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2-local isometries and the reflexivity property of certain spaces of continuous maps

Hiroko Meguro,
Master's Program in Fundamental Sciences
Graduate School of Science and Technology,
Niigata University

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1 Introduction

The studies about local maps were started by Larson, Kadison and Sourour. In 1988, Larson[9] studied local automorphisms of Banach algebra and obtained the first results concerning to local maps. In 1990, Kadison[8] exhibited the results concerning to local derivations on von neumann algebras. Larson and Sourour[10] got the results of local derivations of $B(X)$ for a Banach space X.

The studies of 2-local maps were initiated by Šemrl[13]. He got the results about 2-local automorphisms and 2-local derivations in 1997. Inspired by his results, Molnár[12] started the sudies about 2-local isometries in 2002. He considered the group of all surjective complex linear isometries. If X is locally compact Hausdorrf space, Győry[3] studies that 2-local isometries are complex linear isometries on the ste of all continuous functions vanishing at infinity $C_0(X)$. Hatori, Miura, Oka and Takagi[4] got the results in the case of the uniform algebras in 2007. $C^{(n)}[0,1]$ denotes the set of all n-times continuously differentiable functions on $[0,1]$ with $\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} |f^{(k)}(t)|/k!$. In 2018, Kawamura, Koshimizu and Miura[7] studied about $C^{(n)}[0,1]$. They got the results that 2-local isometries are surjective complex linear isometries on each space. In recent years, the case of surjective real linear isometries are studied. Hosseini[5] studied $C^{(n)}[0,1]$ with $\|f\|_n = \max\{|f(0)|, |f'(0)|, |f^{(2)}(0)|, \cdots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\}$ in 2017. The results about 2-local isometries in the case of real linear isometries is fewer than the case of complex linear isometries. I get the result about surjective real linear isometries. I will prove it.

2 Fundamental definitions

In this paper, $\mathbb{R}$ stands for the set of all real numbers. The symbol $\mathbb{C}$ stands for all complex numbers.

Definition 2.1 (isometry)**.** *Let $(X, d_X), (Y, d_Y)$ be metric spaces. Let T be a map X into Y. If $d_X(x_1, x_2) = d_Y(T(x_1), T(x_2))$ for all points $x_1, x_2 \in X$, then T is called an isometry.*

Note that T is injective if T is an isometry .

Definition 2.2. *Let X be a Banach space. The set of all surjective complex linear isometries on X is denoted by $Iso_{\mathbb{C}}(X)$. The set of all surjective real linear isometries on X is denoted by $Iso_{\mathbb{R}}(X)$.*

Definition 2.3 (2-local isometry)**.** *Let X be a Banach space. Let T be a map on X. If for each pair of elements $f, g \in X$ there exists $T_{f,g} \in Iso_{\mathbb{C}}(X)$ (or $\in Iso_{\mathbb{R}}(X)$) such that $T_{f,g}(f) = T(f)$ and $T_{f,g}(g) = T(g)$ depending on f and g, then T is called a 2-local isometry .*

We note that no continuity, surjectivity nor linearity are assumed for T.

Definition 2.4. *Let $C[0,1]$ denote the set of all complex-valued functions f on the closed interval endowed with the supremum norm*

$$\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}.$$

Then $(C[0,1], \|\cdot\|_\infty)$ is a Banach algebra.

Definition 2.5 (Choquet boundary)**.** *Let X be a locally compact Hausdorff space. Let A be a uniform algebra on X. Define a subset E of X by $E = \{t \in X : f(x) = 1\}$ for some $f \in A$. Then E is called a peak set for A. For every $x \in X$, E_α*

is a peak set for A. If $\{x\} = \bigcap_\alpha E_\alpha$, x is called a weak peak point of A. Define $Ch(A)$ by $Ch(A) = \{x \in X : x \text{ is a weak peak point for } A\}$. Then $Ch(A)$ is called the Choquet boundary of A.

Definition 2.6 (reflexivity)**.** *Let X be a Banach space. We say that $Iso_{\mathbb{R}}(X)$ is 2-local reflexive if every 2-local isometry is in $Iso_{\mathbb{R}}(X)$.*

3 Surjective real linear isometries on $C[0,1]$

In this section, we consider the form of surjective real linear isometries（Theorem 3.1）. This theorem was essentially proved by Ellis[2] or Miura[11]. We note that the Chouque boungary and the Shilov boudary of $C[0,1]$ corresponds to the closed interval $[0,1]$.

Theorem 3.1. *A map T is a surjective real linear isometry on $C[0,1]$ if and only if there exist a continuous function T(1) : $[0,1] \to \{z \in \mathbb{C} : |z| = 1\}$ and a homeomorphism $\varphi : [0,1] \to [0,1]$ such that one of the following equalities*

$$\begin{cases} T(f)(t) = T(1) f \circ \varphi(t) & (f \in C[0,1], t \in [0,1]) \\ T(f)(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0,1], t \in [0,1]). \end{cases}$$

Proof. First, we assume that a map $T : C[0,1] \to C[0,1]$ is a surjective real linear isometry on $C[0,1]$. The Couquet boundary of $C[0,1]$ coincides with the closed interval $[0,1]$. By a theorem of Miura[11] and the connectivity of $[0,1]$, one of the following equalities

$$\begin{cases} Tf(t) = T(1) f \circ \varphi(t) & (f \in C[0,1], t \in [0,1]) \\ Tf(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0,1], t \in [0,1]). \end{cases}$$

Next, we assume that there exist a continuous function T(1) : $[0,1] \to \{z \in \mathbb{C} : |z| = 1\}$ and a homeomorphism $\varphi : [0,1] \to [0,1]$ such that one of the following equalities

$$\begin{cases} T(f)(t) = T(1) f \circ \varphi(t) & (f \in C[0,1], t \in [0,1]) \\ T(f)(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0,1], t \in [0,1]). \end{cases}$$

We infer that T is a surjectve real linear isometry on $C[0,1]$.

$\square$

4 2-local isometries in $C[0,1]$

The studies about 2-local isometries were started by Molnár[12]. If there exists $T_{f,g} \in Iso_{\mathbb{R}}(C[0,1])$ such that $Tf = T_{f,g}f$ and $Tg = T_{f,g}g$ for every pair of elements $f, g \in C[0,1]$, then T is called 2-local isometry.

The following is the main result in this paper.

Theorem 4.1. *Let T be a 2-local isometry on $C[0,1]$. Then T is a 2-local isometry. Thus $Iso_{\mathbb{R}}(C[0,1])$ is 2-local reflexive.*

To prove Theorem 4.1, we can reduce the case of $T(1) = 1$ (Proposition 4.1). When we assume that $T(1) = 1$, for every element $f \in C[0,1]$ there exists an isometry $T_{1,f}$ such that $T(1) = T_{1,f}(1)$. Since $T(1) = 1$, we get $T_{1,f}(1) = 1$. By Theorem 3.1, T satisfies one of the following equalities

$$\begin{cases} Tf(t) = T_{1,f}f(t) = T_{1,f}(1)f \circ \varphi_{1,f}(t) = f \circ \varphi_{1,f}(t) & (f \in C[0,1], t \in [0,1]) \\ Tf(t) = T_{1f}f(t) = T_{1,f}(1)\overline{f \circ \varphi_{1,f}(t)} = \overline{f \circ \varphi_{1,f}(t)} & (f \in C[0,1], t \in [0,1]), \end{cases}$$

where $\varphi_{1,f}$ is a homeomorphism. When we put t_0 such that $\varphi_{1,f}(t) = t_0$, one of the following equalities

$$\begin{cases} Tf(t) = f(t_0) \\ Tf(t) = \overline{f(t_0)}. \end{cases}$$

Proposition 4.1. *Let T be a 2-local isometry on $C[0,1]$. When $T(1) = 1$, T is a 2-local isometry.*

Proof. Let Id be the identity map of $C[0,1]$. Since T is a 2-local isometry, for every $f \in C[0,1]$ there exists $T_{f,Id} \in Iso_{\mathbb{R}}(C[0,1])$ such that $T(f) = T_{f,Id}(f)$ and $TId = T_{f,Id}(Id)$, also there exists $T_{1,Id} \in Iso_{\mathbb{R}}(C[0,1])$ such that $T(1) = T_{1,Id}(1)$ and $T(Id) = T_{1,Id}(Id)$. By Theorem 3.1, $T_{f,Id}$ and $T_{1,Id}$ are represented by

$$\begin{cases} T_{f,Id}g(t) = T_{f,Id}(1)g \circ \varphi_{f,Id}(t) & (g \in C[0,1], t \in [0,1]) \\ \text{or} \\ T_{f,Id}g(t) = T_{f,Id}(1)\overline{g \circ \varphi_{f,Id}(t)} & (g \in C[0,1], t \in [0,1]) \end{cases} \tag{1}$$

$$\begin{cases} T_{1,Id}g(t) = T_{1,Id}(1)g \circ \varphi_{1,Id}(t) & (g \in C[0,1], t \in [0,1]) \\ \text{or} \\ T_{1,Id}g(t) = T_{1,Id}(1)\overline{g \circ \varphi_{1,Id}(t)} & (g \in C[0,1], t \in [0,1]), \end{cases}$$

where $\varphi_{f,Id}$ and $\varphi_{1,Id}$ are homeomorphisms on $[0,1]$ respectively. Since $T_{1,Id}(1) = T(1) = 1$, $T_{1,Id}$ is represented by

$$\begin{cases} T_{1,Id}g(t) = g \circ \varphi_{1,Id}(t) & (g \in C[0,1], t \in [0,1]) \\ \text{or} \\ T_{1,Id}g(t) = \overline{g \circ \varphi_{1,Id}(t)} & (g \in C[0,1], t \in [0,1]). \end{cases} \tag{2}$$

We define a set $E_{t_0 f}$ by $E_{t_0 f} = \left\{ t \in [0,1] : \begin{array}{c} Tf(t) = f(t_0) \\ Tf(t) = \overline{f(t_0)} \end{array} \right\}$ for every $f \in C[0,1]$, $t_0 \in [0,1]$. Now, $E_{t_0 f}$ is a subset of $[0,1]$. By the definition of $E_{t_0 f}$, $E_{t_0 Id}$ is represented by $E_{t_0 Id} = \{t \in [0,1] : T(Id)(t) = Id(t_0)\}$. Since $TId = T_{1,Id}Id$ and (2), we get

$$TId = T_{1,Id}Id = Id \circ \varphi_{1,Id} = \varphi_{1,Id}. \tag{3}$$

We get $E_{t_0 Id} = \{t \in [0,1] : \varphi_{1,Id}(t) = t_0\}$ since (3) and $Id(t_0) = t_0$. Since $\varphi_{1,Id}$ is a homeomorphism, $E_{t_0 Id}$ is a singleton.

We take $b_{t_0} \in [0,1]$ such that $\{b_{t_0}\} = E_{t_0 Id}$. We have $TId(b_{t_0}) = Id(t_0) = t_0$ by $b_{t_0} \in E_{t_0 Id}$ and the definition of $E_{t_0 Id}$. Therefore we obtain

$$\varphi_{1,Id}(b_{t_0}) = t_0 \tag{4}$$

by (3). Forthermore we have

$$\begin{aligned} TId(b_{t_0}) &= T_{f,Id}Id(b_{t_0}) \\ &= T_{f,Id}(1)Id \circ \varphi_{f,Id}(b_{t_0}) \\ &= T_{f,Id}(1)\varphi_{f,Id}(b_{t_0}) \end{aligned} \tag{5}$$

by $TId = T_{f,Id}Id$ and (1). By (5) and $T(Id)(b_{t_0}) = t_0$, we have $T_{f,Id}(1)\varphi_{f,Id}(b_{t_0}) = t_0$. Since $\varphi_{f,Id}(b_{t_0})$ is in $[0,1]$ and t_0 is in $[0,1]$, $T_{fId}(1)(b_{t_0})$ is a real number which is a scalar of modulars 1. we get

$$T_{f,Id}(1)(b_{t_0}) = 1. \tag{6}$$

Therefore we obtain

$$\varphi_{f,Id}(b_{t_0}) = t_0. \tag{7}$$

We consider $E_{t_0 f} = \left\{ t \in [0,1] : \begin{array}{c} Tf(t) = f(t_0) \\ Tf(t) = \overline{f(t_0)} \end{array} \right\}$ for every $f \in C[0,1]$. Since $Tf = T_{f,Id}f$ and (1), we get

$$\begin{cases} Tf(b_{t_0}) = T_{f,Id}(1)(b_{t_0}) f \circ \varphi_{f,Id}(b_{t_0}) \\ Tf(b_{t_0}) = T_{f,Id}(1)(b_{t_0}) \overline{f \circ \varphi_{f,Id}(b_{t_0})}. \end{cases}$$

By (6), we have

$$\begin{cases} Tf(b_{t_0}) = f \circ \varphi_{f,Id}(b_{t_0}) \\ Tf(b_{t_0}) = \overline{f \circ \varphi_{f,Id}(b_{t_0})}. \end{cases}$$

By (7), we have

$$\begin{cases} Tf(b_{t_0}) = f(t_0) \\ Tf(b_{t_0}) = \overline{f(t_0)}. \end{cases}$$

Therefore b_{t_0} is an element of $E_{t_0 f}$. Since f is an arbitrary element of $C[0,1]$, we get $E_{t_0 Id} = \{b_{t_0}\} = \bigcap_{f \in C[0,1]} E_{t_0 f}$.

Let ψ be a map $[0,1]$ into $[0,1]$ such that $\{\psi(t_0)\} = \bigcap_{f \in C[0,1]} E_{t_0 f}$. Since $\{b_{t_0}\} = \bigcap_{f \in C[0,1]} E_{t_0 f}$, we get $\psi(t_0) = b_{t_0}$. By $TId = T_{1,Id}Id$ and (2), we have

$$\begin{aligned} TId(\psi(t_0)) &= T_{1,Id}Id(\psi(t_0)) \\ &= Id\varphi_{1,Id}(\psi(t_0)) \\ &= \varphi_{1,Id}(\psi(t_0)) \\ &= \varphi_{1,Id}(b_{t_0}). \end{aligned}$$

By (4), we get

$$TId(\psi(t_0)) = t_0. \tag{8}$$

We will prove that a map ψ is bijective. Let $x \in [0,1]$ be $x = \varphi_{1,Id}(y)$ for every $y \in [0,1]$. We obtain $b_{\varphi_{1,Id}(y)} = \psi(\varphi_{1,Id}(y)) \in E_{\varphi_{1,Id}(y)Id}$. We get $TId = \varphi_{1,Id}$ by (3). By $TId = \varphi_{1,Id}$ and (8), we get $\varphi_{1,Id}(\psi(\varphi_{1,Id}(y))) = TId(\psi(\varphi_{1,Id}(y))) = \varphi_{1,Id}(y)$. Since $\varphi_{1,Id}$ is a homeomorphism, we get $\psi(\varphi_{1,Id}(y)) = y$. By $x = \varphi_{1,Id}(y)$, y is represented by $\psi(x) = y$. Therefore ψ is surjective.

We take $t_1,\ t_2 \in [0,1]$ and assume that $t_1 \neq t_2$. We notice $\psi(t_1) = b_{t_1} \in E_{t_1 f}$ and $\psi(t_2) = b_{t_2} \in E_{t_2 f}$ $\quad (f \in C[0,1])$. We get $TId(\psi(t_1)) = \varphi_{1,Id}(\psi(t_1))$ by (3). Since we have $TId(\psi(t_1)) = t_1$ by(8), we get $\varphi_{1,Id}(\psi(t_1)) = t_1$. In the same way, we get $\varphi_{1,Id}\psi(t_2) = t_2$. By the assumption $t_1 \neq t_2$, we get $\varphi_{1,Id}(\psi(t_1)) \neq \varphi_{1,Id}(\psi(t_2))$. We obtain $\psi(t_1) \neq \psi(t_2)$. Therefore ψ is injective.

By (4) and (7), we get $\varphi_{1,Id}(b_{t_0}) = \varphi_{f,Id}(b_{t_0})$. Since $b_{t_0} = \psi(t_0) \quad (t_0 \in [0,1])$, we have $\varphi_{1,Id}(\psi(t_0)) = \varphi_{f,Id}(\psi(t_0))$. Since ψ is a bijection, for every $t \in [0,1]$ we represent $\varphi_{1,Id}(t) = \varphi_{f,Id}(t)$. We get

$$\varphi_{1,Id} = \varphi_{f,Id}. \tag{9}$$

Let i be a constant function : $[0,1] \to i$. A map T is represented by

$$\begin{cases} Ti(\psi(t_0)) = i(t_0) = i \\ \text{or} \\ Ti(\psi(t_0)) = \overline{i(t_0)} = -i \end{cases}$$

for every $t_0 \in [0,1]$. Since ψ is bijective and $[0,1]$ is connected, T satisfies either of the cases
(a)T satisfies $Ti = i$ for every $t \in [0,1]$
or
(b)T satisfies $Ti = -i$ for every $t \in [0,1]$.

First, we consider the case (a). We get

$$\begin{aligned} TId &= T_{f,Id}(1) Id \circ \varphi_{f,Id} \\ &= T_{f,Id}(1)\varphi_{f,Id} \end{aligned}$$

for the identity map Id of $C[0,1]$. By the above equation and (3), we get $\varphi_{1,Id} = T_{f,Id}(1)\varphi_{f,Id}$. By (9), we get $T_{f,Id}(1) = 1$. Since (9) and $T_{f,Id}(1) = 1$, and we get

$$\begin{aligned} Tf &= T_{f,Id}(1) f \circ \varphi_{fId} \\ &= f \circ \varphi_{f,Id} \\ &= f \circ \varphi_{,1Id}. \end{aligned}$$

Consequently, in the case (a), T is represented by $Tf = f \circ \varphi_{1,Id}$ for every $f \in C[0,1]$. Next, we consider the case (b). Let U be a map : $C[0,1] \to C[0,1]$ such that $U = \overline{T}$. We notice U is a 2-local isometry. For the constant functions $1, i \in C[0,1]$ we have $U(1) = \overline{T(1)} = 1$ and $U(i) = \overline{T(i)} = \overline{-i} = i$. we apply the case (a) to U, we get $\overline{Tf} = Uf = f \circ \varphi_{1,Id}$. So we get $Tf = \overline{f \circ \varphi_{1,Id}}$. Therefore when $T(1) = 1$, one of the following equalities

$$\begin{cases} Tf(t) = f\varphi_{1,Id}(t) & (f \in C[0,1], t \in [0,1]) \\ Tf(t) = \overline{f\varphi_{1,Id}(t)} & (f \in C[0,1], t \in [0,1]). \end{cases}$$

By Theorem 3.1, T is a surjective real linear isometry on $C[0,1]$. $\square$

Proposition 4.2. *Let T be a 2-local isometry on $C[0,1]$. Then T satisfies* $|T(1)(t)| = 1 \quad (t \in [0,1])$.

Proof. Since T is a 2-local isoetry, for every $f \in C[0,1]$ there exists $T_{f,1} \in Iso_{\mathbb{R}}(C[0,1])$ such that $T_{f,1}(f) = T(f)$ and $T_{f,1}(1) = T(1)$. Since $T_{f,1}$ is an element of $Iso_{\mathbb{R}}(C[0,1])$, there exists $T_{f,1}(1)$ such that $|T_{f,1}(1)| = 1$. By $T_{f,1}(1) = T(1)$, there exists $T(1)$ such that $|T(1)(t)| = 1 \ (t \in [0,1])$. □

Proposition 4.3. *Let T be a 2-local isometry on $C[0,1]$. Define a map S by $S = \overline{T(1)}T$. Then S is a 2-local isometry on $C[0,1]$ such that $S(1) = 1$.*

Proof. Since T is a 2-local isometry, for every pair of elements $f, g \in C[0,1]$ there exist $T_{f,g} \in Iso_{\mathbb{R}}(C[0,1])$ such that $T_{f,g}f = Tf$ and $T_{f,g}g = Tg$. Define a map $S_{f,g}$ by $S_{f,g} = \overline{T(1)}T_{f,g}$. Since $T_{f,g}$ is a real linear isometry, we get that for every $\alpha, \beta \in \mathbb{R}$, $u, v \in C[0,1]$

$$\begin{aligned} S_{f,g}(\alpha u + \beta v) &= \overline{T(1)}T_{f,g}(\alpha u + \beta v) \\ &= \overline{T(1)}(\alpha T_{f,g}(u) + \beta T_{f,g}(v)) \\ &= \alpha\overline{T(1)}T_{f,g}(u) + \beta\overline{T(1)}T_{f,g}(v)) \\ &= \alpha S_{f,g}(u) + \beta S_{f,g}(v). \end{aligned}$$

Consequently, $S_{f,g}$ is a real linear map. We get that for every $u \in C[0,1]$

$$\begin{aligned} \|S_{f,g}(u)\|_\infty &= \|\overline{T(1)}T_{f,g}(u)\|_\infty \\ &= \|T_{f,g}(u)\|_\infty \\ &= \|u\|_\infty. \end{aligned}$$

So $S_{f,g}$ is an isometry. Since $T_{f,g}$ is a surjective real linear isometry on $C[0,1]$, $T_{f,g}$ is bijective. There exists a map $T_{f,g}^{-1}$ which is an inverse of $T_{f,g}$. Define a map v by $v = T_{f,g}^{-1}T(1)u$ for every $u \in C[0,1]$, then v is an element of $C[0,1]$. We get $S_{f,g}(v) = \overline{T(1)}T_{f,g}T_{f,g}^{-1}T(1)u = u$. We notice $S_{f,g}$ is surjentive. Therefore $S_{f,g}$ is a surjective real linear isometry on $C[0,1]$. By the assumption, $S_{f,g} = \overline{T(1)}T_{f,g}$. We have

$$\begin{aligned} S_{f,g}f &= \overline{T(1)}T_{f,g}f \\ &= \overline{T(1)}Tf \\ &= Sf. \end{aligned}$$

By the same way, we get $S_{f,g}g = Sg$. Therefore S is a 2-local isometry. For the constant function $1 \in C[0,1]$ we get $S(1) = \overline{T(1)}T(1) = 1$. $\square$

Proof of Theorem 4.1. Let S be a map $S = \overline{T(1)}T$. By Proposition 4.3, S is a 2-local isometry of $C[0,1]$ such that $S(1) = 1$. We apply Proposition 4.1 to S, S satisfies that one of the following equalities

$$\begin{cases} Sf(t) = f \circ \varphi(t) & (t \in [0,1]) \\ Sf(t) = \overline{f \circ \varphi(t)} & (t \in [0,1]), \end{cases}$$

where φ is a homeomorphism on $[0,1]$. Since $S = \overline{T(1)}T$, we get $T(1)S = T(1)\overline{T(1)}T = T$. Therefore T satisfies that one of the following equalities

$$\begin{cases} Tf(t) = T(1)f \circ \varphi(t) & (f \in C[0,1], t \in [0,1]) \\ Tf(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0,1], t \in [0,1]). \end{cases}$$

By Theorem 3.1, T is a surjective real linear isometry. Therefore $Iso_{\mathbb{R}}(C[0,1])$ is 2-local reflexive. $\square$

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