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Anticipating Quantum Stochastic Integrals for Basic Quantum Martingales

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1 Introduction

Since the quantum stochastic integrals of adapted quantum stochastic processes have been introduced by Hudson and Parthasarathy [10] as a quantum extension of the Itô (stochastic) integral, the quantum stochastic calculus has been studied extensively with wide applications (see [28, 32]).

The Hudson-Parthasarathy quantum stochastic integrals has been extended to the quantum stochastic integrals of nonadapted quantum stochastic processes by Belavkin [3], Lindsay [24] and Attal & Lindsay [2]. Since then the nonadapted quantum stochastic integral has been studied systematically in terms of quantum stochastic gradients by Ji & Obata [16, 18]. Based on the quantum white noise theory [12], the notion of quantum white noise derivatives has been introduced by Ji & Obata (see [14, 15, 16, 17, 19, 20]). The explicit formulas [16] of integrands for quantum stochastic integral representation of quantum martingales [11] has been derived in terms of the quantum white noise derivatives. Also, the notion of quantum stochastic gradients [18] has been introduced based on the notion of the quantum white noise derivatives. Recently, Ji & Sinha [21] studied the quantum stochastic integrals for quadratic quantum noises.

On the other hand, based on the white noise theory [8, 22, 30] introduced by Hida, Kuo & Russek [23] studied anticipating (classical) stochastic integrals by applying the quantum decomposition of a Brownian motion.

In this paper, we study some regularity properties of the quantum Hitsuda–Skorohod integrals as anticipating quantum stochastic integrals. Also, motivated by the results in [23], we discuss new types of anticipating quantum stochastic integrals in terms of pointwisely defined quantum white noise derivatives.

2 Admissible Generalized Operators

2.1 Admissible Rigging of Fock Space

We now review a construction of admissible rigging of Fock space which provides the basic structure of this paper. Let $H = L^2(\mathbb{R}_+, dt)$ be the Hilbert space of complex valued square integrable functions on $\mathbb{R}_+ = [0, \infty)$ with respect to the Lebesque measure dt and let $\Gamma(H)$ be the Fock space over H defined by

$$\Gamma(H) = \left\{ \phi = (f_n)_{n=0}^{\infty} \, ; \, f_n \in H^{\widehat{\otimes} n}, \ \sum_{n=0}^{\infty} n! |f_n|^2 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of H and $|\cdot|$ is the Hilbertian norm on H and $H^{\widehat{\otimes} n}$. For $p \geq 0$, we set

$$\mathcal{G}_p = \left\{ \phi = (f_n)_{n=0}^{\infty} \in \Gamma(H) \, ; \, \| \phi \|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|^2 < \infty \right\}$$

and $\mathcal{G}_{-p}$ to be the completion of $\Gamma(H)$ with respect to the norm $\| \cdot \|_{-p}$ defined by

$$\| \phi \|_{-p}^2 = \sum_{n=0}^{\infty} n! e^{-2pn} |f_n|^2.$$

Then $\{\mathcal{G}_p \, ; \, p \in \mathbb{R}\}$ forms a chain of weighted Fock spaces and so we have

$$\mathcal{G} = \operatorname*{proj\,lim}_{p \to \infty} \mathcal{G}_p \subset \mathcal{G}_p \subset \mathcal{G}_0 = \Gamma(H) \subset \mathcal{G}_{-p} \subset \mathcal{G}^* \cong \operatorname*{ind\,lim}_{p \to \infty} \mathcal{G}_{-p}$$

for $p \geq 0$, where the strong dual space $\Gamma(H)^*$ of $\Gamma(H)$ is identified with $\Gamma(H)$, and the strong dual space $\mathcal{G}^*$ of $\mathcal{G}$ is topologically isomorphic to the inductive limit space $\operatorname{ind\,lim}_{p \to \infty} \mathcal{G}_{-p}$. The canonical $\mathbb{C}$-bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{G}^* \times \mathcal{G}$ takes the form:

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \, \langle f_n, g_n \rangle \, , \qquad \Phi = (f_n) \in \mathcal{G}^*, \quad \phi = (g_n) \in \mathcal{G},$$

where $\langle f_n, g_n \rangle$ is the canonical $\mathbb{C}$-bilinear form on $H^{\otimes n} \times H^{\otimes n}$. Note that $\mathcal{G}$ is a countable Hilbert space but not necessarily a nuclear space. An element in $\mathcal{G}$ is said to be *admissible* or *regular*.

Remark 2.1 Let $E_{\mathbb{R}} \subset H_{\mathbb{R}} \subset E_{\mathbb{R}}^*$ be a Gelfand triple, i.e. $E_{\mathbb{R}}$ is a nuclear space, where $H_{\mathbb{R}} = L^2_{\mathbb{R}}(\mathbb{R}_+, dt)$ is the Hilbert space of real valued square integrable functions on $\mathbb{R}_+$ with respect to dt. Then for the standard Gaussian measure μ on $E_{\mathbb{R}}^*$ characterized by

$$\int_{E_{\mathbb{R}}^*} e^{i\langle x, \xi \rangle} \, d\mu(x) = e^{-\frac{1}{2}|\xi|^2}, \qquad \xi \in E_{\mathbb{R}},$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $E_{\mathbb{R}}^* \times E_{\mathbb{R}}$ again, by the Wiener-Itô-Segal isomorphism, $\Gamma(H)$ is unitarily equivalent with the Hilbert space $L^2(E_{\mathbb{R}}^*, \mu)$ of complex valued square integrable functions on $E_{\mathbb{R}}^*$ with respect to the Gaussian measure μ. In this sense, the elements of $\mathcal{G}$ are considered as admissible Gaussian functionals. The spaces $\mathcal{G}$ and $\mathcal{G}^*$ were introduced by Belavkin [3] and have appeared along with classical and quantum stochastic analysis, see e.g., [1, 4, 7, 11, 13, 14, 24, 25, 26, 33, 34].

2.2 Multiplications of Admissible Gaussian Functionals

Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for finite numbers of n and m. Then the *Wiener product* (or *pointwise multiplication*) $\phi\psi \in \mathcal{G}$ of ϕ and ψ is defined by

$$\phi\psi = (h_n), \qquad h_n = \sum_{l+m=n} \sum_{k=0}^{\infty} k! \binom{l+k}{k} \binom{m+k}{k} f_{l+k} \widehat{\otimes}_k g_{m+k}, \tag{2.1}$$

where $f_{l+k} \widehat{\otimes}_k g_{m+k}$ is the k-contraction of f_{l+k} and g_{m+k}, see [30].

The following lemma is useful to study the continuities of Wiener product of admissible Gaussian functionals and similar estimates can be found in [25] (see also [30, 33]).

Lemma 2.2 *Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for a finite numbers of n and m. Then for any $p, r, s \in \mathbb{R}$ with $r + s > 0$ and*

$$(n+1) \left(\frac{e^{(s-3r)/2} + e^{(r-3s)/2}}{e^{-2p}(r+s)} \right)^n \le c^n \tag{2.2}$$

for some $0 < c < 1$, it holds that

$$\| \phi\psi \|_p^2 \le \frac{1}{1-c} \| \phi \|_r^2 \| \psi \|_s^2 . \tag{2.3}$$

Proof. For given h_n as in (2.1), we obtain that

$$\begin{aligned} n!|h_n|^2 &= n! \left(\sum_{l+m=n} \sum_{k=0}^{\infty} k! \binom{l+k}{k} \binom{m+k}{k} |f_{l+k}||g_{m+k}| \right)^2 \\ &\le n! \left(\sum_{l+m=n} \sum_{k=0}^{\infty} M_{l,m,k} \sqrt{(l+k)!} e^{r(l+k)} |f_{l+k}| \sqrt{(m+k)!} e^{s(m+k)} |g_{m+k}| \right)^2 , \end{aligned} \tag{2.4}$$

where

$$M_{l,m,k} = \frac{e^{-rl-sm}}{l!m!} \frac{\sqrt{(l+k)!(m+k)!}}{k!} e^{-(r+s)k} \le \frac{e^{-rl-sm}}{l!m!} \sqrt{C_{l,m;r+s}},$$

where

$$C_{l,m;q} = \sup_{n \ge 0} \left\{ \frac{(l+n)!}{n!} \frac{(n+m)!}{n!} e^{-2qn} \right\} \le e^q l^l m^m \left(\frac{e^{q/2}}{eq} \right)^{l+m} < \infty \tag{2.5}$$

for $q > 0$ (see e.g., [30]: Section 4.1). Therefore, for any $r \in \mathbb{R}$ and $s \in \mathbb{R}$ with $r + s > 0$, from (2.4), by applying Cauchy-Schwarz inequality we obtain that

$$\begin{aligned} n!|h_n|^2 &\le n! \left(\sum_{l+m=n} \frac{e^{-rl-sm}}{l!m!} \sqrt{C_{l,m;r+s}} \right)^2 \| \phi \|_r^2 \| \psi \|_s^2 \\ &\le \left(\sum_{l+m=n} \frac{\sqrt{n!}}{l!m!} e^{-rl-sm} \sqrt{C_{l,m;r+s}} \right)^2 \| \phi \|_r^2 \| \psi \|_s^2 . \end{aligned} \tag{2.6}$$

By applying a simple inequality $n^n \leq e^n n!$, from (2.5) we see that

$$\sqrt{C_{l,m;r+s}} \leq e^{(r+s)/2}\sqrt{l^l m^m}\left(\frac{e^{(r+s)/2}}{e(r+s)}\right)^{(l+m)/2} \leq e^{(r+s)/2}\sqrt{l!m!}\left(\frac{e^{(r+s)/2}}{r+s}\right)^{(l+m)/2}$$

Therefore, for any $r \in \mathbb{R}$ and $s \in \mathbb{R}$ with $e^{r+s} \geq 2$, from (2.6) we obtain that

$$\begin{aligned}
n!|h_n|^2 &\leq \left(\sum_{l+m=n} \frac{\sqrt{n!}}{l!m!} e^{-rl-sm}\sqrt{C_{l,m;r+s}}\right)^2 \| \phi \|_r^2 \| \psi \|_s^2 \\
&\leq e^{r+s}\left(\sum_{l+m=n} \frac{\sqrt{n!}}{\sqrt{l!m!}} e^{-rl-sm}\left(\frac{e^{(r+s)/2}}{r+s}\right)^{(l+m)/2}\right)^2 \| \phi \|_r^2 \| \psi \|_s^2 \\
&\leq (n+1)\left(\sum_{l+m=n} \frac{n!}{l!m!}\left(\frac{e^{(s-3r)/2}}{r+s}\right)^l \left(\frac{e^{(r-3s)/2}}{r+s}\right)^m\right) \| \phi \|_r^2 \| \psi \|_s^2 \\
&= (n+1)\left(\frac{e^{(s-3r)/2}+e^{(r-3s)/2}}{r+s}\right)^n \| \phi \|_r^2 \| \psi \|_s^2 . \qquad (2.7)
\end{aligned}$$

Therefore, from (2.1) and (2.7) we obtain that

$$\begin{aligned}
\| \phi\psi \|_p^2 &= \sum_{n=0}^{\infty} n! e^{2pn}|h_n|^2 \leq \left[\sum_{n=0}^{\infty}(n+1)\left(\frac{e^{(s-3r)/2}+e^{(r-3s)/2}}{e^{-2p}(r+s)}\right)^n\right] \| \phi \|_r^2 \| \psi \|_s^2 \\
&\leq \frac{1}{1-c} \| \phi \|_r^2 \| \psi \|_s^2 \qquad (2.8)
\end{aligned}$$

for some $0 < c < 1$ satisfying (2.3), which gives the proof. □

The following two theorem are obvious consequences of Lemma 2.2.

Theorem 2.3 ([33]) *The Wiener product of admissible Gaussian functionals is continuous from $\mathcal{G} \times \mathcal{G}$ (equipped with the product topology) onto $\mathcal{G}$. In particular, $\mathcal{G}$ is an algebra with respect to the Wiener product.*

Theorem 2.4 *The Wiener product of admissible white noise functionals is continuous from $\mathcal{G}^* \times \mathcal{G}$ (equipped with the product topology) onto $\mathcal{G}^*$.*

Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for finite numbers of n and m. Then the *Wick product* (or *normal-ordered product*) $\phi \diamond \psi$ of ϕ and ψ is defined by

$$\phi \diamond \psi = (k_n), \qquad k_n = \sum_{l+m=n} f_l \widehat{\otimes} g_m, \qquad (2.9)$$

see [6, 8, 22]

The following lemma is useful to study the continuities of Wick product of admissible Gaussian functionals and similar estimates can be found in [33].

Lemma 2.5 *Let $\phi = (f_n), \psi = (g_n) \in \mathcal{G}$ be given. Suppose that $f_n = 0$ and $g_m = 0$ except for a finite numbers of n and m. Then for any $p, r, s \in \mathbb{R}$ satisfying that*

$$e^{2(p-r)} + e^{2(p-s)} < 1, \tag{2.10}$$

it holds that

$$\| \phi \diamond \psi \|_p^2 \le \| \phi \|_r^2 \| \psi \|_s^2 . \tag{2.11}$$

Proof. For given k_n as in (2.9), we obtain that

$$\begin{aligned} n!|k_n|^2 &= n! \left(\sum_{l+m=n} |f_l||g_m| \right)^2 \\ &\le n! \left(\sum_{l+m=n} \frac{e^{-2rl-2sm}}{l!m!} \right) \left(\sum_{l+m=n} l! e^{2rl} |f_l|^2 m! e^{2sm} |g_m|^2 \right) \\ &\le \left(e^{-2r} + e^{-2s} \right)^n \left(\sum_{l+m=n} l! e^{2rl} |f_l|^2 m! e^{2sm} |g_m|^2 \right), \end{aligned} \tag{2.12}$$

Therefore, for any $p, r, s \in \mathbb{R}$ satisfying (2.10), from (2.12) we obtain that

$$\begin{aligned} \| \phi \diamond \psi \|_p^2 &= \sum_{n=0}^{\infty} n! e^{2pn} |k_n|^2 \le \sum_{n=0}^{\infty} \left(e^{2(p-r)} + e^{2(p-s)} \right)^n \left(\sum_{l+m=n} l! e^{2rl} |f_l|^2 m! e^{2sm} |g_m|^2 \right) \\ &\le \| \phi \|_r^2 \| \psi \|_s^2 , \end{aligned}$$

which gives the proof. □

The following theorem is an obvious consequence of Lemma 2.5.

Theorem 2.6 ([33]) *The Wick product is continuous from $\mathcal{G} \times \mathcal{G}$ (equipped with the product topology) onto $\mathcal{G}$, and from $\mathcal{G}^* \times \mathcal{G}^*$ onto $\mathcal{G}^*$. In particular, $\mathcal{G}$ and $\mathcal{G}^*$ are algebras under the Wick product.*

3 Admissible Generalized Operators

We denote by $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$ equipped with the topology of bounded convergence. An operator in $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is called an *admissible generalized operator* [14] or simply *admissible operator*.

3.1 Integral Kernel Operators

Let l, m be non-negative integers. Let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$ and $\Phi = (f_n)_{n=0}^{\infty} \in \mathcal{G}^*$. For each $n \ge 0$, we put

$$g_{l+n} = \frac{(n+m)!}{n!} (K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}}. \tag{3.1}$$

Then from Lemma 4.1 in [11], for any $p \in \mathbb{R}$ and $q > 0$, we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}(l+n)!e^{2p(l+n)}|g_{l+n}|^2 &\le \|K_{l,m}\|^2 \sum_{n=0}^{\infty}(n+m)!\,\frac{(l+n)!}{n!}\frac{(n+m)!}{n!}e^{2p(l+n)}|f_{n+m}|^2\\ &\le \|K_{l,m}\|^2 e^{2(pl-(p+q)m)}C_{l,m;q}\,|\!|\!|\,\phi\,|\!|\!|_{p+q}^2\,, \qquad (3.2)\end{aligned}$$

where $\|K_{l,m}\|$ is the operator norm and $C_{l,m;q}$ is given as in (2.5). Therefore, we define an linear operator $\Xi_{l,m}(K_{l,m})$ on $\mathcal{G}^*$ by

$$\Xi_{l,m}(K_{l,m})\Phi = (g_{l+n})_{n=0}^{\infty}\,, \qquad \Phi = (f_n)_{n=0}^{\infty} \in \mathcal{G}^*, \qquad (3.3)$$

where g_{l+n} is given as in (3.1). Then for any $p \in \mathbb{R}$ and $q > 0$ it holds that

$$|\!|\!|\,\Xi_{l,m}(K_{l,m})\Phi\,|\!|\!|_p \le \|K_{l,m}\|e^{(pl-(p+q)m)}\sqrt{C_{l,m;q}}\,|\!|\!|\,\Phi\,|\!|\!|_{p+q}\,, \qquad \Phi \in \mathcal{G}^*, \qquad (3.4)$$

which implies that $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}_{p+q}, \mathcal{G}_p)$. The operator $\Xi_{l,m}(K_{l,m})$ is called the *integral kernel operator* with kernel $K_{l,m}$ (see [13, 9, 22, 30]).

Now the following theorem is obvious.

Theorem 3.1 ([11]) *Let l, m be non-negative integers and let $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$. Then it holds that*

$$\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*).$$

Let $\eta \in H$ and let $K_\eta \in \mathcal{L}(H, \mathbb{C})$ be defined by $K_\eta(f) = \langle \eta, f\rangle$ for any $f \in H$. For simple notation, we identify $\eta = K_\eta = K_\eta^*$, where K_η^* is the adjoint operator of K_η with respect to the canonical bilinear form $\langle\cdot, \cdot\rangle$, i.e., $K_\eta^*(a) = a\eta$ for all $a \in \mathbb{C}$. Then the *annihilation operator* $a(\eta)$ and the *creation operator* $a^*(\eta)$ associated with η are defined by $a(\eta) = \Xi_{0,1}(\eta)$ and $a^*(\eta) = \Xi_{1,0}(\eta)$, respectively, and then from Theorem 3.1, it holds that

$$a(\eta), a^*(\eta) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*).$$

It is straightforward to verify the canonical commutation relation:

$$[a(\xi), a(\eta)] = 0, \quad [a^*(\xi), a^*(\eta)] = 0, \quad [a(\xi), a^*(\eta)] = \int_{\mathbb{R}_+}\xi(t)\eta(t)dt = \langle \xi, \eta\rangle \qquad (3.5)$$

for $\xi, \eta \in H$.

The exponential vector ϕ_ξ associated with $\xi \in H$ is defined by $\phi_\xi = (\xi^{\otimes n}/n!)_{n=0}^{\infty}$. Then $\{\phi_\xi\,;\,\xi \in H\}$ spans a dense subspace of $\mathcal{G}$.

Proposition 3.2 ([14, 5]) *Let $\zeta \in H$ be given. Then it holds that*

$$a(\zeta)(\Phi\psi) = (a(\zeta)\Phi)\,\psi + \Phi\,(a(\zeta)\psi)\,, \qquad \Phi \in \mathcal{G}^*, \quad \psi \in \mathcal{G}, \qquad (3.6)$$

$$a(\zeta)(\Phi \diamond \Psi) = (a(\zeta)\Phi) \diamond \Psi + \Phi \diamond (a(\zeta)\Psi)\,, \qquad \Phi, \Psi \in \mathcal{G}^*. \qquad (3.7)$$

Proof. (i) For any $\xi, \eta \in H$, we obtain that

$$\begin{aligned}a(\zeta)(\phi_\xi\phi_\eta) &= a(\zeta)(\phi_{\xi+\eta})e^{\langle\xi,\,\eta\rangle} = \langle\zeta,\,\xi+\eta\rangle\,\phi_{\xi+\eta}e^{\langle\xi,\,\eta\rangle} = \langle\zeta,\,\xi+\eta\rangle\,\phi_\xi\phi_\eta\\ &= (a(\zeta)\phi_\xi)\,\phi_\eta + \phi_\xi\,(a(\zeta)\phi_\eta)\,.\end{aligned}$$

Therefore, by the continuity property $a(\zeta) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*)$ and the fact that exponential vectors span a dense subspace of $\mathcal{G}$ and $\mathcal{G}^*$, we complete the proof.

(ii) The proof is similar to the proof of (i). In fact, we obtain that

$$\begin{aligned} a(\zeta)(\phi_\xi \diamond \phi_\eta) &= a(\zeta)(\phi_{\xi+\eta}) = \langle \zeta,\, \xi+\eta\rangle\, \phi_{\xi+\eta} = \langle \zeta,\, \xi+\eta\rangle\, \phi_\xi \diamond \phi_\eta \\ &= (a(\zeta)\phi_\xi) \diamond \phi_\eta + \phi_\xi \diamond (a(\zeta)\phi_\eta)\,. \end{aligned}$$

□

Let $K \in \mathcal{L}(H, H)$. Then from Theorem 3.1, it holds that

$$\Lambda(K) := \Xi_{1,1}(K) \in \mathcal{L}(\mathcal{G}, \mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*).$$

The operator $\Lambda(K)$ is called the *conservation operator*. for any $p \in \mathbb{R}$ and $q > 0$, from (3.4) we obtain that

$$\|\, \Lambda(K)\Phi \,\|_p \le e^{-q}\sqrt{C_{1,1;q}}\|K\| \,\|\, \Phi \,\|_{p+q}\,, \qquad \Phi \in \mathcal{G}^*. \tag{3.8}$$

3.2 Multiplication Operators

Theorem 3.3 *For any $\Phi \in \mathcal{G}^*$ and $\phi, \psi \in \mathcal{G}$, it holds that*

$$\langle\!\langle \Phi\phi,\, \psi \rangle\!\rangle = \langle\!\langle \Phi,\, \phi\psi \rangle\!\rangle\,. \tag{3.9}$$

PROOF. For given $\Phi = (F_n) \in \mathcal{G}^*$ and any $\xi, \eta \in H$, from (2.1) we obtain that

$$\Phi\phi_\xi = \left(\sum_{l+m=n} \sum_{k=0}^{\infty} \binom{l+k}{k} \left(F_{l+k}\widehat{\otimes}_k \xi^{\otimes k}\right) \otimes \frac{\xi^{\otimes m}}{m!} \right)_{n=0}^{\infty}$$

and

$$\begin{aligned} \langle\!\langle \Phi\phi_\xi,\, \phi_\eta \rangle\!\rangle &= \sum_{n=0}^{\infty} \left\langle \sum_{m=0}^{n} \sum_{k=0}^{\infty} \binom{n-m+k}{k} \left(F_{n-m+k}\widehat{\otimes}_k \xi^{\otimes k}\right) \otimes \frac{\xi^{\otimes m}}{m!},\, \eta^{\otimes n} \right\rangle \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\langle \sum_{k=0}^{\infty} \binom{n+k}{k} \left(F_{n+k}\widehat{\otimes}_k \xi^{\otimes k}\right) \otimes \frac{\xi^{\otimes m}}{m!},\, \eta^{\otimes (n+m)} \right\rangle \\ &= e^{\langle \xi,\, \eta \rangle} \sum_{l=0}^{\infty} \left\langle F_l,\, \sum_{n+k=l} \binom{n+k}{k} \eta^{\otimes n} \widehat{\otimes} \xi^{\otimes k} \right\rangle \\ &= e^{\langle \xi,\, \eta \rangle} \langle\!\langle \Phi,\, \phi_{\xi+\eta} \rangle\!\rangle \\ &= \langle\!\langle \Phi,\, \phi_\eta\phi_\eta \rangle\!\rangle\,. \end{aligned}$$

Since the exponential vectors span a dense subspace of $\mathcal{G}$, by the continuity of the Wiener product (see Theorems 2.3 and 2.4), the proof is immediate. □

Let $\Phi \in \mathcal{G}^*$ be given. Then we consider the Wiener multiplication operator $M_\Phi : \mathcal{G} \to \mathcal{G}^*$ and then from (3.9), $\phi, \psi \in \mathcal{G}$, we obtain that it holds that

$$\langle\!\langle M_\Phi\phi,\, \psi \rangle\!\rangle = \langle\!\langle \Phi\phi,\, \psi \rangle\!\rangle = \langle\!\langle \Phi,\, \phi\psi \rangle\!\rangle\,.$$

Theorem 3.4 ([30]) *For each $\zeta \in H$, $X_\zeta = (0, \zeta, 0, \cdots) \in \mathcal{G}$ as a Wiener multiplication operator is represented as the sum of $a(\zeta)$ and $a^*(\zeta)$, i.e,*

$$X_\zeta = a(\zeta) + a^*(\zeta), \tag{3.10}$$

which is called the quantum decomposition of X_η.

PROOF. Since $X_\zeta = (0, \zeta, 0, \cdots) \in \mathcal{G}$, from (3.9) we obtain that

$$\begin{aligned} \langle X_\zeta \phi_\xi,\, \phi_\eta \rangle &= \langle X_\zeta,\, \phi_\xi \phi_\eta \rangle = \langle X_\zeta,\, \phi_{\xi+\eta} \rangle\, e^{\langle \xi,\, \eta \rangle} = \langle \zeta,\, \xi + \eta \rangle\, e^{\langle \xi,\, \eta \rangle} \\ &= \langle (a(\zeta) + a^*(\zeta))\, \phi_\xi,\, \phi_\eta \rangle\,, \end{aligned}$$

which gives the proof. □

For each $t \geq 0$, put $B_t = X_{\mathbf{1}_{[0,t]}}$. Then $\{B_t\}_{t \geq 0}$ becomes a Brownian motion which is called a *realization of Brownian motion* and so from Theorem 3.4 we have the following quantum decomposition of Brownian motion:

$$B_t = a(\mathbf{1}_{[0,t]}) + a^*(\mathbf{1}_{[0,t]}), \qquad t \geq 0. \tag{3.11}$$

Remark 3.5 The operators $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ on admissible Gaussian functionals play an essential role in the study of quantum martingales and integral representations [11, 14, 16, 17].

4 Quantum White Noise Derivatives

In this section, we briefly review some basic properties of quantum white noise derivatives [14, 15, 16, 17, 18, 20].

4.1 Annihilation and Creation Derivatives

For any admissible operator $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ and $\zeta \in H$, from Theorem 3.1 the commutators

$$[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi a(\zeta), \qquad -[a^*(\zeta), \Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi$$

are well defined as compositions of admissible operators, i.e., belong to $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. We define

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. Both together are referred to as the *quantum white noise derivatives* (*qwn-derivatives* for brevity) of Ξ. By the definitions, it is obvious that

$$\begin{aligned} \left(D_\zeta^+ \Xi^*\right)^* &= ([a(\zeta), \Xi^*])^* = (a(\zeta)\Xi^* - \Xi^* a(\zeta))^* = \Xi a^*(\zeta) - a^*(\zeta)\Xi \\ &= D_\zeta^- \Xi. \end{aligned} \tag{4.1}$$

For each admissible operator $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$, we operator norm of Ξ is denoted by $\|\Xi\|_{p;q}$.

Theorem 4.1 ([14]) *Let $\zeta \in H$ be given. Then $D_\zeta^\pm$ are continuous linear operators from $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ itself.*

Proof. Suppose that $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$. Then for any $r > 0$, by applying (3.4), we obtain that

$$\begin{aligned}\left\|D_\zeta^+ \Xi\right\|_{p-r;q+r} &= \|[a(\zeta), \Xi]\|_{p-r;q+r} = \|a(\zeta)\Xi - \Xi a(\zeta)\|_{p-r;q+r} \\ &\le \|a(\zeta)\|_{q;q+r}\|\Xi\|_{p;q} + \|\Xi\|_{p;q}\|a(\zeta)\|_{p-r;p},\end{aligned}$$

which implies that D_ζ^+ is a continuous linear operator on $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. Similarly, we see that D_ζ^- is a continuous linear operator on $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. □

Proposition 4.2 *For each $\zeta \in H$ and $\Phi \in \mathcal{G}^*$, it holds that*

$$\left(D_\zeta^+ M_\Phi\right)\phi_0 = \left(D_\zeta^- M_\Phi\right)\phi_0 = a(\zeta)\Phi.$$

Proof. We obtain that

$$\begin{aligned}\left(D_\zeta^+ M_\Phi\right)\phi_0 &= \left(a(\zeta)M_\Phi - M_\Phi a(\zeta)\right)\phi_0 = a(\zeta)\Phi, \\ \left(D_\zeta^- M_\Phi\right)\phi_0 &= \left(M_\Phi a^*(\zeta) - a^*(\zeta)M_\Phi\right)\phi_0 = \Phi X_\zeta - a^*(\zeta)\Phi = a(\zeta)\Phi,\end{aligned}$$

where we used the quantum decomposition as $\Phi X_\zeta = X_\zeta \Phi = (a(\zeta) + a^*(\zeta))\Phi$. □

4.2 Pointwise QWN-Derivatives

Let $\phi = (f_n) \in \mathcal{G}$ and $t \in \mathbb{R}_+$ be given. Suppose that $f_n = 0$ except for a finite number of n. We define

$$D_t\phi := (nf_n(t, \cdot))_{n=1}^\infty,$$

where $f_n(t, \cdot) \in H^{\widehat{\otimes}(n-1)}$, and then D is called the *classical stochastic gradient*. The classical stochastic gradient is denoted by ∇ in some literatures see [8, 16, 18, 22, 29]. We now extend the domain of D to the space $\mathcal{G}^*$.

Lemma 4.3 ([16]) *For any $p \in \mathbb{R}$ and $r > 0$ we have*

$$\|D\phi\|^2_{L^2(\mathbb{R}_+, \mathcal{G}_{-p-r})} = \int_{\mathbb{R}_+} \| D\phi(t) \|^2_{-p-r}\, dt \le K(p, r) \| \phi \|^2_{-p}, \qquad \phi \in \mathcal{G}, \tag{4.2}$$

where $K(p, r) = \sup_n (n+1)e^{2p-2rn} < \infty$. In particular, the classical stochastic gradient

$$D : \mathcal{G}_{-p} \to L^2(\mathbb{R}_+, \mathcal{G}_{-p-r}) \cong L^2(\mathbb{R}_+) \otimes \mathcal{G}_{-p-r} \tag{4.3}$$

is a continuous linear map.

Proof. For each $\phi = (f_n)_{n=0}^\infty \in \mathcal{G}$ consisting of continuous functions f_n on $\mathbb{R}_+^n$, we have $D\phi(t) = ((n+1)f_{n+1}(t, \cdot))_{n=0}^\infty$, where the right-hand side has a pointwise meaning. Then we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+} \| D\phi(t) \|^2_{-p-r}\, dt &= \sum_{n=0}^\infty n! e^{-2(p+r)n} \int_{\mathbb{R}_+} |(n+1)f_{n+1}(t, \cdot)|_0^2 dt \\ &= \sum_{n=0}^\infty (n+1)e^{2p-2rn} \times (n+1)! e^{-2p(n+1)} |f_{n+1}|_0^2 \\ &\le K(p, r) \| \phi \|^2_{-p},\end{aligned}$$

which implies the proof of (4.2). $\square$

Put

$$L^2(\mathbb{R}, \mathcal{G}) := \operatorname*{proj\,lim}_{p\to\infty} L^2(\mathbb{R}, \mathcal{G}_p) \cong \operatorname*{proj\,lim}_{p\to\infty} L^2(\mathbb{R}) \otimes \mathcal{G}_p\,,$$
$$L^2(\mathbb{R}_+, \mathcal{G}^*) := \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R}_+, \mathcal{G}_{-p}) \cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R}) \otimes \mathcal{G}_{-p}.$$

Then by Lemma 4.3, the classical stochastic gradient D is a continuous linear map from $\mathcal{G}$ into $L^2(\mathbb{R}_+, \mathcal{G})$ and from $\mathcal{G}^*$ into $L^2(\mathbb{R}_+, \mathcal{G}^*)$.

We see from (4.3) that $D\Phi(t)$ has a meaning as $\mathcal{G}_{-p-r}$-valued L^2-function in $t \in \mathbb{R}_+$. Given $\zeta \in L^2(\mathbb{R}_+)$, the linear map $\mathcal{G}_{p+r} \ni \psi \mapsto \langle\langle D\Phi,\, \zeta \otimes \psi\rangle\rangle$ is continuous. Therefore there exists a unique $\Psi \in \mathcal{G}_{-p-r}$ such that

$$\langle\langle D\Phi,\, \zeta \otimes \psi\rangle\rangle = \langle\langle \Psi,\, \psi\rangle\rangle, \qquad \psi \in \mathcal{G}_{p+r}.$$

It is reasonable to write

$$\Psi = \int_{\mathbb{R}_+} \zeta(t) D\Phi(t)\, dt.$$

As is easily seen, the Schwarz inequality holds:

$$\left|\!\left|\!\left| \int_{\mathbb{R}_+} \zeta(t) D\Phi(t)\, dt \right|\!\right|\!\right|_{-p-r} \le |\zeta|_0 \, |\!|\!| \, D\Phi \, |\!|\!|_{L^2(\mathbb{R}_+, \mathcal{G}_{-p-r})}\,, \tag{4.4}$$

which implies that the map

$$\mathcal{G}_{-p} \ni \Phi \mapsto \int_{\mathbb{R}_+} \zeta(t) D\Phi(t)\, dt \in \mathcal{G}_{-(p+r)}$$

is continuous. On the other hand, for any $\xi \in H$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+} \zeta(t) D\phi_\xi(t)\, dt &= \int_{\mathbb{R}_+} \zeta(t) a_t \phi_\xi\, dt = \int_{\mathbb{R}_+} \zeta(t)\xi(t)\phi_\xi\, dt = \langle \zeta, \xi\rangle \phi_\xi \\ &= a(\zeta)\phi_\xi.\end{aligned}$$

Therefore, we obtain that

$$\int_{\mathbb{R}_+} \zeta(t) D\Phi(t)\, dt = a(\zeta)\Phi, \qquad \Phi \in \mathcal{G}^*, \tag{4.5}$$

see [16].

Remark 4.4 The space $\mathcal{G}^*$ as a domain of the classical gradient D appeared in Aase–Øksendal–Privault–Ubøe [1]. For a standard domain see e.g., Kuo [22], Malliavin [27], Nualart [29].

Let $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ for some $p, q \in \mathbb{R}$. Then for any $r > 0$ and $\phi \in \mathcal{G}$, from (4.2) we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+} |\!|\!| \, \Xi D_t \phi \, |\!|\!|_q^2 \, dt &\le \int_{\mathbb{R}_+} |\!|\!| \, \Xi \, |\!|\!|_{p;q}^2 \, |\!|\!| \, D_t\phi \, |\!|\!|_p^2 \, dt \\ &\le K(-p, r) \, |\!|\!| \, \Xi \, |\!|\!|_{p;q}^2 \, |\!|\!| \, \phi \, |\!|\!|_{p+r}^2\,,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}_+} \| \Xi D_t \|_{p+r;q}^2 \, dt \le K(-p,r) \, \| \Xi \|_{p;q}^2 \, ,$$

and so the map

$$\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q) \ni \Xi \longmapsto \Xi D \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_{p+r}, \mathcal{G}_q)$$

is continuous. Similarly, we obtain that

$$\int_{\mathbb{R}_+} \| D_t \Xi \phi \|_{q-r}^2 \, dt \le K(-q,r) \, \| \Xi \phi \|_q^2 \le K(-q,r) \, \| \Xi \|_{p;q}^2 \, \| \phi \|_p^2 \, ,$$

which implies that

$$\int_{\mathbb{R}_+} \| \Xi D_t \|_{p+r;q}^2 \, dt \le K(-p,r) \, \| \Xi \|_{p;q}^2 \, ,$$

and so the map

$$\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q) \ni \Xi \longmapsto D\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$$

is continuous. Therefore, the *pointwise creation derivative* D_t^+ is defined by

$$D_t^+ \Xi = D_t \Xi - \Xi D_t, \qquad \Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$$

and $D_t^+ \Xi$ is an $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$-valued L^2-function in $t \in \mathbb{R}_+$. Motivated by (4.1), the *pointwise annihilation derivative* D_t^- is defined by

$$D_t^- \Xi = \left(D_t^+ \Xi^* \right)^* , \qquad \Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*),$$

see [16, 18]. In fact, for given $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ and $\zeta \in H$, from (4.5) we obtain that

$$\int_{\mathbb{R}_+} \zeta(t) D_t^+ \Xi dt = D_\zeta^+ \Xi$$

and

$$D_\zeta^- \Xi = \left(D_\zeta^+ \Xi^* \right)^* = \int_{\mathbb{R}_+} \zeta(t) \left(D_t^+ \Xi^* \right)^* dt.$$

Proposition 4.5 *For each $t \ge 0$ and $\Phi \in \mathcal{G}^*$, it holds that*

$$\left(D_t^+ M_\Phi \right) \phi_0 = \left(D_t^- M_\Phi \right) \phi_0 = D_t \Phi.$$

Proof. We obtain that

$$\begin{aligned} \left(D_t^+ M_\Phi \right) \phi_0 &= \left(D_t M_\Phi - M_\Phi D_t \right) \phi_0 = D_t \Phi, \\ \left(D_t^- M_\Phi \right) \phi_0 &= \left(D_t^+ M_\Phi^* \right)^* \phi_0 = \left(D_t^+ M_\Phi \right)^* \phi_0 = \left(M_{D_t \Phi} \right)^* \phi_0 = D_t \Phi, \end{aligned}$$

which gives the proof. □

5 Anticipating Quantum Stochastic Integrals

For each $t \geq 0$, let $\mathcal{F}_t$ be the σ-field generated by $\{B_s\,;\, 0 \leq s \leq t\}$. A one-parameter family $\Phi = \{\Phi_t\}_{t\geq 0} \subset \mathcal{G}^*$ is called a *generalized stochastic process* [4, 11, 31] if there exists a $p \geq 0$ (independent of $t \geq 0$) such that $\Phi_t \in \mathcal{G}_{-p}$ for all $t \geq 0$ and the map $t \mapsto \Phi_t \in \mathcal{G}_{-p}$ is Borel measurable on $\mathbb{R}_+$. A generalized stochastic process $\{\Phi_t = (F_{t;n})\}_{t\geq 0}$ is said to be *adapted* (w.r.t. $\mathcal{F}_t$) if for all $t \geq 0$ and $n \geq 0$, $\mathrm{supp} F_{t;n} \subset [0,t]^n$.

A one-parameter family $\{\Xi_t\}_{t\in\mathbb{R}_+} \subset \mathcal{L}(\mathcal{G},\mathcal{G}^*)$ is called a *quantum stochastic process*. Our approach covers a wide class of classical and quantum stochastic processes in the sense that $\mathcal{G}^*$ and $\mathcal{L}(\mathcal{G},\mathcal{G}^*)$ involve distributions. As examples, for each $t \geq 0$, we put

$$A_t = a(\mathbf{1}_{[0,t]}), \quad A_t^* = a^*(\mathbf{1}_{[0,t]}), \quad \Lambda_t = \Xi_{1,1}(\mathbf{1}_{[0,t]}).$$

For the definition of Λ_t, the indicator function $\mathbf{1}_{[0,t]}$ is considered as a multiplication operator on H, i.e., $\mathbf{1}_{[0,t]}(\xi) = \mathbf{1}_{[0,t]}\xi =: \xi_{[0,t]}$ for any $\xi \in H$. Then for each $t \geq 0$, $A_t, A_t^*, \Lambda_t \in \mathcal{L}(\mathcal{G},\mathcal{G}) \cap \mathcal{L}(\mathcal{G}^*,\mathcal{G}^*)$. The processes $\{A_t\}_{t\geq 0}$, $\{A_t^*\}_{t\geq 0}$ and $\{\Lambda_t\}_{t\geq 0}$ are called the *annihilation*, *creation* and *conservation* (or *gauge*) *processes*, respectively.

5.1 Quantum Hitsuda–Skorohod Integrals

In this section, we study the Hitsuda–Skorohod type quantum stochastic integrals with their regular properties.

Theorem 5.1 *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p,\mathcal{G}_q))$ be a quantum stochastic process. Then there exists an admissible operator, denoted by $\delta^-(\Xi)$, in $\mathcal{L}(\mathcal{G}_{p+r},\mathcal{G}_q)$ for any $r > 0$ such that*

$$\delta^-(\Xi)\phi = \int_{\mathbb{R}_+} \Xi(t)(D_t\phi)\,dt \tag{5.1}$$

for any $\phi \in \mathcal{G}$.

Proof. For any $\phi \in \mathcal{G}$ and $r > 0$, by applying (4.2), we obtain that

$$\begin{aligned}\left\| \int_{\mathbb{R}_+} \Xi(t)(D_t\phi)\,dt \right\|_q &\leq \int_{\mathbb{R}_+} \| \Xi(t) \|_{p;q} \, \| D_t\phi \|_p \, dt \\ &\leq \left(\int_{\mathbb{R}_+} \| \Xi(t) \|_{p;q}^2 \, dt \right)^{1/2} \left(\int_{\mathbb{R}_+} \| D_t\phi \|_p^2 \, dt \right)^{1/2} \\ &\leq \sqrt{K(-p,r)} \left(\int_{\mathbb{R}_+} \| \Xi(t) \|_{p;q}^2 \, dt \right)^{1/2} \| \phi \|_{p+r}^2 ,\end{aligned}$$

which implies that the linear operator

$$\mathcal{G}_{p+r} \ni \phi \longmapsto \int_{\mathbb{R}_+} \Xi(t)(D_t\phi)\,dt \in \mathcal{G}_q$$

is continuous. □

For given $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p,\mathcal{G}_q))$, the admissible operator $\delta^-(\Xi)$ satisfying (5.1) is called the *annihilation integral* of Ξ, see [3, 24, 16, 18].

Remark 5.2 Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then for any $\xi \in H$, we obtain that

$$\delta^-(\Xi)\phi_\xi = \int_{\mathbb{R}_+} \Xi(t)(D_t\phi_\xi)\, dt = \int_{\mathbb{R}_+} \xi(t)\Xi(t)\phi_\xi\, dt = \left(\int_{\mathbb{R}_+} \Xi(t)\, dA_t\right)\phi_\xi,$$

which implies that

$$\delta^-(\Xi) = \int_{\mathbb{R}_+} \Xi(t)\, dA_t$$

on a certain domain. Furthermore, if Ξ is adapted, then $\delta^-(\Xi)$ coincides with the annihilation integral of Hudson-Parthasarathy. For the definition of the adaptedness of quantum stochastic processes, we refer to [11]. Also, for more study on quantum Hitsuda–Skorohod integrals, we refer to [3, 24, 18].

As for a criterion for $\delta^-(\Xi)$ being a bounded operator on $\Gamma(H)$, we have the following corollary. A similar result can be found in [18].

Corollary 5.3 *For any $r > 0$ and $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\mathcal{G}_{-r}, \Gamma(H)))$, the annihilation integral $\delta^-(\Xi)$ is a bounded operator on $\Gamma(H)$.*

Proof. The proof is immediate from Theorem 5.1. □

Theorem 5.4 *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then there exists an admissible operator, denoted by $\delta^+(\Xi)$, in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$ for any $r > 0$ such that*

$$\langle\langle \delta^+(\Xi)\phi,\, \psi\rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Xi(t)\phi,\, D_t\psi\rangle\rangle\, dt \tag{5.2}$$

for $\phi, \psi \in \mathcal{G}$.

Proof. For any $\phi, \psi \in \mathcal{G}$ and $r > 0$, by applying (4.2), we obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}_+} \langle\langle \Xi(t)\phi,\, D_t\psi\rangle\rangle\, dt\right| &\le \int_{\mathbb{R}_+} \|\, \Xi(t)\phi \,\|_q \,\|\, D_t\psi \,\|_{-q}\, dt \\ &\le \left(\int_{\mathbb{R}_+} \|\, \Xi(t) \,\|_{p;q}^2\, dt\right)^{1/2} \left(\int_{\mathbb{R}_+} \|\, D_t\psi \,\|_{-q}^2\, dt\right)^{1/2} \|\, \phi \,\|_p \\ &\le \sqrt{K(q,r)} \left(\int_{\mathbb{R}_+} \|\, \Xi(t) \,\|_{p;q}^2\, dt\right)^{1/2} \|\, \phi \,\|_p \,\|\, \psi \,\|_{-q+r},\end{aligned}$$

which implies that the bilinear form

$$\mathcal{G}_p \times \mathcal{G}_{-q+r} \ni (\phi, \psi) \longmapsto \int_{\mathbb{R}_+} \langle\langle \Xi(t)\phi,\, D_t\psi\rangle\rangle\, dt \in \mathbb{C}$$

is continuous. Therefore, there exists a unique admissible operator $\delta^+(\Xi) \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{p-r})$ such that (5.2) holds. □

For given $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, the admissible operator $\delta^+(\Xi)$ satisfying (5.2) is called the *creation integral* of Ξ, see [3, 24, 16, 18].

As for a criterion for $\delta^+(\Xi)$ being a bounded operator on $\Gamma(H)$, we have the following corollary. A similar result can be found in [18].

Corollary 5.5 *For any $r > 0$ and $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), \mathcal{G}_r))$, the creation integral $\delta^+(\Xi)$ is a bounded operator on $\Gamma(H)$.*

PROOF. The proof is immediate from Theorem 5.4. □

Remark 5.6 The *classical Hitsuda–Skorohod integral* δ is defined as the adjoint map of the classical stochastic gradient D (see [8, 18, 22, 29]), i.e., for given $\Psi \in L^2(\mathbb{R}_+, \mathcal{G}^*)$, the classical Hitsuda–Skorohod integral $\delta(\Psi) \in \mathcal{G}^*$ of Ψ is defined by

$$\langle\langle \delta(\Psi),\, \phi \rangle\rangle = \int_{\mathbb{R}_+} \langle\langle \Psi(t),\, D_t\phi \rangle\rangle\, dt, \qquad \phi \in \mathcal{G}. \tag{5.3}$$

Therefore, by denoting $(\Xi\phi)(t) = \Xi(t)\phi$, from (5.2) we have

$$\delta^+(\Xi)\phi = \delta\left(\Xi\phi\right), \qquad \phi \in \mathcal{G}, \tag{5.4}$$

see [2, 24, 18].

The creation and annihilation integrals are related directly. The following corollary gives a relation between creation and annihilation integrals.

Corollary 5.7 ([18]) *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then it holds that*

$$(\delta^-(\Xi))^* = \delta^+(\Xi^*). \tag{5.5}$$

PROOF. For any $\phi, \psi \in \mathcal{G}$, we obtain that

$$\begin{aligned} \left\langle\!\left\langle \delta^-(\Xi)\phi,\, \psi \right\rangle\!\right\rangle &= \int_{\mathbb{R}_+} \langle\langle \Xi(t)(D_t\phi),\, \psi \rangle\rangle\, dt = \int_{\mathbb{R}_+} \langle\langle \Xi^*(t)\psi,\, (D_t\phi) \rangle\rangle\, dt \\ &= \left\langle\!\left\langle \delta^+(\Xi^*)\psi,\, \phi \right\rangle\!\right\rangle, \end{aligned}$$

which proves (5.5). □

Theorem 5.8 *Let $p, q \in \mathbb{R}$ be given and $\Xi \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process. Then there exists an admissible operator, denoted by $\delta^0(\Xi)$, in $\mathcal{L}(\mathcal{G}_{p+r}, \mathcal{G}_{q-r})$ for any $r > 0$ such that*

$$\left\langle\!\left\langle \delta^0(\Xi)\phi,\, \psi \right\rangle\!\right\rangle = \int_{\mathbb{R}_+} \langle\langle \Xi(t)D_t\phi,\, D_t\psi \rangle\rangle\, dt \tag{5.6}$$

for $\phi, \psi \in \mathcal{G}$.

PROOF. The proof is a simple modification of the proofs of Theorems 5.1 and 5.4. □

For given $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, the admissible operator $\delta^0(\Xi)$ satisfying (5.6) is called the *conservation integral* of Ξ, see [3, 24, 16, 18].

As for a criterion for $\delta^0(\Xi)$ being a bounded operator on $\Gamma(H)$, we have the following corollary. A similar result can be found in [18].

Corollary 5.9 *For any $r > 0$ and $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\mathcal{G}_{-r}, \mathcal{G}_r))$, the conservation integral $\delta^0(\Xi)$ is a bounded operator on $\Gamma(H)$.*

PROOF. The proof is immediate from Theorem 5.8. □

5.2 Extensions of Anticipating Quantum Stochastic Integrals

In this section, motivated by the results in [23], we discuss extensions of the quantum Hitsuda-Skorohod integrals studied in Section 5.1. Based on the quantum white noise calculus [12], we have the following integral representations:

$$A_t = \int_0^t a_s ds, \quad A_t^* = \int_0^t a_s^* ds, \quad \Lambda_t = \int_0^t a_s^* a_s ds,$$

where a_t and a_t^* are the pointwisely defined annihilation and creation operators. On the other hand, the pointwisely defined annihilation operator a_t and the stochastic gradient D_t coincide on a certain domain. Hence, the following informal computations gives motivations for extensions of the quantum Hitsuda-Skorohod integrals: for a given quantum stochastic process $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ of enough regular operators Ξ_t, we may write as

$$\begin{aligned}
\int_0^t \Xi_s dA_s &= \int_0^t \Xi_s D_s ds = \delta^-(\mathbf{1}_{[0,t]}\Xi),\\
\int_0^t (dA_s)\,\Xi_s &= \int_0^t D_s \Xi_s ds = \int_0^t \Xi_s D_s ds + \int_0^t D_s^+ \Xi_s ds = \delta^-(\mathbf{1}_{[0,t]}\Xi) + \int_0^t D_s^+ \Xi_s ds,\\
\int_0^t (dA_s^*)\,\Xi_s &= \int_0^t D_s^* \Xi_s ds = \delta^+(\mathbf{1}_{[0,t]}\Xi),\\
\int_0^t \Xi_s dA_s^* &= \int_0^t \Xi_s D_s^* ds = \delta^+(\mathbf{1}_{[0,t]}\Xi) + \int_0^t D_s^- \Xi_s ds,\\
\int_0^t \Xi_s d\Lambda_s &= \int_0^t \Xi_s D_s^* D_s ds = \delta^0(\mathbf{1}_{[0,t]}\Xi) + \delta^-(\mathbf{1}_{[0,t]} D_\cdot^- \Xi),\\
\int_0^t (d\Lambda_s)\,\Xi_s &= \int_0^t D_s^* D_s \Xi_s ds = \delta^0(\mathbf{1}_{[0,t]}\Xi) + \delta^+(\mathbf{1}_{[0,t]} D_\cdot^+ \Xi).
\end{aligned} \tag{5.7}$$

However, $D_t^\pm \Xi_t$ has no meaning directly. For example, we consider the annihilation process $A_t = a(\mathbf{1}_{[0,t]})$ and then

$$D_t^- A_s = \mathbf{1}_{[0,s]}(t).$$

But the annihilation process A_t can be defined as $a(\mathbf{1}_{[0,t)})$ and then we would have $D_t^- A_s = \mathbf{1}_{[0,s)}(t)$. Therefore, $D_t^- A_t$ cannot be defined in a unique way [23]. From the above example, if we deal with quantum stochastic processes, then it is natural to consider two kinds of pointwisely defined annihilation derivative, $D_{t+}^\pm$ and $D_{t-}^\pm$. Let $\{\Xi_t\}_{s\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ be a quantum stochastic process. We define

$$D_{t+}^\pm \Xi_t = \lim_{s\downarrow t} D_s^\pm \Xi_t, \qquad D_{t-}^\pm \Xi_t = \lim_{s\uparrow t} D_s^\pm \Xi_t,$$

if the limits exist.

Definition 5.10 Let $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ be a quantum stochastic process.

(1) Suppose that $\delta^+(\Xi)$ exists, and $D_{t+}^- \Xi_t$ exists and it is integrable on $\mathbb{R}_+$. Then we define

$$\int_{\mathbb{R}_+} \Xi_t dA_{t+}^* = \delta^+(\Xi) + \int_{\mathbb{R}_+} D_{t+}^- \Xi_t dt. \tag{5.8}$$

(2) Suppose that $\delta^+(\Xi)$ exists, and $D^-_{t-}\Xi_t$ exists and it is integrable on $\mathbb{R}_+$. Then we define

$$\int_{\mathbb{R}_+} \Xi_t dA^*_{t-} = \delta^+(\Xi) + \int_{\mathbb{R}_+} D^-_{t-}\Xi_t dt. \tag{5.9}$$

(3) Suppose that $\delta^+(\Xi)$ exists, and $D^-_{t+}\Xi_t, D^-_{t-}\Xi_t$ exist as integrable functions on $\mathbb{R}_+$. Then we define

$$<\boldsymbol{\alpha}> \int_{\mathbb{R}_+} \Xi_t \circ dA^*_t = \delta^+(\Xi) + \alpha_1 \int_{\mathbb{R}_+} D^-_{t+}\Xi_t dt + \alpha_2 \int_{\mathbb{R}_+} D^-_{t-}\Xi_t dt \tag{5.10}$$

for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, which is called the $<\boldsymbol{\alpha}>$*-creation integral.*

Theorem 5.11 *Let $p, q \in \mathbb{R}$ be given and let $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$ be a quantum stochastic process.*

(1) *Suppose that $D^-_{t+}\Xi_t$ exists and $D^-_{\cdot+}\Xi_\cdot \in L^1(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r}))$ for some $r > 0$. Then the integral $\int_{\mathbb{R}_+} \Xi_t dA^*_{t+}$ exists as an operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$.*

(2) *Suppose that $D^-_{t-}\Xi_t$ exists and $D^-_{\cdot-}\Xi_\cdot \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r}))$ for some $r > 0$. Then the integral $\int_{\mathbb{R}_+} \Xi_t dA^*_{t-}$ exists as an operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$.*

Proof. (1) Since $\Xi \in L^2(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q))$, by Theorem 5.4, the quantum Hitsuda-Skorohod creation integral $\delta^+(\Xi)$ of Ξ exists as an admissible operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-s})$ for any $s > 0$. Also, since, by assumption, $D^-_{t+}\Xi_t$ exists and $D^-_{\cdot+}\Xi_\cdot \in L^1(\mathbb{R}_+, \mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r}))$ for some $r > 0$, for any $\phi \in \mathcal{G}$ we obtain that

$$\left\| \int_{\mathbb{R}_+} D^-_{t+}\Xi_t \phi \, dt \right\|_{q-r} \leq \left(\int_{\mathbb{R}_+} \| D^-_{t+}\Xi_t \|_{p;q-r} \, dt \right) \| \phi \|_p ,$$

which implies that the integral $\int_{\mathbb{R}_+} D^-_{t+}\Xi_t \, dt$ exists as an admissible operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$. Finally, the integral $\int_{\mathbb{R}_+} \Xi_t dA^*_{t+}$ exists as an operator in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_{q-r})$.

(2) The proof is similar to the proof of (1). □

By similar arguments used in Definition 5.10, we can define the quantum stochastic integrals of types given as in (5.7) of which the study will be appear in some other papers.

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