数理解析研究所講究録 第2123巻 2019年 86-100

DISSIPATIVE SCATTERING THEORY: AN OVERVIEW OF RECENT RESULTS

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ABSTRACT. We provide an introduction to the scattering theory of dissipative quantum systems representing the long-time evolution of a system S interacting with another system S' and susceptible of being absorbed by S'. The effective dynamics of S is generated by an operator of the form $H = H_0 + V - \mathrm{i}C^*C$ on the Hilbert space of the pure states of S, where H_0 is the self-adjoint generator of the free dynamics of S, V is symmetric and C is bounded. We define the basic objects of the scattering theory for the pair (H, H_0), next we review recent results on the spectral singularities and the asymptotic completeness of the wave operators.

1. INTRODUCTION

When a physical quantum system interacts with another one, part of its energy may be irreversibly transferred to the other system. This phenomenon of irreversible loss of energy is usually called *quantum dissipation*. In particular, fundamentally, quantum systems cannot be completely isolated from their environment and, therefore, any quantum system experiences quantum dissipation to some extent, due to interactions with the environment.

This note is concerned with the mathematical study of effective or empirical models of quantum dissipation. We consider a quantum system S interacting with another quantum system S'. Our main concern is the understanding of the phenomenon of "capture", i.e., we aim at studying models allowing for the description of both elastic scattering and absorption of S by S' . Such models apply to many physical situations. We focus first on a typical example to fix the ideas.

Consider a neutron targeted onto a complex nucleus. After interacting with the nucleus, the neutron may be elastically scattered off the nucleus or be absorbed by the nucleus, leading to the formation of a compound nucleus. The concept of a compound nucleus was introduced by Bohr [1] in 1936. In 1954, Feshbach, Porter and Weisskopf [10] proposed a model describing the interaction of a neutron with a nucleus, allowing for the description of both elastic scattering and the formation of a compound nucleus. The force exerted by the nucleus on the neutron is modeled by a phenomenological potential of the form $V - \mathrm{i}W$, where V, W are real-valued and $W \geq 0$. The nucleus is supposed to be localized in space, which corresponds to the assumption that V and W are compactly supported or decay rapidly at infinity. On $L^2(\mathbb{R}^3)$, in suitable units, the "pseudo-Hamiltonian" for the neutron is given by

$$H = -\Delta + V - \mathrm{i}W. \tag{1.1}$$

Here, the operator H is called a pseudo-Hamiltonian as it generates a strongly continuous semigroup of contractions. The model of Feshbach, Porter and Weisskopf, or generalizations thereof, is called the *nuclear optical model*.

In [8, 9], the scattering theory for a class of abstract pseudo-Hamiltonians on a Hilbert space $\mathcal{H}$ is studied. In the abstract setting considered in [8, 9], the pseudo-Hamiltonian for the system S is given by

$$H = H_0 + V - \mathrm{i}C^*C,$$

where H_0 is a self-adjoint operator on $\mathcal{H}$ with purely absolutely continuous spectrum, V is symmetric and relatively compact with respect to H_0, and C is bounded and relatively compact with respect to H_0. The operator H_0 is the generator of the (unitary) free dynamics of S. The main purpose is then to study the scattering theory for the pair (H, H_0). Suitable hypotheses on H_0, V and C are formulated in such a way that they can be verified in the particular case where H is given by a dissipative Schrödinger operator of the form (1.1). See the next section for more details.

Prior to [8, 9], mathematical scattering theory for dissipative operators on Hilbert spaces has been considered by many authors (see, e.g., [2, 3, 7, 12, 13, 21] and references therein). In these references, in particular, the existence of the wave operators associated to H and H_0 is established under various conditions. In this note, following [8, 9], we go one step beyond by considering the *asymptotic completeness* of the wave operators. We review recents results established in [8, 9] showing that, under suitable assumptions, asymptotic completeness is equivalent to the absence of *spectral singularities* embedded into the essential spectrum of H.

The paper is organized as follows. In Section 2, we introduce the main objects involved in dissipative scattering theory and we recall their basic properties. Section 3 is devoted to the notions of spectral singularities and asymptotic completeness.

2. Mathematical setting

As mentioned in the introduction, we consider a quantum system S interacting with another quantum system S' and susceptible of being absorbed by S'. Hence, in the main example we have in mind, S is a neutron and S' is a nucleus. The pure states of S correspond to the normalized vectors in a complex Hilbert space $\mathcal{H}$. The effective dynamics of S is supposed to be generated by a pseudo-Hamiltonian acting on $\mathcal{H}$, of the form

$$H = H_0 + V - \mathrm{i}C^*C = H_V - \mathrm{i}C^*C,$$

where

- H_0 is a self-adjoint operator on $\mathcal{H}$ corresponding to the generator of the free dynamics of S,
- $V - \mathrm{i}C^*C$ is an effective interaction term due to the presence of S', with V symmetric and, of course, $C^*C \geq 0$.

In this section, we will state other abstract assumptions on the operators H_0, V and C which were introduced in [8, 9] in order to establish various results on the spectral and scattering theories for the pair (H, H_0). In the next section, we will recall that those abstract assumptions are fulfilled in our main example, namely the nuclear optical model. In this model $H_0 = -\Delta$ on $L^2(\mathbb{R}^3)$ and V and C are multiplication operators by functions $\mathbb{R}^3 \ni x \mapsto V(x) \in \mathbb{R}$, $\mathbb{R}^3 \ni x \mapsto C(x) \in \mathbb{C}$ decaying at ∞.

To shorten notations below, the resolvents of the operators H_0, H_V and H are denoted by

$$R_0(z) = (H_0 - z)^{-1}, \quad R_V(z) = (H_V - z)^{-1}, \quad R(z) = (H - z)^{-1},$$

for any z in the resolvent set of the corresponding operator.

2.1. **Basic assumptions and consequences.** The set of bounded operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. The following basic assumptions are made:

Hypothesis 1 (Basic assumptions)**.**

(i) $H_0 \geq 0$ *(or, more generally,* H_0 *is self-adjoint and semi-bounded from below),*
(ii) V *is symmetric and relatively compact with respect to* H_0*,*
(iii) $C \in \mathcal{L}(\mathcal{H})$ *and* C *is relatively compact with respect to* H_0*.*

We recall that an operator A on $\mathcal{H}$ is called *dissipative* if, for all $u \in \mathcal{D}(A)$, $\mathrm{Im}(\langle u, Au\rangle) \leq 0$. Moreover, A is called *maximal dissipative* if A is dissipative and has no proper dissipative extension. Hypothesis 1 has the following simple consequences.

Proposition 2.1. *Suppose that Hypothesis 1 holds. Then*

(1) $H_V = H_0 + V$ *is a self-adjoint operator on* $\mathcal{H}$ *with domain* $\mathcal{D}(H_V) = \mathcal{D}(H_0)$.
(2) $H = H_V - \mathrm{i}C^*C$ *is a maximal dissipative operator on* $\mathcal{H}$ *with domain* $\mathcal{D}(H) = \mathcal{D}(H_0)$.
(3) *The operator* $-\mathrm{i}H$ *generates a strongly continuous group* $\{e^{-\mathrm{i}tH}\}_{t\in\mathbb{R}}$ *such that*

$$\big\|e^{-\mathrm{i}tH}\big\| \leq 1 \ \text{ if } t \geq 0, \qquad \big\|e^{-\mathrm{i}tH}\big\| \leq e^{\|C^*C\||t|} \ \text{ if } t \leq 0.$$

In particular, $-\mathrm{i}H$ *generates the strongly continuous semigroup of contractions* $\{e^{-\mathrm{i}tH}\}_{t\geq 0}$.
(4) *The adjoint of* H *is*

$$H^* = H_0 + V + \mathrm{i}C^*C,$$

with domain $\mathcal{D}(H^*) = \mathcal{D}(H_0)$. *Moreover,* $\mathrm{i}H^*$ *generates of a strongly continuous group* $\{e^{\mathrm{i}tH^*}\}_{t\in\mathbb{R}}$ *such that* $\{e^{\mathrm{i}tH^*}\}_{t\geq 0}$ *is a semigroup of contractions.*

Proof. (1) is a simple consequence of the Kato-Rellich Theorem together with the fact that V is symmetric and relatively compact with respect to H_0, and hence infinitesimally small with respect to H_0 (see, e.g., [17]).

To prove (2), we begin by observing that H is dissipative since, for all $u \in \mathcal{D}(H) = \mathcal{D}(H_0)$,

$$\mathrm{Im}(\langle u, Hu\rangle) = -\|Cu\|^2 \leq 0.$$

To verify that H is maximal dissipative, by a theorem of Phillips [15], it then suffices to show that $\mathrm{Ran}(H - \mathrm{i}\lambda) = \mathcal{H}$ for some $\lambda > 0$. We write

$$H - \mathrm{i}\lambda = (H_V - \mathrm{i}\lambda)\big(\mathrm{Id} - \mathrm{i}(H_V - \mathrm{i}\lambda)^{-1}C^*C\big).$$

Since H_V is self-adjoint, $H_V - \mathrm{i}\lambda : \mathcal{D}(H_0) \to \mathcal{H}$ is invertible for all $\lambda > 0$. Moreover we have that

$$\big\|(H_V - \mathrm{i}\lambda)^{-1}C^*C\big\| \leq \lambda^{-1}\|C^*C\|,$$

and hence $\mathrm{Id} - \mathrm{i}(H_V - \mathrm{i}\lambda)^{-1}C^*C : \mathcal{H} \to \mathcal{H}$ is invertible for $\lambda > \|C^*C\|$. This shows that $H - \mathrm{i}\lambda : \mathcal{D}(H_0) \to \mathcal{H}$ is invertible for $\lambda > \|C^*C\|$ and hence concludes the proof of (2).

(3) Since H_V is self-adjoint, $-\mathrm{i}H_V$ generates a strongly continuous unitary group $\{e^{-\mathrm{i}tH_V}\}_{t\in\mathbb{R}}$. Hence, since C^*C is bounded, a perturbation argument (see, e.g., [4, Theorem 11.4.1]) shows that $-\mathrm{i}H$ generates a strongly continuous group $\{e^{-\mathrm{i}tH}\}_{t\in\mathbb{R}}$ such that $\|e^{-\mathrm{i}tH}\| \leq e^{\|C^*C\||t|}$ for all $t \in \mathbb{R}$. The fact that $e^{-\mathrm{i}tH}$ is a contraction for $t \geq 0$ is a consequence of the fact that H is maximal dissipative together with the Hille-Yosida Theorem (see e.g. [4, Theorem 8.3.2]).

(4) Standard arguments show that the adjoint of H is given by $H^* = H_0 + V + \mathrm{i}C^*C$ with domain $\mathcal{D}(H^*) = \mathcal{D}(H_0)$. One then verifies, in the same way as for $-\mathrm{i}H$, that $\mathrm{i}H^*$ generates of a strongly continuous group $\{e^{\mathrm{i}tH^*}\}_{t\in\mathbb{R}}$ such that $\{e^{\mathrm{i}tH^*}\}_{t\geq 0}$ is a semigroup of contractions □

The contraction semigroup $\{e^{-\mathrm{i}tH}\}_{t\geq 0}$ has the interpretation of a dynamics in the following sense. If $u_0 \in \mathcal{H}$, $\|u_0\| = 1$, represents the initial state of the quantum system S at time $t = 0$, then the state of S at a positive time $t > 0$ is given by $\|u_t\|^{-1}u_t$, with $u_t := e^{-\mathrm{i}tH}u_0$. Here it should be noted that $\|u_t\| \leq 1$ for all $t \geq 0$ since $e^{-\mathrm{i}tH}$ is a contraction, and that $u_t \neq 0$ since $e^{-\mathrm{i}tH}$ is invertible. Moreover, the semigroup property implies that, for all $u_0 \in \mathcal{H}$, the map $[0, \infty) \ni t \mapsto \|e^{-\mathrm{i}tH}u_0\|$ is decreasing. Hence one can define the probabilities of elastic scattering and absorption as follows.

Definition 1. *Suppose that Hypothesis 1 holds and let* $u_0 \in \mathcal{H}$, $\|u_0\| = 1$. *The probability of elastic scattering of the system* S, *initially in the state* u_0, *is defined by*

$$p_{\mathrm{scatt}}(u_0) := \lim_{t\to\infty} \big\|e^{-\mathrm{i}tH}u_0\big\|^2.$$

The probability of absorption of the system S, initially in the state u_0, is

$$p_{\mathrm{abs}}(u_0) := 1 - \lim_{t\to\infty} \left\| e^{-\mathrm{i}tH} u_0 \right\|^2.$$

In the case where $p_{\mathrm{scatt}}(u_0) > 0$, and if u_0 is orthogonal to the bound states of S (see the next section for the definition of a bound state), a property which is expected to holds, sometimes called weak asymptotic completeness, is that there exists a scattering state $u_+ \in \mathcal{H}$ such that $\|u_+\| = p_{\mathrm{scatt}}(u_0)$ and

$$\lim_{t\to\infty} \left\| e^{-\mathrm{i}tH} u_0 - e^{-\mathrm{i}tH_0} u_+ \right\| = 0.$$

This property will be discussed below in relation with the existence of wave operators.

2.2. **Spectrum and spectral subspaces of $\mathcal{H}$.** Since H is maximal dissipative – or equivalently $-\mathrm{i}H$ generates a strongly continuous semigroup of contractions – another application of the Hille-Yosida Theorem shows that the spectrum of H satisfies

$$\sigma(H) \subset \{z \in \mathbb{C},\, \mathrm{Im}(z) \le 0\}.$$

In the remainder of this section, we review the definitions of important spectral subspaces of H.

2.2.1. *The space of bound states.* If $\mathcal{D}$ is a subset of $\mathcal{H}$, we denote by $\overline{\mathcal{D}}$ its closure.

Definition 2. *Suppose that Hypothesis 1 holds. The space of bound states of H is defined as the closure of the vector space spanned by all eigenvectors of H corresponding to real eigenvalues, i.e.*

$$\mathcal{H}_{\mathrm{b}}(H) := \overline{\mathrm{Span}\{u \in \mathcal{D}(H),\, \exists \lambda \in \mathbb{R},\, Hu = \lambda u\}}.$$

Similarly,

$$\mathcal{H}_{\mathrm{b}}(H^*) := \overline{\mathrm{Span}\{u \in \mathcal{D}(H^*),\, \exists \lambda \in \mathbb{R},\, H^*u = \lambda u\}}.$$

In the particular case were H is self-adjoint, i.e. $C = 0$, we see that the space of bound states identifies with the pure point spectral subspace of H usually denoted by $\mathcal{H}_{\mathrm{pp}}(H)$. In the general case, $\mathcal{H}_{\mathrm{b}}(H)$ and the pure point spectral subspace of the self-adjoint part of H are related as follows.

Proposition 2.2. *Suppose that Hypothesis 1 holds. Then*

$$\mathcal{H}_{\mathrm{b}}(H) = \mathcal{H}_{\mathrm{b}}(H^*) \subset \mathcal{H}_{\mathrm{pp}}(H_V) \cap \mathrm{Ker}(C).$$

Proof. See [8, Lemma 3.1]. □

2.2.2. *Discrete and essential spectra.* The discrete and essential spectra of H may be defined as follows. We recall that an operator A on $\mathcal{H}$ with domain $\mathcal{D}(A)$ is called Fredholm if $\mathrm{Ran}(A - \lambda\mathrm{Id})$ is closed, $\dim \mathrm{Ker}(A - \lambda\mathrm{Id}) < \infty$ and $\mathrm{codim}\, \mathrm{Ran}(A - \lambda\mathrm{Id}) < \infty$.

Definition 3 (Discrete spectrum)**.** *Suppose that Hypothesis 1 holds. The discrete spectrum of H, denoted by $\sigma_{\mathrm{disc}}(H)$, is the set of isolated eigenvalues of H with finite algebraic multiplicity. In other words, $\lambda \in \sigma_{\mathrm{disc}}(H)$ if λ is an isolated point in $\sigma(H)$, there exists $u \in \mathcal{D}(H) \setminus \{0\}$ such that $Hu = \lambda u$ and* $\dim \mathrm{Ker}(H - \lambda\mathrm{Id}) < \infty$.

Definition 4 (Essential spectrum)**.** *Suppose that Hypothesis 1 holds. The essential spectrum of H, denoted by $\sigma_{\mathrm{ess}}(H)$, is the set of $\lambda \in \mathbb{C}$ such that $H - \lambda\mathrm{Id}$ is not Fredholm.*

We mention that other possible definitions of the essential spectrum may be found in the literature (see, e.g., [6, Section IX]) but these different definitions coincide in our context [6, Theorem IX.1.6]. The discrete and essential spectra of H are related as follows.

Proposition 2.3. *Suppose that Hypothesis 1 holds. Then*

$$\sigma_{\mathrm{ess}}(H_0) = \sigma_{\mathrm{ess}}(H_V) = \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H^*) = \sigma(H) \setminus \sigma_{\mathrm{disc}}(H) = \sigma(H^*) \setminus \sigma_{\mathrm{disc}}(H^*).$$

Proof. The first two equalities are consequences of the facts that V and C^*C are relatively compact perturbations of H_0 (see e.g. [4, Theorem 11.2.6]). The last equality is proven e.g. in [6, Theorem IX.1.6]. □

Summing up, the spectrum of H is of the form pictured in Figure 1.

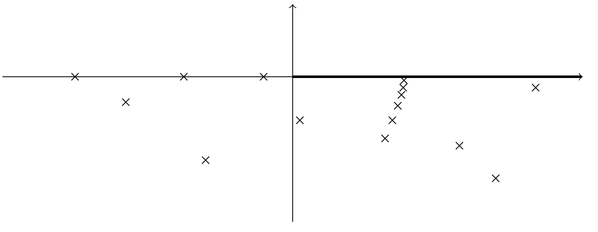

FIGURE 1. **Spectrum of H.** The spectrum of H is contained in the lower half-plane. The essential spectrum of H coincides with that of H_0 and is contained in $[0, \infty)$. The discrete spectrum of H consists of isolated eigenvalues of finite algebraic multiplicities which may accumulate at any point of the essential spectrum.

2.2.3. *Riesz projections.*

Definition 5. *Suppose that Hypothesis 1 holds and let $\lambda \in \sigma_{\mathrm{disc}}(H)$. The Riesz projection corresponding to λ, denoted by π_λ, is defined by*

$$\pi_\lambda := \frac{1}{2\mathrm{i}\pi} \int_\gamma (z\mathrm{Id} - H)^{-1} \mathrm{d}z,$$

where γ is a circle centered at λ, oriented counterclockwise and such that λ is the only point of $\sigma(H)$ contained in the interior of γ.

Given $\lambda \in \sigma_{\mathrm{disc}}(H)$, we recall that a vector $u \in \mathcal{H}$ is called a generalized eigenvector corresponding to λ if there exists a positive integer k such that $u \in \mathcal{D}(H^k)$ and $(H - \lambda)^k u = 0$. As is well-known, the range of the Riesz projection π_λ coincides with the vector space spanned by all generalized eigenvectors corresponding to λ.

Proposition 2.4. *Suppose that Hypothesis 1 holds and let $\lambda \in \sigma_{\mathrm{disc}}(H)$. Then π_λ is a projection such that* $\dim \mathrm{Ran}(\pi_\lambda) < \infty$ *and*

$$\mathrm{Ran}(\pi_\lambda) = \big\{ u \in \mathcal{D}(H^k),\, (H - \lambda)^k u = 0, \ \textit{for some } k \in \mathbb{N},\, 1 \le k \le \dim \mathrm{Ran}(\pi_\lambda) \big\}.$$

Proof. See, e.g., [4, Theorem 1.5.4]. □

In the particular case where λ is a *real* isolated eigenvalue of H, one can prove that the only possible generalized eigenvectors corresponding to λ are eigenvectors in the usual sense.

Proposition 2.5. *Suppose that Hypothesis 1 holds and let $\lambda \in \sigma_{\mathrm{disc}}(H) \cap \mathbb{R}$. Then*

$$\mathrm{Ran}(\pi_\lambda) = \{ u \in \mathcal{D}(H),\, (H - \lambda) u = 0 \}.$$

Proof. See [8, Lemma 3.3]. □

Of course, one can define Riesz projections in the same way for H^* and verify that statements analogous to Propositions 2.4–2.5 hold for H^*.

2.2.4. *The dissipative space.*

Definition 6. *Suppose that Hypothesis 1 holds. The dissipative space, or space of decaying states of H, is defined by*

$$\mathcal{H}_{\mathrm{d}}(H) := \big\{ u \in \mathcal{H},\ \lim_{t\to\infty} \big\| e^{-\mathrm{i}tH} u \big\| = 0 \big\}.$$

Likewise,

$$\mathcal{H}_{\mathrm{d}}(H^*) := \big\{ u \in \mathcal{H},\ \lim_{t\to\infty} \big\| e^{\mathrm{i}tH^*} u \big\| = 0 \big\}.$$

Recalling that the probability of absorption of an initial state u has been defined in Definition 1, we see that $u_0 \in \mathcal{H}$, $\|u_0\| = 1$, belongs to $\mathcal{H}_{\mathrm{d}}(H)$ if and only if $p_{\mathrm{abs}}(u_0) = 1$. The dissipative space contains all generalized eigenvectors of H corresponding to eigenvalues with a strictly negative imaginary part. More precisely, we introduce the following definition.

Definition 7. *Suppose that Hypothesis 1 holds. The subspace $\mathcal{H}_{\mathrm{p}}(H)$ is the closure of the vector space spanned by all generalized eigenvectors of H corresponding to an eigenvalue with a strictly negative imaginary part,*

$$\mathcal{H}_{\mathrm{p}}(H) := \overline{\big\{ u \in \mathrm{Ran}(\pi_\lambda),\ \lambda \in \sigma_{\mathrm{disc}}(H),\ \mathrm{Im}(\lambda) < 0 \big\}}.$$

Likewise,

$$\mathcal{H}_{\mathrm{d}}(H^*) := \overline{\big\{ u \in \mathrm{Ran}(\pi_\lambda),\ \lambda \in \sigma_{\mathrm{disc}}(H^*),\ \mathrm{Im}(\lambda) > 0 \big\}}.$$

We then have the following proposition.

Proposition 2.6. *Suppose that Hypothesis 1 holds. Then*

$$\mathcal{H}_{\mathrm{p}}(H) \subseteq \mathcal{H}_{\mathrm{d}}(H) \subseteq \mathcal{H}_{\mathrm{b}}(H)^\perp, \quad \mathcal{H}_{\mathrm{p}}(H^*) \subseteq \mathcal{H}_{\mathrm{d}}(H^*) \subseteq \mathcal{H}_{\mathrm{b}}(H)^\perp.$$

Proof. First, we prove that $\mathcal{H}_{\mathrm{p}}(H) \subseteq \mathcal{H}_{\mathrm{d}}(H)$. Let $\lambda \in \sigma_{\mathrm{disc}}(H)$, $\mathrm{Im}(\lambda) < 0$ and let $u \in \mathrm{Ran}(\pi_\lambda)$. Let $k = \dim \mathrm{Ran}(\pi_\lambda) < \infty$. We compute

$$\big\| e^{-\mathrm{i}tH} u \big\| = e^{t\mathrm{Im}(\lambda)} \big\| e^{-\mathrm{i}t(H-\lambda)} u \big\| = e^{t\mathrm{Im}(\lambda)} \Big\| \sum_{j=0}^{k-1} \frac{(-\mathrm{i}t)^j}{j!} (H-\lambda)^j u \Big\| \to 0, \quad t \to \infty,$$

since $\mathrm{Im}(\lambda) < 0$. Hence $u \in \mathcal{H}_{\mathrm{d}}(H)$.

Next, we prove that $\mathcal{H}_{\mathrm{d}}(H) \subseteq \mathcal{H}_{\mathrm{b}}(H)^\perp$. Let $u \in \mathcal{H}_{\mathrm{d}}(H)$ and let v be an eigenvector of H^* corresponding to a real eigenvalue. We have that

$$\big| \langle v, u \rangle \big| = \big| \langle e^{\mathrm{i}tH^*} v, e^{-\mathrm{i}tH} u \rangle \big| = \big| \langle v, e^{-\mathrm{i}tH} u \rangle \big| \le \|v\| \big\| e^{-\mathrm{i}tH} u \big\| \to 0, \quad t \to \infty.$$

Hence u is orthogonal to all eigenvectors of H^* corresponding to real eigenvalues, and therefore $u \in \mathcal{H}_{\mathrm{b}}(H^*)^\perp$. Since $\mathcal{H}_{\mathrm{b}}(H) = \mathcal{H}_{\mathrm{b}}(H^*)$ by Proposition 2.2, this concludes the proof.

The proof of $\mathcal{H}_{\mathrm{p}}(H^*) \subseteq \mathcal{H}_{\mathrm{d}}(H^*)$ and $\mathcal{H}_{\mathrm{d}}(H^*) \subseteq \mathcal{H}_{\mathrm{b}}(H)^\perp$ are analogous. □

2.2.5. *The absolutely continuous spectral subspace.* Now, we turn to a possible definition of an "absolutely continuous spectral subspace" for the non-self-adjoint operator H, following Davies [3].

Definition 8. *Suppose that Hypothesis 1 holds. The absolutely continuous spectral subspace of H is defined by*

$$\mathcal{H}_{\mathrm{ac}}(H) := \overline{M(H)},$$

where

$$M(H) := \Big\{ u \in \mathcal{H},\ \exists \mathrm{c}_u > 0,\ \forall v \in \mathcal{H},\ \int_0^\infty \big| \langle e^{-\mathrm{i}tH} u, v \rangle \big|^2 \mathrm{d}t \le \mathrm{c}_u \|v\|^2 \Big\}.$$

The absolutely continuous spectral subspace of H^* is defined similarly, replacing $\mathrm{e}^{-\mathrm{i}tH}$ by $\mathrm{e}^{\mathrm{i}tH^*}$ in the definition above.

In the particular case where H is self-adjoint, the definition of $\mathcal{H}_{\mathrm{ac}}(H)$ coincides with the usual one based on the nature of the spectral measures of H. Moreover, if H is self-adjoint, $M(H)$ is closed and hence $\mathcal{H}_{\mathrm{ac}}(H) = M(H)$. Another possible definition of an absolutely continuous spectral subspace of H follows from the theory of unitary dilations of non-self-adjoint operators, see e.g., [14]. The relevance of Definition 8 may be supported by the following result.

Proposition 2.7. *Suppose that Hypothesis 1 holds. Then*

$$\mathcal{H}_{\mathrm{ac}}(H) = \mathcal{H}_{\mathrm{b}}(H)^{\perp}.$$

In particular,

$$\mathcal{H}_{\mathrm{d}}(H) \subseteq \mathcal{H}_{\mathrm{ac}}(H) = \mathcal{H}_{\mathrm{ac}}(H^*).$$

Proof. The fact that $\mathcal{H}_{\mathrm{ac}}(H) = \mathcal{H}_{\mathrm{b}}(H)^{\perp}$ is proven in [3]. The second equation is a direct consequence of Propositions 2.2 and 2.6. □

We mention that another natural – and relevant – definition for the absolutely continuous spectral subspace of H would be the orthogonal complement of all generalized eigenstates of H, namely

$$\tilde{\mathcal{H}}_{\mathrm{ac}}(H) := \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H)\big)^{\perp}.$$

According to the previous proposition and Definition 8, we then have that

$$\tilde{\mathcal{H}}_{\mathrm{ac}}(H) := \overline{\tilde{M}(H)}, \quad \tilde{M}(H) := \Big\{ u \in \mathcal{H}_{\mathrm{p}}(H)^{\perp}, \, \exists \mathrm{c}_u > 0, \, \forall v \in \mathcal{H}, \, \int_0^\infty \big|\langle e^{-\mathrm{i}tH} u, v\rangle\big|^2 \mathrm{d}t \le \mathrm{c}_u \|v\|^2 \Big\}.$$

2.3. **The wave and scattering operators.** In this section we define the central objects in the scattering theory for the pair (H, H_0), namely, the wave operators, the scattering operator and the scattering matrices. We begin by introducing hypotheses insuring that these objects are well-defined.

2.3.1. *Hypotheses.* Recall that, given a self-adjoint operator A on $\mathcal{H}$, $\mathcal{H}_{\mathrm{pp}}(A)$, $\mathcal{H}_{\mathrm{ac}}(A)$ and $\mathcal{H}_{\mathrm{sc}}(A)$ denote the pure point, absolutely continuous and singular continuous spectral subspaces of A, respectively. Likewise, $\sigma_{\mathrm{pp}}(A)$, $\sigma_{\mathrm{ac}}(A)$ and $\sigma_{\mathrm{sc}}(A)$ denote the pure point, absolutely continuous and singular continuous spectra of A.

Our first hypothesis concerns the spectra of the self-adjoint operators H_0 and H_V (recall that H_0 and H_V have the same essential spectrum, assuming Hypothesis 1).

Hypothesis 2 (Spectra of H_0 and H_V)**.**

(i) *The spectrum of H_0 is purely absolutely continuous, i.e., $\sigma_{\mathrm{ac}}(H_0) = \sigma(H_0)$, $\sigma_{\mathrm{pp}}(H_0) = \emptyset$, $\sigma_{\mathrm{sc}}(H_0) = \emptyset$.*

(ii) *H_V has no singular spectrum, no embedded eigenvalues, and only finitely many eigenvalues counting multiplicity, i.e., $\sigma_{\mathrm{sc}}(H_V) = \emptyset$, $\sigma_{\mathrm{pp}}(H_V) \subset \mathbb{R} \setminus \sigma(H_0)$ and $\dim \mathcal{H}_{\mathrm{pp}}(H_V) < \infty$.*

We denote by $\Pi_{\mathrm{ac}}(H_V)$ the orthogonal projection onto $\mathcal{H}_{\mathrm{pp}}(H_V)$. The symbol s-lim stands for strong limit. Our second hypothesis concerns the unitary wave operators associated to the self-adjoint pair (H_V, H_0) (in the statement of Hypothesis 3 below, it is tacitly assumed that Hypothesis 2 holds).

Hypothesis 3 (Wave operators for (H_0, H_V))**.** *The wave operators*

$$W_\pm(H_V, H_0) := \operatorname*{s-lim}_{t \to \pm\infty} e^{\mathrm{i}tH_V} e^{-\mathrm{i}tH_0}, \quad W_\pm(H_0, H_V) := \operatorname*{s-lim}_{t \to \pm\infty} e^{\mathrm{i}tH_0} e^{-\mathrm{i}tH_V} \Pi_{\mathrm{ac}}(H_V),$$

exist and are asymptotically complete, i.e.,

$$\operatorname{Ran} W_\pm(H_V, H_0) = \mathcal{H}_{\mathrm{ac}}(H_V) = \mathcal{H}_{\mathrm{pp}}(H_V)^{\perp}, \quad \operatorname{Ran} W_\pm(H_0, H_V) = \mathcal{H}.$$

In our third assumption, we require that the operator C be relatively smooth with respect to H_V in the sense of Kato [12].

Hypothesis 4 (Relative smoothness of C with respect to H_V)**.** *There exists* $\mathrm{c}_V > 0$ *such that, for all* $u \in \mathcal{H}_{\mathrm{ac}}(H)$,

$$\int_{\mathbb{R}} \left\| C e^{-\mathrm{i}tH_V} u \right\|^2 \mathrm{d}t \leq \mathrm{c}_V \|u\|^2.$$

In the remainder of this section, we study the wave and scattering operators for the pair (H, H_0), assuming that Hypotheses 1–4 hold.

2.3.2. *The wave operators* $W_-(H, H_0)$ *and* $W_+(H^*, H_0)$. Assuming that H_0 has purely absolutely continuous spectrum, the wave operators $W_-(H, H_0)$ and $W_+(H^*, H_0)$ in dissipative scattering theory are defined in the same way as in unitary scattering theory, namely

$$W_-(H, H_0) := \underset{t\to\infty}{\text{s-lim}}\, e^{-\mathrm{i}tH} e^{\mathrm{i}tH_0}, \quad W_+(H^*, H_0) := \underset{t\to\infty}{\text{s-lim}}\, e^{\mathrm{i}tH^*} e^{-\mathrm{i}tH_0},$$

where, recall, s-lim stands for strong limit.

The existence and basic properties of $W_-(H, H_0)$ and $W_+(H^*, H_0)$ are stated in the following proposition.

Proposition 2.8. *Suppose that Hypotheses 1–4 hold. Then* $W_-(H, H_0)$ *and* $W_+(H^*, H_0)$ *exist and are injective contractions. Moreover,*

$$e^{-\mathrm{i}tH} W_-(H, H_0) = W_-(H, H_0) e^{-\mathrm{i}tH_0}, \quad e^{-\mathrm{i}tH^*} W_+(H^*, H_0) = W_+(H^*, H_0) e^{-\mathrm{i}tH_0},$$

for all $t \in \mathbb{R}$, *and*

$$\overline{\operatorname{Ran} W_-(H, H_0)} = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H^*)\big)^\perp \subseteq \mathcal{H}_{\mathrm{ac}}(H),$$
$$\overline{\operatorname{Ran} W_+(H^*, H_0)} = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H)\big)^\perp \subseteq \mathcal{H}_{\mathrm{ac}}(H).$$

Proof. See [8]. □

2.3.3. *The wave operators* $W_+(H_0, H)$ *and* $W_-(H_0, H^*)$. Recall that $\mathcal{H}_{\mathrm{ac}}(H)$ and $\mathcal{H}_{\mathrm{ac}}(H^*)$ are defined in Definition 8. We denote by $\Pi_{\mathrm{ac}}(H)$, respectively $\Pi_{\mathrm{ac}}(H^*)$, the orthogonal projection onto the absolutely continuous spectral subspace of H, respectively H^*. The wave operators $W_+(H_0, H)$ and $W_-(H_0, H^*)$ are defined by

$$W_+(H_0, H) := \underset{t\to\infty}{\text{s-lim}}\, e^{\mathrm{i}tH_0} e^{-\mathrm{i}tH} \Pi_{\mathrm{ac}}(H), \quad W_-(H_0, H^*) := \underset{t\to\infty}{\text{s-lim}}\, e^{-\mathrm{i}tH_0} e^{\mathrm{i}tH^*} \Pi_{\mathrm{ac}}(H^*).$$

Using unitarity of $e^{\mathrm{i}tH_0}$, we see that the existence of $W_+(H_0, H)$ is equivalent to the property of weak asymptotic completeness mentioned above, in the following sense: for all $u_0 \in \mathcal{H}_{\mathrm{b}}(H)^\perp = \mathcal{H}_{\mathrm{ac}}(H)$, setting $u_+ := W_+(H_0, H)u_0$, we have that $\|e^{-\mathrm{i}tH} u_0 - e^{-\mathrm{i}tH_0} u_+\| \to 0$, as $t \to \infty$.

The existence and basic properties of $W_+(H_0, H)$ and $W_-(H_0, H^*)$ are stated in the following proposition.

Proposition 2.9. *Suppose that Hypotheses 1–4 hold. Then* $W_+(H_0, H)$ *and* $W_-(H_0, H^*)$ *exist and are contractions. Moreover,*

$$W_+(H_0, H)^* = W_+(H^*, H_0), \quad W_-(H_0, H^*)^* = W_-(H, H_0).$$

In particular,

$$\operatorname{Ker} W_+(H_0, H) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H)\big)^\perp, \quad \operatorname{Ker} W_-(H_0, H^*) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H^*)\big)^\perp,$$

and $W_+(H_0, H)$ *and* $W_-(H_0, H^*)$ *have dense ranges.*

Proof. See [8]. □

We mention that similar results can be obtained using the Kato-Birman theory of trace-class perturbations instead of relatively smooth perturbations, see [2].

2.3.4. *The scattering operators.* In dissipative scattering theory, the scattering operators are defined by

$$S(H, H_0) = [W_+(H^*, H_0)]^* W_-(H, H_0), \quad S(H^*, H_0) = [W_-(H^*, H_0)]^* W_+(H^*, H_0).$$

Combining Propositions 2.8 and 2.9, we arrive at the following result.

Proposition 2.10. *Suppose that Hypotheses 1–4 hold. Then $S(H,, H_0)$ and $S(H^*, H_0)$ exist and are contractions. Moreover,*

$$e^{-\mathrm{i}tH_0} S(H, H_0) = S(H, H_0) e^{-\mathrm{i}tH_0}, \quad e^{-\mathrm{i}tH_0} S(H^*, H_0) = S(H^*, H_0) e^{-\mathrm{i}tH_0},$$

for all $t \in \mathbb{R}$ and we have that

$$S(H, H_0)^* = S(H^*, H_0).$$

An important question, both mathematically and physically, concerns the invertibility of the scattering operators. Regarding this question, we can state the following proposition (see the next section for more precise results).

Proposition 2.11. *Suppose that Hypotheses 1–4 hold. Then the following conditions are equivalent:*

(1) *$S(H, H_0)$ and $S(H^*, H_0)$ are invertible in $\mathcal{L}(\mathcal{H})$.*
(2) *The range of the wave operators $W_-(H, H_0)$ and $W_+(H^*, H_0)$ are given by*

$$\mathrm{Ran}\, W_-(H, H_0) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H^*)\big)^{\perp}, \quad \mathrm{Ran}\, W_+(H^*, H_0) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H)\big)^{\perp}.$$

Proof. See [8]. □

2.3.5. *The scattering matrices.* We recall that the multiplicity of the spectrum of a self-adjoint operator is defined *via* the spectral theorem (see, e.g., [16, Section VII]). To study the scattering matrices, it is convenient to add the following condition to Hypothesis 2(i).

Hypothesis 5 (Multiplicity of $\sigma(H_0)$)**.** *The spectrum of H_0 has a constant multiplicity (which may be infinite).*

To simplify the notations below, we set

$$\Lambda := \sigma(H_0).$$

Assuming Hypotheses 2(i) and 5, the spectral theorem ensures that there exists a unitary mapping from $\mathcal{H}$ to a direct integral of Hilbert spaces,

$$\mathcal{F}_0 : \mathcal{H} \to \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) \mathrm{d}\lambda,$$

such that $\mathcal{F}_0 H_0 \mathcal{F}_0^*$ acts as multiplication by λ on each Hilbert space $\mathcal{H}(\lambda)$. Moreover, since $\sigma(H_0)$ has a constant multiplicity, say $k \in \mathbb{N} \cup \{+\infty\}$, all spaces $\mathcal{H}(\lambda)$ can be identified with a fixed Hilbert space $\mathcal{M}$. Hence $\mathcal{F}_0$ becomes an operator

$$\mathcal{F}_0 : \mathcal{H} \to \int_{\Lambda}^{\oplus} \mathcal{M}\, \mathrm{d}\lambda = L^2(\Lambda; \mathcal{M}),$$

where $\dim \mathcal{M} = k$ and $L^2(\Lambda; \mathcal{M})$ is the space of square integrable functions from Λ to $\mathcal{M}$, (see e.g. [23, Chapter 0, Section 1.3]). Note that in the case where $\mathcal{H} = L^2(\mathbb{R}^3)$ and $H_0 = -\Delta$, the Hilbert space $\mathcal{M}$ is given by $\mathcal{M} = L^2(S^2)$, where S^2 stands for the sphere in $\mathbb{R}^3$.

Using that the scattering operator $S(H,H_0)$ commutes with H_0, by Proposition 2.10, one can verify that

$$\mathcal{F}_0 S(H,H_0)\mathcal{F}_0^* = \int_\Lambda^\oplus S(\lambda)\mathrm{d}\lambda.$$

The bounded operators

$$S(\lambda) \in \mathcal{L}(\mathcal{M}),$$

defined for a.e. $\lambda \in \Lambda$, are called the scattering matrices (for the pair (H,H_0)).

One can define in the same way the scattering matrices $S^*(\lambda)$ for the pair (H^*,H_0) by the relation

$$\mathcal{F}_0 S(H^*,H_0)\mathcal{F}_0^* = \int_\Lambda^\oplus S^*(\lambda)\mathrm{d}\lambda.$$

Under the conditions of Proposition 2.11, we then have that

$$[S(\lambda)]^* = S^*(\lambda),$$

for a.e. $\lambda \in \Lambda$. We set

$$\mathcal{F}_\pm := \mathcal{F}_0 W_\pm^*(H_V,H_0) : \mathcal{H} \to L^2(\Lambda;\mathcal{M}).$$

Given $s \geq 0$, an interval X and a Hilbert space $\mathcal{H}$, we denote by $\mathrm{C}^s(X;\mathcal{H})$ the set of Hölder continuous $\mathcal{H}$-valued functions on X of order s.

In order to insure that the map $\lambda \mapsto S(\lambda)$ is continuous, it is convenient to require that the operators V and C are strongly smooth with respect to H_0 and H_V, respectively, in the following sense.

Hypothesis 6 (Strong smoothness of V with respect to H_0)**.**

(i) *There exist an auxiliary Hilbert space $\mathcal{G}$ and operators $G : \mathcal{H} \to \mathcal{G}$ and $K : \mathcal{G} \to \mathcal{G}$ such that $V = G^*KG$, with $G(H_0^{1/2}+1)^{-1} \in \mathcal{L}(\mathcal{H};\mathcal{G})$ and $K \in \mathcal{L}(\mathcal{G})$.*
(ii) *For all $z \in \mathbb{C}$, $\mathrm{Im}(z) \neq 0$, $GR_0(z)G^*$ is compact.*
(iii) *The operator G is strongly H_0-smooth with exponent $s_0 \in (\frac{1}{2},1)$ on any compact set $X \Subset \Lambda$, i.e.*

$$\mathcal{F}_0[G\mathbb{1}_X(H_0)]^* : \mathcal{G} \to \mathrm{C}^{s_0}(X;\mathcal{M}) \text{ is continuous.}$$

Hypothesis 7 (Strong smoothness of C with respect to H_V)**.**

(i) *The operator C is strongly H_V-smooth with exponent $s \in (0,1)$ on any compact set $X \Subset \Lambda$, i.e.*

$$\mathcal{F}_\pm[C\mathbb{1}_X(H_V)]^* : \mathcal{H} \to \mathrm{C}^{s}(\Lambda;\mathcal{H}) \text{ is continuous.}$$

(ii) *For all $z \in \mathbb{C}$, $\mathrm{Im}(z) \neq 0$, $CR_V(z)C^*$ is compact.*
(iii) *The map*

$$\mathring{\Lambda} \in \lambda \mapsto C\big(R_V(\lambda+i0) - R_V(\lambda-i0)\big)C^* \in \mathcal{L}(\mathcal{H}),$$

is bounded.

We refer to [22, 23] for details on the theory of strongly smooth operators.

In the statement below, $S^\sharp$ stands for S or S^*. Based on a generalization of Kuroda's representation formula, the following result was established in [9].

Proposition 2.12. *Suppose that Hypotheses 1–7 hold. Then, for all $\lambda \in \mathring{\Lambda}$, $S^\sharp(\lambda)$ is a contraction and $S^\sharp(\lambda) - \mathrm{Id}$ is compact. If, in addition, $\dim \mathcal{M} = +\infty$, then for all $\lambda \in \mathring{\Lambda}$, $\|S^\sharp(\lambda)\| = 1$ and, in particular, $\|S(H,H_0)\| = 1 = \|S(H^*,H_0)\|$.*

Proof. See [9]. □

3. Spectral singularities and asymptotic completeness

Our next concern is to study more precisely the invertibility of the scattering matrices and operator. Invertibility of $S(\lambda)$ is a strongly relevant physical property since it shows that to any incoming state at energy λ corresponds a unique outgoing state and vice versa. In Section 3.1, we explain that non-invertibility of $S(\lambda)$ is equivalent to the presence of a *spectral singularity* at energy λ. Section 3.2 is devoted to the property of *asymptotic completeness* of the wave operators.

3.1. **Spectral singularities.** Recall that, under our assumptions and notations, the essential spectrum of H is given by $\sigma_{\mathrm{ess}}(H) = \sigma(H_0) = \Lambda$. We recall the notion of a spectral singularity introduced in [8, 9], distinguishing points in the interior of Λ and points in the boundary $\Lambda \setminus \mathring{\Lambda}$.

Definition 9.

(i) *Let $\lambda \in \mathring{\Lambda}$. We say that λ is a regular spectral point of H if there exists a compact interval $K_\lambda \subset \mathbb{R}$ whose interior contains λ, such that K_λ does not contain any accumulation point of eigenvalues of H, and such that the limits*

$$CR(\mu - \mathrm{i}0)C^* := \lim_{\varepsilon \downarrow 0} CR(\mu - \mathrm{i}\varepsilon)C^*$$

exist uniformly in $\mu \in K_\lambda$ in the norm topology of $\mathcal{L}(\mathcal{H})$. If λ is not a regular spectral point of H, we say that λ is a spectral singularity of H.

(ii) *Let $\lambda \in \Lambda \setminus \mathring{\Lambda}$. We say that λ is a regular spectral point of H if there exists a compact interval $K_\lambda \subset \mathbb{R}$ whose interior contains λ, such that all $\mu \in K_\lambda \cap \mathring{\Lambda}$ are regular in the sense of* (i) *and such that the map*

$$K_\lambda \cap \mathring{\Lambda} \ni \mu \mapsto CR(\mu - \mathrm{i}0)C^* \in \mathcal{L}(\mathcal{H})$$

is bounded.

(iii) *If Λ is right-unbounded, we say that $+\infty$ is regular if there exists $m > 0$ such that all $\mu \in [m, \infty) \cap \mathring{\Lambda}$ are regular in the sense of* (i) *and such that the map*

$$[m, \infty) \cap \mathring{\Lambda} \ni \mu \mapsto CR(\mu - \mathrm{i}0)C^* \in \mathcal{L}(\mathcal{H})$$

is bounded.

Note that our definition of a regular spectral point is local. One can rephrase this definition saying that λ is a regular spectral point of H if the limiting absorption principle for H holds in a neighborhood of λ, for the weighted resolvent $CR(z)C^*$, for values of the spectral parameter z in the lower half-plane. It should be noted that we do not need to require that the limiting absorption principle hold for values of the spectral parameter in the upper half-plane: This is due to the fact that H is supposed to be dissipative. We also mention that there is a natural definition of a spectral singularity for the adjoint operator H^*, such that λ is a spectral singularity of H if and only if λ is a spectral singularity of H^*.

In the case where $H = -\Delta + V - \mathrm{i}W$ on $L^2(\mathbb{R}^3)$, with V and W bounded and compactly supported, a spectral singularity of H corresponds to a resonance embedded in the essential spectrum $[0, \infty)$ (see, e.g., [5] for the theory of resonances for Schrödinger operators, and [8] for a comparison between the notions of resonances and spectral singularities).

The next theorem provides several characterizations of a spectral singularity $\lambda \in \mathring{\Lambda}$. It is based, in particular, on a generalization of Kuroda's representation formula to the context of dissipative scattering theory.

Theorem 3.1. *Suppose that Hypotheses 1–7 hold. Let $\lambda \in \mathring{\Lambda}$. Then the following conditions are equivalent:*

(1) *λ is a regular spectral point of H.*
(2) *λ is not an accumulation point of eigenvalues of H located in $\lambda - \mathrm{i}(0,\infty)$ and the limit*

$$CR(\mu - \mathrm{i}0)C^* = \lim_{\varepsilon \downarrow 0} CR(\mu - \mathrm{i}\varepsilon)C^*$$

exists in the norm topology of $\mathcal{L}(\mathcal{H})$.
(3) *The operator* $\mathrm{Id} - \mathrm{i}CR_V(\lambda - \mathrm{i}0)C^*$ *is invertible in $\mathcal{L}(\mathcal{H})$.*
(4) *The scattering matrix $S(\lambda)$ is invertible in $\mathcal{L}(\mathcal{M})$.*

Proof. See [9]. □

In general, it is a difficult problem to identify explicitly the spectral singularities of a given dissipative operator. Nevertheless, one can show that the set of spectral singularities is not too large in the following sense.

Proposition 3.2. *Suppose that Hypotheses 1–7 hold. Then the set of spectral singularities of H is a closed subset of Λ of Lebesgue measure* 0*.*

Proof. See [9]. □

Recall from Propositions 2.8 and 2.11 that the scattering operators $S(H, H_0)$ and $S(H^*, H_0)$ are invertible if and only if the wave operators $W_-(H, H_0)$ and $W_+(H, H_0)$ have closed ranges. The following proposition shows that the study of spectral singularities is also relevant in order to answer the question of the invertibility of the scattering operators.

Proposition 3.3. *Suppose that Hypotheses 1–7 hold. Suppose in addition that $\Lambda \setminus \mathring{\Lambda}$ is finite and that all $\lambda \in \Lambda \setminus \mathring{\Lambda}$ are regular in the sense of Definition 9 (if Λ is right-unbounded, we also assume that $+\infty$ is regular). Then the following conditions are equivalent:*

(1) *$S(H, H_0)$ is invertible in $\mathcal{L}(\mathcal{H})$,*
(2) *$S(H^*, H_0)$ is invertible in $\mathcal{L}(\mathcal{H})$,*
(3) *H has no spectral singularities in $\mathring{\Lambda}$.*

Proof. See [9]. □

To conclude this section, we propose the following definition of the "order" of a spectral singularity of H. It will be relevant in the next section.

Definition 10. *We say that $\lambda \in \mathring{\Lambda}$ is a spectral singularity of H of order $\nu \in \mathbb{N}^*$ if λ is a spectral singularity of H and there exists a compact interval K_λ, whose interior contains λ, such that the limits*

$$\lim_{\varepsilon \downarrow 0} (\mu - \lambda)^\nu CR(\mu - \mathrm{i}\varepsilon)C^*$$

exist uniformly in $\mu \in K_\lambda$ in the norm topology of $\mathcal{L}(\mathcal{H})$.

As mentioned above, if one considers the nuclear optical model $H = -\Delta + V - \mathrm{i}C^*C$ with bounded and compactly supported potentials V and C, then a spectral singularity corresponds to a resonance in the usual sense (see, e.g., [5]). One can then verify that the order of a spectral singularity in the sense of Definition 10 corresponds to the multiplicity of the corresponding resonance, see [8, Section 6].

3.2. **Asymptotic completeness.** We are interested in this section in the property of asymptotic completeness of the wave operators. In our context, this property can be defined as follows.

Definition 11. *The wave operators $W_-(H,H_0)$ and $W_+(H^*,H_0)$ are said to be asymptotically complete if their ranges coincide with the orthogonal complements of all generalized eigenstates of H and H^*, respectively. In other words,*

$$\mathrm{Ran}(W_-(H,H_0)) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\big)^{\perp}, \quad \mathrm{Ran}(W_+(H^*,H_0)) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H)\big)^{\perp}.$$

With the alternative definition $\tilde{\mathcal{H}}_{\mathrm{ac}}(H)$ of the absolutely continuous spectral subspace of H suggested at the end of Section 2.2, we see that asymptotic completeness means that $\mathrm{Ran}(W_+(H,H_0)) = \tilde{\mathcal{H}}_{\mathrm{ac}}(H^*)$ and $\mathrm{Ran}(W_+(H^*,H_0)) = \tilde{\mathcal{H}}_{\mathrm{ac}}(H)$.

In [8], asymptotic completeness is proven under the following further assumption.

Hypothesis 8.

(i) *H has at most finitely many (discrete) eigenvalues.*

(ii) *H has at most finitely many spectral singularities in $\mathring{\Lambda}$ and each spectral singularity is of finite order.*

(iii) *$\Lambda \setminus \mathring{\Lambda}$ is finite and all $\lambda \in \Lambda \setminus \mathring{\Lambda}$ are regular Moreover, if Λ is right-unbounded, then $+\infty$ is regular.*

We then have the following result.

Theorem 3.4. *Suppose that Hypotheses 1–8 hold. Then*

$$\mathcal{H}_{\mathrm{p}}(H) = \mathcal{H}_{\mathrm{d}}(H), \quad \mathcal{H}_{\mathrm{p}}(H^*) = \mathcal{H}_{\mathrm{d}}(H^*).$$

Moreover,

$$W_-(H,H_0) \text{ and } W_+(H^*,H_0) \text{ are asymptotically complete}$$
$$\iff \quad H \text{ has no spectral singularities in } \mathring{\Lambda}.$$

If these equivalent conditions are satisfied, then

(1) *There is an H-invariant direct sum decomposition*

$$\mathcal{H} = \big\{\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H)\big\} \oplus \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\big)^{\perp},$$

and the restriction of H to $\big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\big)^{\perp}$ is similar to H_0. An analogous statement holds for H^*.

(2) *The wave operators $W_+(H_0,H)$ and $W_-(H_0,H^*)$ are surjective and their kernels are given by*

$$\mathrm{Ker}\, W_+(H_0,H) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H)\big)^{\perp}, \quad \mathrm{Ker}\, W_-(H_0,H^*) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\big)^{\perp},$$

(3) *The scattering operators $S(H,H_0)$ and $S(H^*,H_0)$ are bijective.*

Proof. See [8, 9]. □

3.3. **Application to the nuclear optical model.** Now, we describe the main consequences of the abstract results previously stated to the nuclear optical model mentioned in the introduction. We refer to [8, 9] for details showing that the abstract hypotheses 1–8 are indeed satisfied in the case of the nuclear optical model, under the conditions on the potentials imposed in the following theorems.

In this section, on $L^2(\mathbb{R}^3)$, we set

$$H_0 = -\Delta, \quad H_V = -\Delta + V(x), \quad H = H_V - \mathrm{i}W(x).$$

Recall that the sphere in $\mathbb{R}^3$ is denoted by S^2.

Theorem 3.5. *Suppose that*

(i) V *is real-valued,* $V \in \mathrm{C}^2(\mathbb{R}^3)$ *and there exists* $\rho > 3$ *such that, for all* $|\alpha| \le 2$, $\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\rho-|\alpha|})$, $|x| \to \infty$,

(ii) W *is non-negative,* $W(x) > 0$ *on a non-trivial open set and there exists* $\delta > 2$ *such that* $W(x) = \mathcal{O}(\langle x \rangle^{-\delta})$, $|x| \to \infty$,

(iii) 0 *is neither an eigenvalue nor a resonance of* H_V.

Then, for all $\lambda > 0$,

$$S(\lambda) \text{ is invertible in } \mathcal{L}(L^2(S^2)) \iff \lambda \text{ is not a spectral singularity of } H.$$

Moreover,

$$S(H,H_0) \text{ is invertible in } \mathcal{L}(L^2(\mathbb{R}^3)) \iff H \text{ has no spectral singularities in } (0,\infty),$$

and if these conditions hold, then $\operatorname{Ran} W_-(H,H_0) = \mathcal{H}_{\mathrm{d}}(H^*)^\perp$.

Proof. See [8, 9]. □

The set of bounded and compactly supported potentials from $\mathbb{R}^3$ to $\mathbb{C}$ is denoted by $L^\infty_{\mathrm{c}}(\mathbb{R}^3)$. If we suppose that V and W belong to $L^\infty_{\mathrm{c}}(\mathbb{R}^3)$, we have in addition the following more precise results.

Theorem 3.6. *Suppose that*

(i) V *is real-valued and* $V \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$.

(ii) W *is non-negative,* $W(x) > 0$ *on a non-trivial open set and* $W \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$.

(iii) 0 *is neither an eigenvalue nor a resonance of* H_V.

Then, $\mathcal{H}_{\mathrm{p}}(H) = \mathcal{H}_{\mathrm{d}}(H)$. *Moreover,*

$$\begin{aligned} W_-(H,H_0) \text{ is asymptotically complete } &\iff \operatorname{Ran} W_-(H,H_0) = \mathcal{H}_{\mathrm{p}}(H^*)^\perp \\ &\iff H \text{ has no spectral singularities in } (0,\infty). \end{aligned}$$

If these conditions hold, then

(1) $S(H,H_0)$ *is invertible in* $\mathcal{L}(L^2(\mathbb{R}^3))$,

(2) *For all* $\lambda > 0$, $S(\lambda)$ *is invertible in* $\mathcal{L}(L^2(S^2))$,

(3) *The restriction of* H *to* $\mathcal{H}_{\mathrm{p}}(H^*)^\perp$ *is similar to* H_0.

Proof. See [8, 9]. □

We mention that the fact that $\mathcal{H}_{\mathrm{b}}(H) = \{0\}$ in the context of the present section follows from unique continuation arguments. Moreover, it is proven in [19] that 0 cannot be a spectral singularity of H. On the other hand, for any $\lambda > 0$, one can construct smooth and compactly supported potentials V and W such that λ is a spectral singularity of H (see [20]).

Acknowledgements. I am grateful to S. Richard for hospitality at the University of Nagoya, and to F. Hiroshima for the invitation to give a talk at the conference *Mathematical aspects of quantum fields and related topics*, RIMS, Kyoto, July 2018. I warmly thank J. Fröhlich and F. Nicoleau for fruitful collaborations.

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