{2(1-\gamma)} \end{pmatrix}. \tag{3.16}$$

We remark that this matrix $\tilde{L}_0(V)$ is not the relaxation matrix $\tilde{L}(V)$ for the system (3.14). In fact, we can compute $\tilde{L}(V) = -D_V\tilde{H}(V)$ directly by using (3.15) and obtain

$$\tilde{L}(V) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2-6\gamma+3\gamma^2}{2(1-\gamma)^2} \end{pmatrix}. \tag{3.17}$$

Now we linearize the symmetric system (3.14) at the constant state $\bar{V} = (\bar{\rho}, \bar{u}, 0)^T$ corresponding to the constant equilibrium state $\bar{W} = (\bar{\rho}, \bar{\rho}\bar{u}, 0)^T \in \mathcal{E}$. We have

$$\tilde{A}^0 V_t + \tilde{A}V_x + \tilde{L}V = 0, \tag{3.18}$$

where $\tilde{A}^0$, $\tilde{A}$ and $\tilde{L}$ are constant matrices given by

$$\tilde{A}^0 = \begin{pmatrix} \frac{p'(\bar{\rho})}{\bar{\rho}} & 0 & 0 \\ 0 & \bar{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \frac{p'(\bar{\rho})}{\bar{\rho}}\bar{u} & p'(\bar{\rho}) & 0 \\ p'(\bar{\rho}) & \bar{\rho}\bar{u} & -2 \\ 0 & -2 & \bar{u} \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\tilde{L}(\bar{V}) = \tilde{L}_0(\bar{V}) = \mathrm{diag}(0, 0, 1)$.

We check the stability condition for the system (3.18).

Claim 3. *The linearized system* (3.18) *corresponding to* (1.2) *satisfies the stability condition in the sense of Definition 2.4.*

Proof. Let $\phi = (\phi_1, \phi_2, \phi_3)^T \in \mathbb{R}^3$ and suppose that $\tilde{L}\phi = 0$ and $\mu\tilde{A}^0\phi + \tilde{A}\phi = 0$ for some $\mu \in \mathbb{R}$. Then it follows from $\tilde{L}\phi = 0$ that $\phi_3 = 0$. For this ϕ the equality $\mu\tilde{A}^0\phi + \tilde{A}\phi = 0$ gives

$$\frac{p'(\bar{\rho})}{\bar{\rho}}(\mu + \bar{u})\phi_1 + p'(\bar{\rho})\phi_2 = 0,$$
$$p'(\bar{\rho})\phi_1 + \bar{\rho}(\mu + \bar{u})\phi_2 = 0, \qquad -2\phi_2 = 0.$$

From the third equation we have $\phi_2 = 0$. Substituting it into the second equation, we get $\phi_1 = 0$. Consequently, we have $\phi = 0$. Thus we have verified the stability condition. $\square$

We have checked Claims 1, 2 and 3 for our model system (1.2). As the consequence we can apply the general theory reviewed in the previous section to the system (1.2). In particular, as an application of Theorems 2.8 and 2.9, we have the following result on the global existence and decay of solutions.

Theorem 3.1 (Complex fluids)**.** *Consider the initial value problem for* (1.2) *with the initial data* (ρ_0, u_0, γ_0). *Let* $(\bar{\rho}, \bar{u}, 0)$ *be a constant state with* $\bar{\rho} > 0$ *and* $\bar{u} \in \mathbb{R}$.

(i) *Let* $s \geq 2$. *Suppose that* $(\rho_0 - \bar{\rho}, u_0 - \bar{u}, \gamma_0) \in H^s$ *and put* $E_0 = \|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, \gamma_0)\|_{H^s}$. *If* E_0 *is suitably small, then the initial value problem for* (1.2) *has a unique global solution satisfying* $(\rho - \bar{\rho}, u - \bar{u}, \gamma) \in C^0([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. *This solution satisfies the uniform estimate*

$$\|(\rho - \bar{\rho}, u - \bar{u}, \gamma)(t)\|_{H^s}^2 + \int_0^t \|\partial_x(\rho, u)(\tau)\|_{H^{s-1}}^2 + \|\gamma(\tau)\|_{H^s}^2 \, d\tau \leq C E_0^2$$

for $t \geq 0$. *Moreover, the solution* (ρ, u, γ) *converges to the constant state* $(\bar{\rho}, \bar{u}, 0)$ *as* $t \to \infty$, *namely, we have* $\|\partial_x^l(\rho - \bar{\rho}, u - \bar{u}, \gamma)(t)\|_{L^\infty} \to 0$ *for* $t \to \infty$, *where* $0 \leq l \leq s-1$.

(ii) *Let* $s \geq 3$. *Suppose that* $(\rho_0 - \bar{\rho}, u_0 - \bar{u}, \gamma_0) \in H^s \cap L^1$ *and put* $E_1 = \|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, \gamma_0)\|_{H^s \cap L^1}$. *If* E_1 *is small, then the global solution* u *obtained in* (i) *satisfies the following decay estimates:*

$$\|\partial_x^k(\rho - \bar{\rho}, u - \bar{u}, \gamma)(t)\|_{L^2} \leq C E_1 (1+t)^{-n/4-k/2}, \qquad 0 \leq k \leq s-1,$$
$$\|\partial_x^k \gamma(t)\|_{L^2} \leq C E_1 (1+t)^{-n/4-(k+1)/2}, \qquad 0 \leq k \leq s-2.$$

Remark 4. The global existence result in Theorem 2.9 (i) was proved by Hua and Yong [7] in a case where $\bar{u} = 0$. Their proof is based on the straightforward computations.

Remark 5. We can prove a similar global existence result in (i) for initial data in the Besov space $B_{2,1}^{3/2}$. Also we can show the decay estimate similar to the one in (ii) for initial data in $B_{2,1}^{3/2} \cap \dot{B}_{2,\infty}^{-1/2}$. For the details, we refer the readers to [18, 19, 20].

Acknowledgments : This work was partially supported by JSPS KAKENHI Grant Number JP18H01131.

References

[1] K. Beauchard and E. Zuazua, *Large time asymptotics for partially dissipative hyperbolic systems*, Arch. Rational Mech. Anal., **199** (2011), 177-227.

[2] S. Bianchini, B. Hanouzet and R. Natalini, *Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Comm. Pure Appl. Math., **60** (2007), 1559-1622.

[3] G.-Q. Chen, C.D. Levermore and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math., **47** (1994), 787-830.

[4] K.O. Friedrichs and P.D. Lax, *Systems of conservation equations with a convex extension*, Proc. Nat. Acad. Sci. USA, **68** (1971), 1686-1688.

[5] S.K. Godunov, *An interesting class of quasilinear systems*, Dokl. Akad. Nauk SSSR, **139** (1961), 521-523.

[6] B. Hanouzet and R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Rational Mech. Anal., **169** (2003), 89-117.

[7] X. Huo and W.-A. Yong, *Structural stability of a 1D compressible viscoelastic fluid model*, J. Differential Equations, **261** (2016), 1264-1284.

[8] S. Kawashima, *Global existence and stability of solutions for discrete velocity models of the Boltzmann equation*, Recent Topics in Nonlinear PDE, Lecture Notes in Num. Appl. Anal., **6**, Kinokuniya, Tokyo, 1983, 59-85.

[9] S. Kawashima, *Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics*, Doctoral Thesis, Kyoto University, 1984. http://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/97887

[10] S. Kawashima, *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. Roy. Soc. Edinburgh, **106A** (1987), 169-194.

[11] S. Kawashima and Y. Shizuta, *On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws*, Tôhoku Math. J., **40** (1988), 449-464.

[12] S. Kawashima and Y. Ueda, *Mathematical entropy and Euler-Cattaneo-Maxwell system*, Analysis and Applications, **14** (2016), 101-143.

[13] S. Kawashima and W.-A. Yong, *Dissipative structure and entropy for hyperbolic systems of balance laws*, Arch. Rational Mech. Anal., **174** (2004), 345-364.

[14] S. Kawashima and W.-A. Yong, *Decay estimates for hyperbolic balance laws*, ZAA (J. Anal. Appl.), **28** (2009), 1-33.

[15] H.C. Ötinger, Beyond Equilibrium Thermodynamics, John Wiley & Sons, 2005.

[16] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J., **14** (1985), 249-275.

[17] T. Umeda, S. Kawashima and Y. Shizuta, *On the decay of solutions to the linearized equations of electro-magneto-fluid dynamics*, Japan J. Appl. Math., **1** (1984), 435-457.

[18] J. Xu and S. Kawashima, *Global classical solutions for partially dissipative hyperbolic systems of balance laws*, Arch. Rational Mech. Anal., **211** (2014), 513-553.

[19] J. Xu and S. Kawashima, *The optimal decay estimates on the framework of Besov spaces for generally dissipative systems*, Arch. Rational Mech. Anal., **218** (2015), 275-315.

[20] J. Xu and S. Kawashima, *Frequency-localization Duhamel principle and its application to the optimal decay of dissipative systems in low dimensions*, J. Differential Equations, **261** (2016), 2670-2701.

[21] W.-A. Yong, *Entropy and global existence for hyperbolic balance laws*, Arch. Rational Mech. Anal., **172** (2004), 247-266.