Global bifurcation structure of a limiting system to the SKT competition model with cross-diffusion *

Shoji Yotsutani †
Department of Applied Mathematics and Informatics, Ryukoku University
Seta, Otsu, 520-2194, Japan

1 Introduction

This is a joint work with Yuan Lou (The Ohio State University), Wei-Ming Ni (The Chinese University of Hong Kong and University of Minnesota), Tatsuki Mori (Osaka University), and Shota Yamakawa (Ryukoku University).

We have been interested in the cross-diffusion system

\[(P)\]
\[
\begin{aligned}
    u_t &= \Delta \left[(d_1 + \alpha_{11}u + \alpha_{12}v)u + u(a_1 - b_1u - c_1v)\right] \text{ in } \Omega \times (0, \infty), \\
    v_t &= \Delta \left[(d_2 + \alpha_{21}u + \alpha_{22}v)v + v(a_2 - b_2u - c_2v)\right] \text{ in } \Omega \times (0, \infty), \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, \infty), \\
    u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{ in } \Omega,
\end{aligned}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $\nu$ is the outward unit normal vector on $\partial \Omega$.

This mathematical model was proposed by Shigesada, Kawasaki and Teramoto [8] in 1979 to investigate segregation phenomena of two competing species with each other in the same habitat area. Here, $u = u(x, t)$ and $v = v(x, t)$ represent the densities of two competing species, $d_1$ and $d_2$ are their diffusion coefficients, $a_1$ and $a_2$ denote the intrinsic growth rates of these two species, $b_1$ and $c_2$ account

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† E-mail addresses: shoji@math.ryukoku.ac.jp
for intra-specific competitions while $b_2$ and $c_1$ account for inter-specific competitions. The constants $\alpha_{11}$ and $\alpha_{22}$ represent intra-specific population pressures, also known as self-diffusion rates, and $\alpha_{12}$ and $\alpha_{21}$ are the coefficients of inter-specific population pressures, also known as cross-diffusion rates.

The effect of cross-diffusion affects the population pressure between two different kinds. It is an interesting problem to see whether this effect may give rise to a spatial segregation or not, and clarify its mechanism.

We should remark that it is well known that the important quantities involving the constants $a_i, b_i, c_i \ (i = 1, 2)$ are only

$$A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}. \quad (1.5)$$

It seems natural to consider the following two cases separately: the ”strong competition” case $B < C$ and the ”weak competition” case $C < B$. The behavior of solution in case $B < C$ is very different from $C > B$.

We refer to [7] and [8] for further details of this model.

A lot of research works are done by the singular perturbation method, which started from a theoretical research by Mimura [5]. Kan-on [1] obtained some criteria on the stability of those non-constant solutions of (P). However, it is not easy to clarify the global structure of stationary solutions and stability of stationary solutions.

Lou and Ni [2], [3] started to investigate N-dimensional case and general diffusion coefficients. To investigate the cross-diffusion effects, let us put $\alpha_{11} = \alpha_{21} = \alpha_{22} = 0$ and $r := \alpha_{12}/d_1$. We have

\[
\begin{align*}
\text{(TP}_r^N) \quad \begin{cases} 
 u_t &= d_1 \Delta [(1 + rv)u] + u(a_1 - b_1 u - c_1 v) \quad \text{in } \Omega \times (0, \infty), \\
 v_t &= d_2 \Delta v + v(a_2 - b_2 u - c_2 v) \quad \text{in } \Omega \times (0, \infty), \\
 \frac{\partial u}{\partial v} &= \frac{\partial v}{\partial v} = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
 u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega,
\end{cases}
\end{align*}
\]

where $u = u(x, t)$ and $v = v(x, t)$. Then, the stationary problem of (TP}_r^N is

\[
\begin{align*}
\text{(SN}_r^N) \quad \begin{cases} 
 d_1 \Delta [(1 + rv)u] + u(a_1 - b_1 u - c_1 v) &= 0 \quad \text{in } \Omega, \\
 d_2 \Delta v + v(a_2 - b_2 u - c_2 v) &= 0 \quad \text{in } \Omega, \\
 \frac{\partial u}{\partial v} &= \frac{\partial v}{\partial v} = 0 \quad \text{on } \partial \Omega, \\
 u \geq 0, \quad v \geq 0 \quad \text{in } \Omega,
\end{cases}
\end{align*}
\]

where $u = u(x)$ and $v = v(x)$. 

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They obtained limiting systems as $r \to \infty$ for $(TP_N^r)$ and $(SN^r)$. One of limiting systems as $r \to \infty$ are as follows. The time-dependent limiting system is

\[
(TP_N^\infty) \begin{cases}
\frac{\partial}{\partial t} \int_\Omega \frac{\tau}{v} dx = \int_\Omega \frac{\tau}{v} \left( a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx \quad \text{in } (0, \infty), \\
\frac{\partial v}{\partial t} = d_2 \Delta v + v(\tau(a_2 - c_2 v) - b_2 \tau) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial v} = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
v(0, t) = v_0(x) > 0 \quad \text{in } \Omega,
\end{cases}
\]

where $v = v(x, t)$ and $\tau = \tau(t)$ are unknown positive functions, and $\tau(t)/v(x, t)$ corresponds to $u(x, t)$. The stationary limiting system is

\[
(S_N^\infty) \begin{cases}
\int_\Omega \frac{\tau}{v} \left( a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx = 0, \\
\frac{d_2}{v} \Delta v + v(\tau(a_2 - c_2 v) - b_2 \tau) = 0 \quad \text{in } \Omega, \\
\frac{\partial v}{\partial v} = 0 \quad \text{on } \partial \Omega, \\
v(x) > 0, \quad \text{in } \Omega,
\end{cases}
\]

where $v = v(x)$ is an unknown positive function, $\tau$ is an unknown positive constant.

For one-dimension $\Omega := (0, 1)$, the limiting system corresponding $(TP_{N}^\infty)$ and $(SN_{N}^\infty)$ are

\[
(TP_1^\infty) \begin{cases}
\frac{\partial}{\partial t} \left( \int_0^1 \frac{\tau}{v} dx \right) = \int_0^1 \frac{\tau}{v} \left( a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx \quad \text{in } (0, 1) \times (0, \infty), \\
\frac{\partial v}{\partial t} = d_2 v_{xx} + v(\tau(a_2 - c_2 v) - b_2 \tau) \quad \text{in } (0, 1), \\
v_x(0, t) = 0, \quad v_x(1, t) = 0, \quad \text{in } (0, \infty), \\
v(x, 0) = v_0(x) > 0, \quad \text{in } (0, 1),
\end{cases}
\]

and

\[
(S_1^\infty, \text{general}) \begin{cases}
\int_0^1 \frac{\tau}{v} \left( a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx = 0, \\
d_2 v_{xx} + v(\tau(a_2 - c_2 v) - b_2 \tau) = 0 \quad \text{in } (0, 1), \\
v_x(0) = 0, \quad v_x(1) = 0, \quad \text{in } (0, \infty), \\
v(x) > 0 \quad \text{in } (0, 1).
\end{cases}
\]

Lou, Ni and Yotsutani [4] obtained existence and non-existence of non-constant steady state solutions, the asymptotic shape of solutions, and almost clarified the structure of solutions of $(S_1^\infty, \text{general})$. 3
In what follows, we concentrate on the monotone increasing case $v_x(x) > 0$ to understand the essence of structure of $(S^1_{\infty, \text{general}})$.

Now, we introduce a $(S^1_{\infty})$ as follows:

$$
(S^1_{\infty}) \begin{cases}
\int_0^1 \tau \left( a_1 - b_1 \frac{\tau}{v} - c_1 v \right) dx = 0, \\
d_2 v_{xx} + v(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in} \ (0,1), \\
v_x(0) = 0, \quad v_x(1) = 0, \\
v(x) > 0, \quad v_x(x) > 0 \quad \text{in} \ (0,1).
\end{cases}
$$

(1.30) \hspace{6cm} (1.31) \hspace{6cm} (1.32) \hspace{6cm} (1.33)

2 \hspace{1cm} Results

We first explain results in [4] for $(S^1_{\infty})$. As for the existence and non-existence, the following theorems are obtained:

**Theorem A** (Existence, weak competition). *Suppose that $C \leq B$.*

(i) If $B \leq A$ then there exists a solution $(v, \tau)$ of $(S^1_{\infty})$.

(ii) If $(B + 3C)/4 < A < B$, then there exists a solution of $(S^1_{\infty})$ for $d_2 \in (0, \frac{2A - (B + C)}{B - C} \cdot \frac{a_2}{\pi^2})$.

**Theorem B** (Non-Existence, weak competition). *Suppose that $C \leq B$.*

(i) If $d_2 > a_2/\pi^2$, then there exists no solution of $(S^1_{\infty})$.

(ii) If $(B + 3C)/4 < A < B$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of $(S^1_{\infty})$ for $d_2 \in (d_2^*, a_2/\pi^2)$.

(iii) If $A \leq (B + 3C)/4$, there exists no solution of $(S^1_{\infty})$.
Figure 1 shows the existence and non-existence region of solutions of \((S_{1\infty})\) in the case \(C \leq B\) assured by theorems A and B. Here, horizontal axis is \(A\), vertical axis is \(d_2\). For the case \(d_2\) sufficiently close to 0 and \((B + 3C)/4 < A < (B + C)/2\), existence and non-existence of solutions of \((S_{1\infty})\) are not clear.

Figure 2 shows the existence and non-existence region of solutions of \((S_{1\infty})\) in the case \(B < C\) assured by theorems C and D. For the case \(0 < d_2 < ((B + C - 2A)/(C - B)) \cdot (a_2/\pi^2)\) and \(B < A < (B + C)/2\), existence and non-existence of solutions of \((S_{1\infty})\) also are not clear.

**Theorem C** (Existence, strong competition). Suppose that \(B < C\). If
\[
\max\left\{ 0, \frac{B + C - 2A}{C - B} \cdot \frac{a_2}{\pi^2} \right\} < d_2 < \frac{a_2}{\pi^2},
\]
then there exists a solution \((v, \tau)\) of \((S_{1\infty})\).

**Figure 2:** Existence and non-existence of solutions of \((S_{1\infty})\) for \(B < C\).

**Theorem D** (Non-Existence, strong competition). Suppose that \(B < C\).

(i) If \(d_2 > a_2/\pi^2\), then there exists no solution of \((S_{1\infty})\).

(ii) If \(B \leq A < (B + C)/2\), then there exists a \(d_2^* = d_2^*(A, B, C, a_2) > 0\) such that there exists no solution of \((S_{1\infty})\) for \(d_2 \in (0, d_2^*)\).

(iii) If \(A < B\), there exists no solution of \((S_{1\infty})\).

In [9], Lou, Ni and Yotsutani conjectured that the situation of existence, non-existence and the uniqueness drastically changes at \(C = (7/3)B\). For the case \(B < C \leq (7/3)B\), the uniqueness seems to hold as shown in Figures 3 and 4. Recently, we have found a mathematical proof of this case.
Figure 3: $C = B$.

Figure 4: Existence and non-existence of solutions of $(S^1_\infty)$ for $B < C \leq (7/3)B$.

Figure 5: Existence and non-existence of solutions of $(S^1_\infty)$ for $C > (7/3)B$.

On the other hand, for the case $C > (7/3)B$, the existence region becomes wider as shown in Figure 5. In [6], Mori, Suzuki and Yotsutani have obtained precise numerical results with the stability and instability for this case.

As explained above, existence, non-existence and multiplicity of solutions for the case $B \leq C$ are precisely understood.
However, it is not clarified the case $C < B$. Therefore, we investigate this case. Figure 6 show existence, non-existence and multiplicity of non-constant solutions for $(S_1^1)$ obtained by numerical computation.

![Diagram showing existence and non-existence of solutions](image)

Figure 6: $0 < C < B$.

### 3 Representation of solutions

We explain the representation of solutions of $(S_1^1)$, since it is very efficient for investigating the solution structure of $(S_1^1)$.

Let us introduce a notation. Jacobi’s elliptic function $\text{sn}(x, k)$ defined by

$$
\text{sn}^{-1}(z, k) = \int_0^z \frac{d\xi}{\sqrt{1 - k^2\xi^2}\sqrt{1 - \xi^2}} \tag{3.1}
$$

for $-1 \leq z \leq 1$. The complete elliptic integrals of the first, second and third kind are defined by

$$
K(k) := \int_0^1 \frac{d\xi}{\sqrt{1 - k^2\xi^2}\sqrt{1 - \xi^2}}, \quad E(k) := \int_0^1 \sqrt{1 - k^2\xi^2} \frac{d\xi}{\sqrt{1 - \xi^2}}, \quad \Pi(\nu, k) := \int_0^1 \frac{d\xi}{(1 + \nu\xi^2)\sqrt{1 - k^2\xi^2}\sqrt{1 - \xi^2}} \tag{3.2}
$$

and

$$
\Pi(\nu, k) := \int_0^1 \frac{d\xi}{(1 + \nu\xi^2)\sqrt{1 - k^2\xi^2}\sqrt{1 - \xi^2}} \tag{3.3}
$$

for $0 \leq k < 1$ and $-1 < \nu$, respectively.
In what follows in (S\(^1_{\infty}\)), we will concentrate on the case
\[ b_1 = 1 \quad \text{and} \quad a_2 = b_2 = c_2 = 1. \] (3.4)

In fact, we get from (S\(^1_{\infty}\)).
\[
\int_0^1 \frac{1}{\bar{v}} \left( \frac{A}{B} - \frac{\bar{\tau}}{\bar{v}} - \frac{C}{B} \bar{v} \right) dx = 0, \tag{3.5}
\]

\[
\bar{d}_2 \bar{v}_{xx} + \bar{v}(1 - \bar{v}) - \bar{\tau} = 0 \quad \text{in} \ (0,1), \tag{3.6}
\]

\[
\bar{v}_x(0) = 0, \quad \bar{v}_x(1) = 0, \tag{3.7}
\]

\[
\bar{v}(x) > 0, \quad \bar{v}_x(x) > 0 \quad \text{in} \ (0,1) \tag{3.8}
\]

by employing the following change of variables
\[
\bar{v} := \frac{c_2}{a_2} v, \quad \bar{\tau} := \frac{b_2 c_2}{a_2} \tau, \quad \bar{d}_2 := \frac{d_2}{a_2}. \tag{3.9}
\]

Thus, without lose of generality, we may consider the case \( b_1 = 1 \) and \( a_2 = b_2 = c_2 = 1 \).

Now, we introduce an auxiliary problem to investigate \( (S^1_{\infty}) \) with \( b_1 = a_2 = b_2 = c_2 = 1 \). Let \( d_2 > 0 \) be given. Unknowns are a function \( v = v(x) \) and a constant \( \tau > 0 \).

\[
(E) \begin{cases} 
\bar{d}_2 v_{xx} + v(1 - v) - \tau = 0 & \text{in} \ (0,1), \\
v(x) > 0 \text{ in } [0,1] \text{ and } v_x(x) > 0 & \text{in} \ (0,1), \\
v_x(0) = 0, \ v_x(1) = 0 \text{ and } \tau > 0.
\end{cases} \tag{3.10}
\]

Exact solutions of \( (E) \) are given in the following proposition.

**Proposition 3.1.** \( (E) \) has a solution if and only if \( d_2 \in (0, 1/\pi^2) \). All solutions \((v(x), \tau) \) of \( (E) \) are represented by
\[
v(x; d_2, h) = \alpha + (\beta - \alpha) \text{sn}^2(K(\sqrt{h})x, \sqrt{h}), \tag{3.13}
\]

\[
\tau(d_2, h) = \frac{\alpha \beta + \beta \gamma + \gamma \alpha}{3} = \frac{1}{4} - 4 d_2^2 (h^2 - h + 1) K(\sqrt{h})^4, \tag{3.14}
\]

where
\[
\alpha = \frac{1}{2} - 2 d_2 K(\sqrt{h})^2 (h + 1), \tag{3.15}
\]

\[
\beta = \frac{1}{2} + 2 d_2 K(\sqrt{h})^2 (2h - 1), \tag{3.16}
\]

\[
\gamma = \frac{1}{2} + 2 d_2 K(\sqrt{h})^2 (2 - h). \tag{3.17}
\]

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Here \( \hat{h} \) is the unique solution of an equation

\[
(h + 1)K(\sqrt{h})^2 = \frac{1}{4d_2}
\]  

(3.18)

in \( h \), \( K(\sqrt{h}) \) is the complete elliptic integral of the 1st kind, and \( \text{sn}(\cdot, \cdot) \) is Jacobi’s elliptic function.

Now, we note that (1.30) with \( b_1 = 1 \) is rewritten as

\[
\frac{\tau}{\int_0^1 \frac{1}{v^2} dx + c_1} \int_0^1 \frac{1}{v} dx = a_1.
\]  

(3.19)

Thus, let us define a function \( \tilde{a}_1(h; d_2, c_1) \) by

\[
\tilde{a}_1(h; d_2, c_1) := \frac{\tau}{\int_0^1 \frac{1}{v(x; d_2, h)^2} dx + c_1} \int_0^1 \frac{1}{v(x; d_2, h)} dx.
\]  

(3.20)

\( \tilde{a}_1(h; d_2, c_1) \) is explicitly given in the following proposition.

**Proposition 3.2.** Let \( d_2 \in (0, 1/\pi^2) \), \( h \in (0, \hat{h}(d_2)) \). It holds that

\[
\tilde{a}_1(h; d_2, c_1) = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{6\alpha\beta\gamma \Pi \left( \frac{\beta - \alpha}{\alpha}, \sqrt{h} \right)} \cdot \left( (\gamma - \alpha)\alpha E(\sqrt{h}) - \alpha\gamma K(\sqrt{h}) + (\alpha\beta + \beta\gamma + \gamma\alpha) \Pi \left( \frac{\beta - \alpha}{\alpha}, \sqrt{h} \right) \right) + \frac{\alpha K(\sqrt{h})c_1}{\Pi \left( \frac{\beta - \alpha}{\alpha}, \sqrt{h} \right)}.
\]  

(3.21)

where \( \alpha, \beta \) and \( \gamma \) are defined by (3.15), (3.16) and (3.17) respectively. Here, \( K(\cdot), E(\cdot) \) and \( \Pi(\cdot, \cdot) \) are the complete elliptic integral of the 1st, 2nd and 3rd kind, respectively.
We explain the reason that the existence and non-existence regions change at $c_1 = 7/3$ ($C/B = 7/3$). We obtain
\[
\tilde{a}_1(h; d_2, c_1) = \frac{1}{2} \left( d_2 \pi^2(1 - c_1) + (1 + c_1) \right) + \tilde{a}_{1,2} \cdot h^2 + \cdots, \tag{3.22}
\]
by Taylor’s expansion of (3.21) in $h$, where
\[
\tilde{a}_{1,2} := \frac{3d_2 \pi^2}{64(1 - \pi^2 d_2)^2} \left( (35 + 13c_1)\pi^2 d_2^2 - 14\pi^2(c_1 - 1)d_2 + (c_1 - 1) \right). \tag{3.23}
\]
We check the sign of the coefficient $\tilde{a}_{1,2}$. We get $d_2 = d_+$ and $d_-$ by solving
\[
(35 + 13c_1)\pi^2 d_2^2 - 14\pi^2(c_1 - 1)d_2 + (c_1 - 1) = 0, \tag{3.24}
\]
where
\[
d_+ := \frac{7(c_1 - 1) + 2\sqrt{3(c_1 - 1)(3c_1 - 7)}}{\pi^2(35 + 13c_1)}, \tag{3.25}
\]
and
\[
d_- := \frac{7(c_1 - 1) - 2\sqrt{3(c_1 - 1)(3c_1 - 7)}}{\pi^2(35 + 13c_1)}. \tag{3.26}
\]
Thus,
\[
\begin{align*}
\tilde{a}_{1,2} < 0 \quad &\text{for} \quad 0 < c_1 < 1, \quad 0 < d_2 < d_+, \quad (3.27) \\
\tilde{a}_{1,2} \geq 0 \quad &\text{for} \quad 1 \leq c_1 \leq 7/3, \quad 0 < d_2 < 1/\pi^2, \quad (3.28) \\
\tilde{a}_{1,2} \geq 0 \quad &\text{for} \quad c_1 > 7/3, \quad d_+ \leq d_2 < 1/\pi^2, \quad (3.29) \\
\tilde{a}_{1,2} < 0 \quad &\text{for} \quad c_1 > 7/3, \quad d_- < d_2 < d_+, \quad (3.30) \\
\tilde{a}_{1,2} \geq 0 \quad &\text{for} \quad c_1 > 7/3, \quad 0 < d_2 \leq d_. \quad (3.31)
\end{align*}
\]
Therefore, the behavior of $\tilde{a}_1(h, d_2, c_1)$ near $h = 0$ is drastically change at $c_1 = 1$ and $c_1 = 7/3$.

References


