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Methods of improving correction term estimates in the BCS model with imaginary magnetic field

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1 Introduction

In this article we present what the speaker of the lecture with the same title wrote on the black board together with his explanations and answers to questions raised during the lecture, which was held from 11:00 to 11:50, July 2nd, 2018, in Room 111, RIMS, Kyoto University. We intend not to explain far more than the lecture. This is because we believe that the parallel statements would be good review of the lecture and those who want full explanations should read the original papers [9], [10]. However, we also plan to present some different information from the lecture. This is because he gave a few incorrect explanations which need correction.

The contents of this article as well as the lecture are based on the papers [9], [10]. We will state a theorem and lemmas without proofs. The proofs are found in these papers.

2 The BCS model

We study a many-electron system governed by the BCS model, which was proposed by Bardeen, Cooper and Schrieffer in 1957 ([2]). Let us start by defining the model. Let d, L be positive integers. Imposing periodic boundary conditions, we define the spatial lattice Γ by $\Gamma := (\mathbb{Z}/L\mathbb{Z})^d$. Though there are many ways to generalize, let us focus on a simple free Hamiltonian which describes free electrons hopping between nearest-neighbor sites.

$$H_0 := \sum_{\mathbf{x}\in\Gamma}\sum_{\sigma\in\{\uparrow,\downarrow\}}\left(\sum_{j=1}^{d}(\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}+\mathbf{e}_j,\sigma}+\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}-\mathbf{e}_j,\sigma})-\mu\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}\right), \tag{2.1}$$

where $\mathbf{e}_j$ $(j=1,\cdots,d)$ are the standard basis of $\mathbb{R}^d$, $\mu(\in\mathbb{R})$ is chemical potential and $\psi^*_{\mathbf{x}\sigma}$, $\psi_{\mathbf{x}\sigma}$ are Fermionic creation operator, annihilation operator, respectively. On the other hand, the interacting part of the whole Hamiltonian is defined by

$$V := \frac{U}{L^d}\sum_{\mathbf{x},\mathbf{y}\in\Gamma}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\uparrow},$$

where U is a real negative parameter controlling the strength of long range attraction between Cooper pairs. Since it is simply a product of 2 Cooper pair operators, the operator V is sometimes called the reduced BCS interaction. The BCS model H is defined by $H := H_0 + V$, which is a self-adjoint operator on the Fermionic Fock space $F_f(L^2(\Gamma \times \{\uparrow, \downarrow\}))$. Since it has the reduced BCS interaction, the Hamiltonian H is sometimes called the reduced BCS model.

Our aim is to prove existence of superconducting phase transition. Let us recall two canonical characteristics of superconducting phase. In the following $\beta(\in \mathbb{R}_{>0})$ denotes inverse temperature.

Definition 2.1. We say that Spontaneous Symmetry Breaking (SSB) occurs in the system if for any $\mathbf{x} \in \mathbb{Z}^d$

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in\mathbb{R}}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta(H+F)}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow})}{\mathrm{Tr}\, e^{-\beta(H+F)}}$$

converges to a non-zero value. Here the operator F is defined by

$$F := \gamma \sum_{\mathbf{x}\in\Gamma} (\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow} + \psi_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\uparrow}), \quad \gamma \in \mathbb{R}.$$

Definition 2.2. We say that Off-Diagonal Long Range Order (ODLRO) occurs in the system if

$$\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{\mathrm{Tr}(e^{-\beta H}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\uparrow})}{\mathrm{Tr}\, e^{-\beta H}}$$

converges to a non-zero value. Here for a function $f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ and $c \in \mathbb{C}$ we write $\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty} f(\mathbf{x}, \mathbf{y}) = c$ if

$$\forall \varepsilon \in \mathbb{R}_{>0} \exists M \in \mathbb{R}_{>0} (\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d \wedge \|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d} \ge M \to |f(\mathbf{x}, \mathbf{y}) - c| < \varepsilon)$$

One fundamental question we face is the following. Do SSB and ODLRO occur in the BCS model ? It is still a fair remark that we have not achieved a consensus on this question. Despite a long history of mathematical research around the BCS model, we can hardly find a paper answering this question at a rigorous level. Let us refer to [4] where SSB and ODLRO are proved in the BCS model without hopping, so-called the strong coupling limit of the BCS model, by a C*-algebraic approach. We find that there are many papers on quasi-spin formulations of the BCS model, which are reduction of the BCS model into quantum spin systems. See e.g. the review article [5] for the preceding papers on the quasi-spin formulations. Another active mathematical approach to the BCS theory is analysis of the BCS energy functional, which is a functional of generalized one-particle density matrices. Let us refer to the review article [6] by the developers of this approach. However, we should add that equivalence between SSB and ODLRO in a quasi-spin formulation or in the minimizer of the BCS energy functional and those in the BCS model is not rigorously established yet. There are a few papers studying BCS-type interactions in Grassmann integral formulation ([12], [13]). In these papers the reduced BCS interaction is approximated at the level of Grassmann algebra and thus the results do not directly imply SSB and ODLRO in the BCS model. Though we consider a different model, our analysis is also based on Grassmann integral formulation.

3 The BCS model with imaginary magnetic field

Let us consider the operator $H + i\theta S_z$, where $\theta \in \mathbb{R}$ and S_z is the z-component of the spin operator defined by

$$S_z := \frac{1}{2}\sum_{\mathbf{x}\in\Gamma}(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow} - \psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow}).$$

The operator $H + i\theta S_z$ should be formally considered as the BCS model interacting with an imaginary magnetic field. Though it is not trivial, by symmetries and periodicity we may assume that $\theta \in [0, 2\pi/\beta]$ without losing generality. The following theorem is a part of the main theorem of [10].

Theorem 3.1. *Assume that* $\theta \in [0, 2\pi/\beta)$, $d = 3, 4$, $\mu = 2d$ *and* $\beta \geq 1$. *Then there exists a constant* $c \in \mathbb{R}_{>0}$ *independent of any parameter such that* (i), (ii) *hold for any* $U \in (-c, 0)$.

(i) There exists $L_0 \in \mathbb{N}$ *such that*

$$\mathrm{Tr}\, e^{-\beta(H+i\theta S_z+F)} \in \mathbb{R}_{>0}, \quad (\forall L \in \mathbb{N} \text{ with } L \geq L_0,\ \gamma \in [0,1]).$$

(ii) There exists $\delta(U,\beta) \in \mathbb{R}_{>0}$ *depending only on* U, β *such that if* $|\theta/2 - \pi/\beta| < \delta(U,\beta)$, *then SSB and ODLRO occur in the system governed by* $H + i\theta S_z$, *i.e. for any* $\mathbf{w} \in \mathbb{Z}^d$

$$\lim_{\substack{\gamma\searrow 0\\ \gamma\in\mathbb{R}}}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(H+i\theta S_z+F)}\psi^*_{\mathbf{w}\uparrow}\psi^*_{\mathbf{w}\downarrow})}{\mathrm{Tr}\, e^{-\beta(H+i\theta S_z+F)}},$$

$$\lim_{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^d}\to\infty}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\mathrm{Tr}(e^{-\beta(H+i\theta S_z)}\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\uparrow})}{\mathrm{Tr}\, e^{-\beta(H+i\theta S_z)}}$$

converge to non-zero values.

Remark 3.2. Since $H + i\theta S_z + F$ is not self-adjoint, the claim (i) is not trivial.

Remark 3.3. We have to assume that $\theta \neq 2\pi/\beta$, which is a loss of generality. For $\theta = 2\pi/\beta$ the free partition function may vanish (see (4.1)) and accordingly the denominator of the free covariance may be zero. Since the free covariance is a central object in this approach, we have to ensure its well-definedness and thus exclude this case.

Remark 3.4. In [10] the same results are proved for many other free Hamiltonians including the nearest-neighbor hopping model on the honeycomb lattice with zero chemical potential. Degeneracy of free Fermi surface, which is equal to zero set of free dispersion relation, is one particular property that these models have in common. Also, in [10] SSB and ODLRO are proved for $\beta \in (0, 1)$. However, in this case an additional (β, θ)-dependent condition must be imposed on $|U|$. On the other hand, in [9] the free Hamiltonian is defined as in (2.1) and SSB and ODLRO are proved for any $d \in \mathbb{N}$, $\mu \in (-2d, 2d)$. Though the free Fermi surface is naturally non-degenerate in this case, we have to impose complex (β, θ)-dependent restrictions on $|U|$ instead.

Remark 3.5. During the talk the speaker did not mention the dependency of "δ" on β in the claim (ii). In fact it does depend on β and for this reason we cannot prove SSB and ODLRO for $\theta = 0$, i.e. in the BCS model without imaginary magnetic field. The claim (ii) guarantees that for any $U \in (-c, 0)$, $\beta \geq 1$ we can choose $\theta \in [0, 2\pi/\beta)$ so that $|\theta/2 - \pi/\beta| < \delta(U, \beta)$ and thus SSB and ODLRO occur.

4 Grassmann integral formulation

Let us assume that $\theta \in [0, 2\pi/\beta)$ in the following. The theorem is proved by analyzing Grassmann Gaussian integral formulation of the thermal expectations. Let us review basic notions of finite-dimensional Grassmann Gaussian integral. Take $h \in (2/\beta)\mathbb{N}$ and set

$$[0,\beta)_h := \left\{0, \frac{1}{h}, \frac{2}{h}, \cdots, \beta - \frac{1}{h}\right\}.$$

The set $[0,\beta)_h$ is a discretization of $[0,\beta)$ with the mesh size $1/h$. Moreover, set

$$I_0 := \{1,2\} \times \Gamma \times [0,\beta)_h, \quad I := I_0 \times \{1,-1\}.$$

The finite set I is the index set of Grassmann algebra on which our model is formulated. Let W be the complex vector space spanned by the abstract basis $\{\psi_X\}_{X\in I}$. For $n \in \mathbb{N}$ let $\bigwedge^n W$ denote the n-fold anti-symmetric tensor product of W and set $\bigwedge^0 W := \mathbb{C}$. Then we set

$$\bigwedge W := \bigoplus_{n=0}^{\sharp I} \bigwedge^n W.$$

We call it Grassmann algebra generated by $\{\psi_X\}_{X\in I}$. For a covariance $C : I_0^2 \to \mathbb{C}$ the Grassmann Gaussian integral $\int \cdot d\mu_C(\psi)$ is a linear functional on $\bigwedge W$ defined by

$$\begin{aligned}
&\int 1 d\mu_C(\psi) := 1,\\
&\int \overline{\psi}_{X_1} \cdots \overline{\psi}_{X_m} \psi_{Y_n} \cdots \psi_{Y_1} d\mu_C(\psi) := \begin{cases} \det(C(X_i, Y_j))_{\substack{1\le i\le m\\ 1\le j\le m}} & \text{if } m = n,\\ 0 & \text{else,}\end{cases}\\
&(\forall X_1, \cdots, X_m, Y_1, \cdots, Y_n \in I_0)
\end{aligned}$$

and by linearity and anti-symmetry. Here we set $\overline{\psi}_X := \psi_{(X,1)}$, $\psi_X := \psi_{(X,-1)}$ for $X \in I_0$.

To present our Grassmann Gaussian integral formulation of the normalized partition function, we need to prepare some more notations. The momentum lattice Γ^* is defined by $\Gamma^* := (\frac{2\pi}{L}\mathbb{Z}/2\pi\mathbb{Z})^d$. The free dispersion relation $e(\cdot) : \mathbb{R}^d \to \mathbb{R}$ is given by

$$e(\mathbf{k}) := 2\sum_{j=1}^{d} \cos k_j - \mu, \quad \mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{R}^d.$$

Define the Grassmann polynomials $\hat{V}_s(\psi)$, $\hat{V}_v(\psi)$, $\hat{V}(\psi) \in \bigwedge W$ by

$$\begin{aligned}
\hat{V}_s(\psi) &:= -\frac{U}{L^d h} \sum_{\mathbf{x}\in\Gamma} \sum_{s\in[0,\beta)_h} \overline{\psi}_{\mathbf{1x}s} \psi_{\mathbf{1x}s},\\
\hat{V}_v(\psi) &:= -\frac{U}{L^d h^2} \sum_{\mathbf{x},\mathbf{y}\in\Gamma} \sum_{s,t\in[0,\beta)_h} \left(h 1_{s=t} - \frac{1}{\beta}\right) \overline{\psi}_{\mathbf{1x}s} \psi_{\mathbf{2x}s} \overline{\psi}_{\mathbf{2y}t} \psi_{\mathbf{1y}t},\\
\hat{V}(\psi) &:= \hat{V}_s(\psi) + \hat{V}_v(\psi),
\end{aligned}$$

where for a proposition P, $1_P := 1$ if P is true, $1_P := 0$ otherwise. For $\phi \in \mathbb{C}$ the parameterized covariance $C(\phi) : I_0^2 \to \mathbb{C}$ is defined by

$$C(\phi)(\rho\mathbf{x}s, \eta\mathbf{y}t) := \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in M_h} e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle + i\omega(s-t)} h^{-1} \left(I_2 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{1}{h}E(\phi)(\mathbf{k})} \right)^{-1} (\rho, \eta),$$

where M_h is a finite subset of the Matsubara frequencies defined by

$$M_h := \left\{ \omega \in \frac{\pi}{\beta}(2\mathbb{Z}+1) \;\Big|\; |\omega| < \pi h \right\},$$

I_2 denotes the 2×2 unit matrix and

$$E(\phi)(\mathbf{k}) := \begin{pmatrix} e(\mathbf{k}) & \overline{\phi} \\ \phi & -e(\mathbf{k}) \end{pmatrix}.$$

We need Grassmann integral formulation of the thermal expectations to prove SSB and ODLRO. However, we only state the formulation of the partition function for conciseness. The thermal expectations are formulated in a similar way.

Lemma 4.1.

$$\frac{\operatorname{Tr} e^{-\beta(H+i\theta S_z)}}{\operatorname{Tr} e^{-\beta(H_0+i\theta S_z)}} = \frac{\beta L^d}{\pi |U|} \int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{-\frac{\beta L^d}{|U|}|\phi|^2} \frac{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2) + \cosh(\beta\sqrt{e(\mathbf{k})^2 + |\phi|^2}))}{\prod_{\mathbf{k}\in\Gamma^*}(\cos(\beta\theta/2) + \cosh(\beta e(\mathbf{k})))}$$
$$\cdot \lim_{\substack{h\to\infty \\ h\in(2/\beta)\mathbb{N}}} \int e^{\hat{V}(\psi)} d\mu_{C(\phi)}(\psi),$$

where $\phi := \phi_1 + i\phi_2 \in \mathbb{C}$, $|\phi| := \sqrt{\phi_1^2 + \phi_2^2}$.

Remark 4.2. For well-definedness we have to know that $\operatorname{Tr} e^{-\beta(H_0+i\theta S_z)} \neq 0$. This is true, since

$$\operatorname{Tr} e^{-\beta(H_0+i\theta S_z)} = e^{-\beta\sum_{\mathbf{k}\in\Gamma^*} e(\mathbf{k})} 2^{L^d} \prod_{\mathbf{k}\in\Gamma^*} \left(\cos\left(\frac{\beta\theta}{2}\right) + \cosh(\beta e(\mathbf{k})) \right) \tag{4.1}$$

and $\theta \in [0, 2\pi/\beta)$.

5 Analysis

As the title indicates, the main aim of this article as well as the lecture is to describe essential parts of the proof of the theorem. Above all we have to prove that the function

$$U \mapsto \log\left(\int e^{\hat{V}(\psi)} d\mu_{C(\phi)}(\psi) \right) \tag{5.1}$$

can be analytically continued into a domain

$$\{U \in \mathbb{C} \mid |U| < c\}, \tag{5.2}$$

where c is a positive constant independent of any parameter. Since

$$\int e^{\hat{V}(\psi)} d\mu_{C(\phi)}(\psi)$$

is a polynomial of U whose constant term is 1, the function (5.1) is analytic in a neighborhood of the origin. If we follow a routine of single-scale analysis, we can prove that the function (5.1) can be analytically continued into a domain

$$\left\{ U \in \mathbb{C} \ \Big|\ |U| < c(\beta) \left| \frac{\theta}{2} - \frac{\pi}{\beta} \right|^{d+1} \right\}, \tag{5.3}$$

where $c(\beta)(\in \mathbb{R}_{>0})$ depends on β, does not depend on θ. In order to prove SSB and ODLRO, we have to guarantee that the gap equation governing the order parameter admits a positive solution for $U(\in \mathbb{R}_{<0})$ belonging to a domain into which the function (5.1) is analytically continued. Since the gap equation has no positive solution for U belonging to the domain (5.3), all we can prove as a result of the routine is non-existence of SSB and ODLRO. On the contrary, the gap equation can have a positive solution for U belonging to the domain (5.2). Therefore SSB and ODLRO follow if the function (5.1) is analytically continued into (5.2).

Set

$$\bigwedge_{even} W := \bigoplus_{n=1}^{\sharp I/2} \bigwedge^{2n} W,$$

which is a subspace of $\bigwedge W$. Observe that for any $f(\psi) \in \bigwedge_{even} W$ there uniquely exist anti-symmetric functions $f_m : I^m \to \mathbb{C}$ $(m = 2, 4, \cdots, \sharp I)$ such that

$$f(\psi) = \sum_{m=2}^{\sharp I} 1_{m\in 2\mathbb{N}} \left(\frac{1}{h}\right)^m \sum_{(X_1,\cdots,X_m)\in I^m} f_m(X_1, \cdots, X_m)\psi_{X_1}\cdots\psi_{X_m}.$$

Let us define the norm $\|\cdot\|$ on $\bigwedge_{even} W$ as follows.

$$\|f\| := \sum_{m=2}^{\sharp I} 1_{m\in 2\mathbb{N}} \sup_{X_0\in I} \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{X}\in I^{m-1}} |f_m(X_0, \mathbf{X})|.$$

The norm $\|\cdot\|$ and its variant have been considered convenient in the recent development of multi-scale analysis of many-Fermion systems. We can compute $\|\hat{V}_s\|$ as follows. The anti-symmetric kernel function of $\hat{V}_s(\psi)$ is that

$$((\rho, \mathbf{x}, s, \xi), (\eta, \mathbf{y}, t, \zeta)) \mapsto \frac{-Uh}{2L^d} 1_{(\rho,\mathbf{x},s)=(\eta,\mathbf{y},t)} 1_{\rho=1}(1_{(\xi,\zeta)=(1,-1)} - 1_{(\xi,\zeta)=(-1,1)}).$$

Thus

$$\|\hat{V}_s\| = \sup_{(\rho,\mathbf{x},s,\xi)\in I} \frac{|U|}{2L^d} \sum_{(\eta,\mathbf{y},t,\zeta)\in I} 1_{(\rho,\mathbf{x},s)=(\eta,\mathbf{y},t)} 1_{\rho=1} |1_{(\xi,\zeta)=(1,-1)} - 1_{(\xi,\zeta)=(-1,1)}| = \frac{|U|}{2L^d}. \tag{5.4}$$

It is helpful that $\|\hat{V}_s\|$ is bounded by L^{-d}, which is negligibly small as we eventually send $L \to \infty$. On the other hand, the norm $\|\hat{V}_v\|$ cannot be bounded by L^{-d}. Instead, it has a particular vanishing property that

$$\int \hat{V}_v(\psi) f(\psi) d\mu_C(\psi) = 0 \tag{5.5}$$

for any $f(\psi) \in \bigwedge W$ and $C : I_0^2 \to \mathbb{C}$ satisfying that

$$C(\rho \mathbf{x} s, \eta \mathbf{y} t) = C(\rho \mathbf{x} 0, \eta \mathbf{y} 0), \quad (\forall (\rho, \mathbf{x}, s), (\eta, \mathbf{y}, t) \in I_0). \tag{5.6}$$

The property (5.5) can be confirmed as follows.

$$\begin{aligned}\int \hat{V}_v(\psi) f(\psi) d\mu_C(\psi) &= -\frac{U}{L^d h^2} \sum_{\mathbf{x},\mathbf{y}\in\Gamma} \sum_{s,t\in[0,\beta)_h} \left(h 1_{s=t} - \frac{1}{\beta} \right) \int \overline{\psi}_{1\mathbf{x}0} \psi_{2\mathbf{x}0} \overline{\psi}_{2\mathbf{y}0} \psi_{1\mathbf{y}0} d\mu_C(\psi) \\ &= 0.\end{aligned}$$

This implies that if the covariance C satisfies (5.6),

$$\int e^{\hat{V}(\psi)} d\mu_C(\psi) = \int e^{\hat{V}_s(\psi)} d\mu_C(\psi).$$

Moreover, by the norm bound (5.4) the function

$$U \mapsto \log \left(\int e^{\hat{V}_s(\psi)} d\mu_C(\psi) \right)$$

can be analytically continued into a domain of the form (5.2). This is because possibly very heavy contribution from C is absorbed by L^{-d}, not by $|U|$. It is clear that we should make use of this mechanism. However, the same argument as above does not immediately apply, since the actual covariance $C(\phi)$ does not satisfy (5.6). We overcome the problem by decomposing $C(\phi)$. We define $C_j : I_0^2 \to \mathbb{C}$ $(j = 1, 2)$ by

$$\begin{aligned}&C_1(\rho \mathbf{x} s, \eta \mathbf{y} t) \\ &:= \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega \in M_h \setminus \{\pi/\beta\}} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y}\rangle + i(\omega - \frac{\pi}{\beta})(s-t)} h^{-1} \left(I_2 - e^{-\frac{i}{h}(\omega - \frac{\theta}{2}) + \frac{1}{h} E(\phi)(\mathbf{k})} \right)^{-1} (\rho, \eta), \\ &C_2(\rho \mathbf{x} s, \eta \mathbf{y} t) := \frac{1}{\beta L^d} \sum_{\mathbf{k}\in\Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y}\rangle} h^{-1} \left(I_2 - e^{-\frac{i}{h}(\frac{\pi}{\beta} - \frac{\theta}{2}) + \frac{1}{h} E(\phi)(\mathbf{k})} \right)^{-1} (\rho, \eta).\end{aligned}$$

Observe that

$$C(\phi)(\rho \mathbf{x} s, \eta \mathbf{y} t) = e^{i\frac{\pi}{\beta}(s-t)} (C_1(\rho \mathbf{x} s, \eta \mathbf{y} t) + C_2(\rho \mathbf{x} s, \eta \mathbf{y} t)), \quad ((\rho, \mathbf{x}, s), (\eta, \mathbf{y}, t) \in I_0)$$

and C_2 satisfies (5.6). If $|U|$ is sufficiently small, the following transformation is justified.

$$\begin{aligned}\int e^{\hat{V}(\psi)} d\mu_{C(\phi)}(\psi) &= \int e^{\hat{V}(\psi)} d\mu_{C_1 + C_2}(\psi) = \int\int e^{\hat{V}(\psi + \psi^1)} d\mu_{C_1}(\psi^1) d\mu_{C_2}(\psi) \\ &= \int e^{R(\psi)} d\mu_{C_2}(\psi),\end{aligned} \tag{5.7}$$

$$R(\psi) := \log\left(\int e^{\hat{V}(\psi+\psi^1)} d\mu_{C_1}(\psi^1)\right).$$

In the first equality we used the fact that $\hat{V}(\psi)$ is invariant under the transform

$$\psi_{(\rho,\mathbf{x},s,\xi)} \to e^{-i\xi\frac{\pi}{\beta}s}\psi_{(\rho,\mathbf{x},s,\xi)}, \quad ((\rho,\mathbf{x},s,\xi)\in I).$$

Though it is not at all a trivial procedure, we can decompose $R(\psi)$ as follows. $R(\psi) = R_0 + R_s(\psi) + R_v(\psi)$, where $R_0 \in \mathbb{C}$, $R_s(\psi)$, $R_v(\psi) \in \bigwedge_{even} W$,

$$|R_0| \le c(\beta,\theta)|U|, \quad \|R_s\| \le c(\beta,\theta)|U|L^{-d}$$

and $R_v(\psi)$ satisfies (5.5). Therefore,

$$\int e^{R(\psi)} d\mu_{C_2}(\psi) = e^{R_0}\int e^{R_s(\psi)} d\mu_{C_2}(\psi).$$

In fact the covariance C_2 has an intense infrared singularity. However, its heavy contribution is absorbed by the factor L^{-d} binding $\|R_s\|$. As the result, the function

$$U \mapsto R_0 + \log\left(\int e^{R_s(\psi)} d\mu_{C_2}(\psi)\right)$$

is shown to be analytically continued into a domain of the form (5.2). Going back to the equality (5.7), we reach the desired conclusion on the function (5.1). We should add that these are largely simplified explanations of the proof of the theorem. The full proof requires many but finite slices of the covariance $C(\phi)$ and iteration of the above argument over all the sliced covariances.

Remark 5.1. One natural question is whether the BCS model with imaginary magnetic field has any relevance to contemporary physical research. At this stage the most relevant interpretation is that loss of analyticity of the free energy density

$$\lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{1}{L^d}\log(\mathrm{Tr}\, e^{-\beta H + itS_z}) \tag{5.8}$$

with the real parameter t is an indication of dynamical phase transition at finite temperature. The function $\mathrm{Tr}\, e^{-\beta H+itS_z}/\,\mathrm{Tr}\, e^{-\beta H}$ is a finite-temperature analogue of the overlap amplitude $\langle\psi_0, e^{itS_z}\psi_0\rangle$, where ψ_0 is the ground state of H. Loss of analyticity of the function

$$t \mapsto \lim_{\substack{L\to\infty \\ L\in\mathbb{N}}} \frac{1}{L^d}\log|\langle\psi_0, e^{itS_z}\psi_0\rangle|^2$$

has been interpreted as indication of Dynamical Quantum Phase Transition (DQPT) since the proposal in [8]. Non-analyticity of the function (5.8) with t can be seen as a straightforward generalization of DQPT to finite temperatures. We can refer to the recent physical articles [1], [3], [7], [14], [15] and so on for dynamical phase transitions at finite temperature. We should add that the free energy density (5.8) is explicitly computed in [9], [10] and its non-analyticity with t directly follows from the results in [10, Section 2].

Remark 5.2. Extending the external magnetic field into complex plane has been an important subject of mathematical physics since the pioneering study by Lee and Yang ([11], [16]). Observe that for any $w \in \mathbb{C}$

$$\mathrm{Tr}\, e^{-\beta(H+wS_z)} = e^{\frac{\beta L^d}{2}w} \sum_{n=0}^{2L^d} (\mathrm{Tr}_{(n-L^d)/2}\, e^{-\beta H})(e^{-\frac{\beta}{2}w})^n, \tag{5.9}$$

where for $m \in \{-L^d, -L^d+1, \cdots, L^d\}$, $\mathrm{Tr}_{m/2}\, e^{-\beta H}$ denotes the trace of $e^{-\beta H}$ over the eigenspace of S_z associated with the eigenvalue $m/2$. The Lee-Yang zeros for the BCS model are defined as zeros of the polynomial

$$\sum_{n=0}^{2L^d} (\mathrm{Tr}_{(n-L^d)/2}\, e^{-\beta H}) z^n.$$

Complete determination of the Lee-Yang zeros is beyond what this article can offer instantly. At least the equalities (4.1), (5.9) tell us that if $U = 0$, $\theta = 2\pi/\beta + 4\pi m/\beta$ ($m \in \mathbb{Z}$) and $e(\mathbf{k})$ vanishes at some $\mathbf{k} \in \Gamma^*$,

$$0 = \mathrm{Tr}\, e^{-\beta(H+i\theta S_z)} = (-1)^{L^d} \sum_{n=0}^{2L^d} (\mathrm{Tr}_{(n-L^d)/2}\, e^{-\beta H})(-1)^n.$$

Thus, $z = -1$ is a Lee-Yang zero. Note that $\{z \in \mathbb{C} \mid |z| = 1,\ z \neq -1\} = \{e^{i\beta\theta/2},\ e^{-i\beta\theta/2} \mid \theta \in [0, 2\pi/\beta)\}$. Thus there is no Lee-Yang zero in $\{z \in \mathbb{C} \mid |z| = 1,\ z \neq -1\}$ in the case $U = 0$. We can also extract some information from Theorem 3.1 (i) about the case $U \neq 0$. Assume that d, μ, β and U satisfy the same conditions as in Theorem 3.1. Then for any $\theta \in [0, 2\pi/\beta)$ there exists $L_0 \in \mathbb{N}$ such that

$$\mathrm{Tr}\, e^{-\beta(H+i\theta S_z)} = \mathrm{Tr}\, e^{-\beta(H-i\theta S_z)} > 0, \quad (\forall L \in \mathbb{N} \text{ with } L \geq L_0).$$

Thus it follows from (5.9) that for any $z \in \mathbb{C}\backslash\{-1\}$ with $|z| = 1$ there exists $L_0 \in \mathbb{N}$ such that z is not a Lee-Yang zero for any $L \in \mathbb{N}$ with $L \geq L_0$.

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