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楕円関数を用いて表される周期解をもつある遅延微分方程式

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概要．本論文では、Duffing 型の非線型常微分方程式を用いて、以下の遅延微分方程式の周期解の存在について考察する。

$$\frac{d}{dt}x(t) = rx(t)\left(1 - \int_0^1 x(t-s)ds\right).$$

本稿で考察する遅延微分方程式は、ヤコビの楕円関数で表される周期 2 の周期解を持つ。証明のアイデアは、Kaplan and York (1974) によるものである。

1. Introduction

The delay differential equation

$$\frac{d}{dt}z(t) = f\left(z(t-1)\right), \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, has been extensively studied in the literature [3, 4, 9, 8, 12]. Assuming that f is an odd function, in the paper [5], Kaplan and Yorke constructed a periodic solution of the equation (1.1) via a Hamiltonian system of ordinary differential equations. See also Chapter XV of [1]. In this paper we follow the approach by Kaplan and Yorke [5]: we find a periodic solution of a differential equation with distributed delay, considering a system of ordinary differential equations.

In this paper we study the existence of a periodic solution of the following delay differential equation

$$\frac{d}{dt}x(t) = rx(t)\left(1 - \int_0^1 x(t-s)ds\right), \tag{1.2}$$

where r is a positive parameter, $r > 0$. The delay differential equation (1.2) can be seen as a variant of the Hutchinson-Wright equation. The author's motivation to study (1.2) is that the equation appears as a limiting case of an infectious disease model with temporary immunity. For the equation (1.2), the existence of periodic solutions does not seem to be well understood. The periodicity, which may explain the recurrent disease dynamics, is a trigger of this study. Differently from the discrete delay case, the distributed delay is an obstacle, when one tries to construct a suitable Poincare map to find a periodic solution, but see [6, 11].

The existence of the periodic solution is proven, solving a corresponding ordinary differential equation, which turns out to be equivalent to the Duffing equation. The periodic solution, explicitly expressed in terms of the Jacobi elliptic functions, appears at $r = \frac{\pi^2}{2}$, as the positive equilibrium ($x = 1$) loses stability via Hopf bifurcation. We refer the reader to [7] for detail.

2. Preliminary

Observe that, defining

$$y(t) = \int_0^1 x(t-s)ds - 1,\ t \geq 0,$$

the delay differential equation (1.2) is equivalent to the following system of delay differential equations

$$\frac{d}{dt}x(t) = -rx(t)y(t), \tag{2.1a}$$

$$\frac{d}{dt}y(t) = x(t) - x(t-1) \tag{2.1b}$$

with the following initial condition

$$x(\theta) = \phi(\theta),\ \theta \in [-1, 0],$$
$$y(0) = \int_0^1 \phi(-s)ds - 1.$$

Assume that for (1.2) there exists a periodic solution of period 2. Denote by $x^*(t)$ the periodic solution, i.e., $x^*(t) = x^*(t-2)$. Then we let

$$x_1(t) = x^*(t),\ y_1(t) = \int_0^1 x^*(t-s)ds - 1,$$
$$x_2(t) = x^*(t-1),\ y_2(t) = \int_1^2 x^*(t-s)ds - 1.$$

We are interested in the positive periodic solution. The periodic solution satisfies the following system of ordinary differential equations

$$\frac{d}{dt}x_1(t) = -rx_1(t)y_1(t), \tag{2.2a}$$

$$\frac{d}{dt}y_1(t) = x_1(t) - x_2(t), \tag{2.2b}$$

$$\frac{d}{dt}x_2(t) = -rx_2(t)y_2(t), \tag{2.2c}$$

$$\frac{d}{dt}y_2(t) = x_2(t) - x_1(t). \tag{2.2d}$$

The initial condition is

$$x_1(0) = a > 0,\ x_2(0) = b > 0, \tag{2.3a}$$

$$y_1(0) = y_2(0) = 0, \tag{2.3b}$$

where a and b will be determined later ($a = x^*(0) = x^*(2),\ b = x^*(-1) = x^*(1)$), so that $x_1(t) = x_1(t+2)$ holds.

From (2.2) one sees that

$$y_1(t) + y_2(t) = 0, \tag{2.4a}$$

$$x_1(t)x_2(t) = ab \tag{2.4b}$$

hold for any $t \geq 0$. Thus one sees that the periodic solution satisfies the following properties

$$\int_0^2 x^*(t-s)ds = 2,\ x^*(t)x^*(t-1) = \mathrm{Const},\ t \in \mathbb{R}. \tag{2.5}$$

3. Integrable ordinary differential equations

The system (2.2) with (2.4) is reduced to the following system of ordinary differential equations

$$\frac{d}{dt}x(t) = -rx(t)y(t), \tag{3.1a}$$

$$\frac{d}{dt}y(t) = x(t) - ab\frac{1}{x(t)}, \tag{3.1b}$$

dropping the indices from x_1 and y_1 (cf. (2.1)). The initial condition of (3.1) is

$$x(0) = a, \tag{3.2a}$$

$$y\,(0) = 0 \tag{3.2b}$$

(see (2.3)).

命題 1. *It holds that*

$$x(t) + ab\frac{1}{x(t)} + \frac{r}{2}y^2(t) = a+b,\ t \in \mathbb{R} \tag{3.3}$$

for the solution of the equation (3.1) with the initial condition (3.2).

Differentiating the both sides of the equation (3.1b), we obtain

$$\begin{aligned} \frac{d^2}{dt^2}y(t) &= -\left(1 + ab\frac{1}{x^2(t)}\right) rx(t)y(t) \\ &= -ry(t)\left(x(t) + ab\frac{1}{x(t)}\right). \end{aligned}$$

Using the identity (3.3) in Proposition 1, we derive the Duffing equation:

$$\frac{d^2}{dt^2}y(t) = -ry(t)\left(a+b-\frac{r}{2}y^2(t)\right) \tag{3.4}$$

with the following initial condition

$$y(0) = 0, \tag{3.5a}$$

$$\frac{d}{dt}y(0) = x(0) - ab\frac{1}{x(0)} = a-b. \tag{3.5b}$$

Denote by sn the Jacobi elliptic sine function. It is known that the solution of the Duffing equation (3.4) is given by

$$y(t) = \alpha \mathrm{sn}(\beta t, k), \tag{3.6}$$

where α, β and k are functions of a and b defined by

$$\alpha(a,b) = \sqrt{\frac{2}{r}}\left(\sqrt{a}-\sqrt{b}\right),\ \beta(a,b) = \sqrt{\frac{r}{2}}\left(\sqrt{a}+\sqrt{b}\right), \tag{3.7}$$

$$k(a,b) = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}, \tag{3.8}$$

To simplify the notation, we occasionally drop (a, b) from α, β and k.

We then obtain the exact solution of the system (3.1) with the initial condition (3.2).

命題 2. *The solution of the equations (3.1) with the initial condition (3.2) is expressed as*

$$x(t) = a\left(\frac{1-k}{dn(\beta t, k) - k\,cn(\beta t, k)}\right)^2 = a\left(\frac{dn(\beta t, k) + k\,cn(\beta t, k)}{1+k}\right)^2, \tag{3.9}$$

$$y(t) = \alpha sn(\beta t, k), \tag{3.10}$$

where α, β and k are defined in (3.7) and (3.8).

4. Periodic solution of period 2

In this section we will determine a, the initial value for the x component of the system (3.1), so that, for the solution given in Proposition 2, the period is 2 and the integral constant becomes -1. The periodic solution finally solves the delay differential equation (1.2).

Let us introduce the complete elliptic integrals of the first kind and of the second kind. Those are respectively given as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta,$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}d\theta$$

for $0 \leq k < 1$. The Jacobi elliptic functions sn and cn are periodic functions with period $4K(k)$, i.e.,

$$\mathrm{sn}\,(t,k) = \mathrm{sn}\,(t+4K\,(k)\,,k)\,,\ \ \mathrm{cn}\,(t,k) = \mathrm{cn}\,(t+4K\,(k)\,,k)\,,\ \ t\in\mathbb{R}$$

and dn is periodic with period $2K(k)$.

In the following theorem we have two conditions so that the period of the solution given in Proposition 2 is two.

定理 3. *Assume that the following two conditions hold*

$$\sqrt{\frac{r}{2}}\left(\sqrt{a}+\sqrt{b}\right) = 2K(k), \tag{4.1}$$

$$\left(\sqrt{a}+\sqrt{b}\right)\sqrt{\frac{2}{r}}E(k) - \sqrt{ab} = 1. \tag{4.2}$$

Then, for the solution of the equation (3.1) with the initial condition (3.2), it holds that

$$(x(t), y\,(t)) = (x(t+2), y(t+2)) \tag{4.3}$$

and that

$$y(t) = \int_{t-1}^{t} x(s)ds - 1 \tag{4.4}$$

for any $t \in \mathbb{R}$.

The conditions (4.1) and (4.2) ensure the existence of a periodic solution of period 2 for the system of ordinary differential equations (3.1), satisfying (4.4). The periodic solution obtained in Theorem 3 is also a periodic solution of the delay differential equation (1.2). Our remaining task is to interpret the conditions (4.1) and (4.2) in terms of the parameter r in the equation (1.2).

Eliminating a and b from the conditions (4.1) and (4.2), we obtain the following equality

$$r = L(k),\ 0 \le k < 1, \tag{4.5}$$

where

$$L(k) := 2K(k)\left(2E(k) - K(k)\left(1-k^2\right)\right).$$

Now we show that the equation (4.5) has a unique root.

補題 4. *The function L is a strictly increasing function with*

$$L(0) = \frac{\pi^2}{2} < \lim_{k \to 1-0} L(k) = \infty.$$

Then, a and b are determined by the following Proposition.

命題 5. *There exist $a > 0$ and $b > 0$ such that the two conditions (4.1) and (4.2) in Theorem 3 hold if and only if $r > \frac{\pi^2}{2}$. In particular, a and b are given as*

$$\left[\begin{array}{c} a \\ b \end{array}\right] = \frac{K(k)}{2E(k) - K(k)\left(1-k^2\right)} \left[\begin{array}{c} (1+k)^2 \\ (1-k)^2 \end{array}\right] = \frac{2K(k)^2}{r} \left[\begin{array}{c} (1+k)^2 \\ (1-k)^2 \end{array}\right], \tag{4.6}$$

where $k = L^{-1}(r),\ r > \frac{\pi^2}{2}$.

Finally we introduce the following theorem

定理 6. *Let $r > \frac{\pi^2}{2}$. Then the delay differential equation (1.2) has a periodic solution of period 2. The periodic solution is expressed as in (3.9), where a and b are determined in Proposition 5.*

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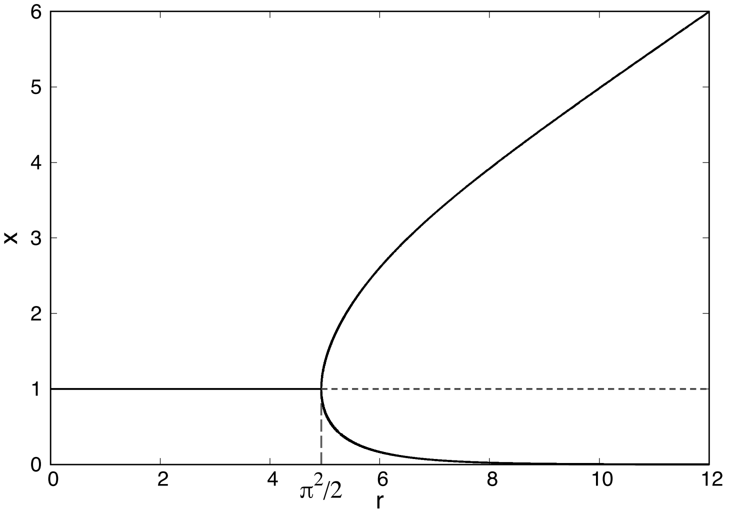

図 4.1. Bifurcation of the equilibrium. The equilibrium $x = 1$ is asymptotically stable for $r < \frac{\pi^2}{2}$ and is unstable for $r > \frac{\pi^2}{2}$. At $r = \frac{\pi^2}{2}$ a Hopf bifurcation occurs and the periodic solution appears.

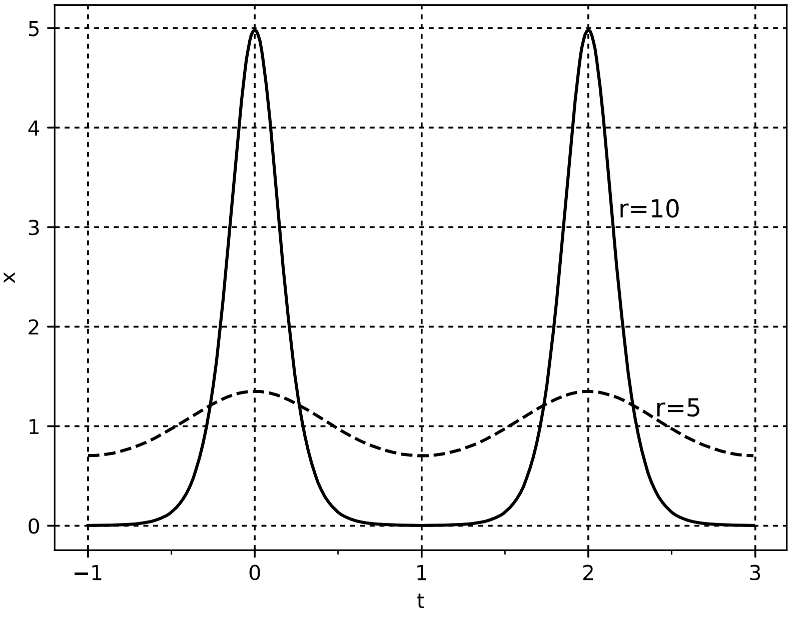

図 4.2. Time profile of the periodic solution for $r = 5$ and $r = 10$.

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