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On the 2-local property for operators on the space of some functions

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1 Introduction

Šemrl introduced the 2-locality [1] and defined 2-local automorphisms and 2-local derivations. For a given algebra A, a (not necessarily linear nor multiplicative) map $T: A \longrightarrow A$ is said to be a 2-local automorphism (resp. 2-local derivation) if for any $a, b \in A$, there exists an automorphism (resp. derivation) $T_{a,b}$ on Asuch that $T(a) = T_{a,b}(a)$ and $T(b) = T_{a,b}(b)$. Šemrl proved that every 2-local automorphism (resp. 2-local derivation) of $B(H)$, the algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H, is an automorphism (resp. derivation). After that, Molnár extended the concept of 2-locality to isometries [2] and studied 2-local isometries of $B(H)$. If X is a Banach space, a (not necessarily surjective nor linear) map $T: X \longrightarrow X$ is called a 2-local isometry if for any $x, y \in X$, there exists a surjective complex-linear isometry $T_{x,y}$ on Xsuch that $T(x) = T_{x,y}(x)$ and $T(y) = T_{x,y}(y)$. Molnár showed that every 2-local isometry is a complex-linear isometry. Motivated by this result, Győry studied 2-local isometries of the function space $C_0(X)$ [3], where $C_0(X)$ denotes the Banach algebra of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space X. Győry proved that there exists a 2-local isometry which is a non-surjective complex-linear isometry on $C_0(X)$ for some X. Győry also proved that every 2-local isometry of $C_0(X)$ is a surjective complex-linear isometry in the case of X is first countable σ-compact Hausdorff space. We refer to other result [4, 5, 6, 7, 8]. By the Mazur-Ulam theorem [9], every surjective isometry between normed spaces which preserves the origin is real-linear. So it would be interesting to consider real-linear isometries. We motivated by a Kawamura, Koshimizu and Miura's research of 2-local isometries of $C^n([0,1])$ equiped with the C-norm (or Σ-norm) [10] (notatations and the statement are in section 3) and studied 2-local isometries of $C^1([0,1])$ as the 2-locality of the group of all surjective "real-linear" isometries. We proved that every 2-local isometries (as the 2-local property for the group of surjective real-linear isometries) of $C^1([0,1])$ equiped with the C-norm (or Σ-norm) is actually a surjective real-linear isometry (Hosseini studied in the case of different type norms [11]).

2 2-local isometry

In this section, we prepare some definitions. We use the following notations for the given Banach space X.

$\mathrm{Iso}_{\mathbb{C}}(X) := \{T : X \longrightarrow X \mid T\text{:surjective complex-linear isometry}\}$

$\mathrm{Iso}_{\mathbb{R}}(X) := \{T : X \longrightarrow X \mid T\text{:surjective real-linear isometry}\}$

By these notations, we can rewrite the definition of 2-local isometry as follows.

Definition 2.1 (2-local isometry(Molnár)). *Let X be a Banach space. Then, a map $T : X \longrightarrow X$ is a 2-local isometry if the following holds.*

$$\forall x, y \in X\ \exists T_{x,y} \in Iso_{\mathbb{C}}(X)\ s.t.\ T(x) = T_{x,y}(x)\ \wedge\ T(y) = T_{x,y}(y)$$

We want to consider 2-local isometries as the 2-local property for the group of all surjective "real-linear" isometries in the main theorem. To emphasize it, we write "2-local $\mathrm{Iso}_{\mathbb{R}}(X)$" or "2-local $\mathrm{Iso}_{\mathbb{C}}(X)$" instead of "2-local isometry".

3 2-local isometries of the space of continuously differentiable functions

We need some preparations before writing the statement of a result of Kawamura, Koshimizu and Miura on 2-local isometries of $C^n([0,1])$ equiped with the C-norm (or Σ-norm). We denoted by $C^n([0,1])$ the space of all complex-valued n-times continuously differentiable functions on the closed unit interval $[0,1]$. The C-norm and Σ-norm on $C^n([0,1])$ are as follows.

$$\|f\|_C := \sup_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}|}{k!} \ (f \in C^n([0,1])), \quad \|f\|_\Sigma := \sum_{k=0}^{n} \frac{\|f^{(k)}(t)\|_\infty}{k!} \ (f \in C^n([0,1]))$$

Both norms make $C^n([0,1])$ into a Banach algebra. Kawamura, Koshimizu and Miura studied 2-local isometries of this Banach algebra [10].

Theorem 3.1 (Kawamura, Koshimizu and Miura). *Let A be a Banach algebra $(C^n([0,1]), \|\cdot\|_C)$. Then every* 2-local $\mathrm{Iso}_{\mathbb{C}}(A)$ *T is of the following form:*

$$T(f) = c[f\circ\pi]^{\varepsilon} \quad (f \in C^n([0,1])),$$

where $c \in \mathbb{T}$, $\pi \in \{id, 1-id\}$, $\varepsilon \in \{\pm 1\}$ and $[f]^{\varepsilon} := Re(f) + i\varepsilon Im(f)$.

So every 2-local $\mathrm{Iso}_{\mathbb{C}}((C^n([0,1]), \|\cdot\|_C))$ map is in $\mathrm{Iso}_{\mathbb{C}}((C^n([0,1]), \|\cdot\|_C))$. They proved that this statement is also true for the Σ-norm in the case of $n = 1$. Motivated by this result, we studied 2-local $\mathrm{Iso}_{\mathbb{R}}(C^n([0,1]))$ and proved that every 2-local $\mathrm{Iso}_{\mathbb{R}}(C^1([0,1]))$ map is in $\mathrm{Iso}_{\mathbb{R}}(C^1([0,1]))$, where the norm is the C-norm or Σ-norm.

4 Main theorem

We want to prove that every 2-local $\mathrm{Iso}_{\mathbb{R}}(A)$ for A with the C-norm (or Σ-norm) is actually a surjective real-linear isometry. Hereinafter, we denotes $C^1([0,1])$ equiped with the norm is C-norm or Σ-norm by A.

Theorem 4.1 (Main theorem). *Every* 2-local $\mathrm{Iso}_{\mathbb{R}}(A)$ *map S is of the form*

$$S(f) = c[f\circ\pi]^{\varepsilon} \quad (\forall f \in A),$$

where $c \in \mathbb{T}$, $\varepsilon \in \{\pm 1\}$ and $\pi \in \{1, 1-id\}$.

We applying two theorems in the proof of the main theorem. First one is a theorem by Kawamura, Koshimizu and Miura [12]. This theorem gave the form of maps of $\mathrm{Iso}_{\mathbb{R}}(C^1([0,1]))$.

Theorem 4.2 (Kawamura, Koshimizu and Miura). *Let B be $(C^n([0,1]), \|\cdot\|_C)$ or $(C^1([0,1]), \|\cdot\|_\Sigma)$. If $S : B \longrightarrow B$ is a surjective real-linear map, then there exists*

$c \in \mathbb{T}$, $\pi \in \{1, 1-id\}$ *and* $\varepsilon \in \{\pm 1\}$ *such that* $S(f) = c[f \circ \pi]^{\varepsilon}$ *for every* $f \in B$.

Second one is a theorem by Li, Peralta, Wang and Wang [13]. This theorem is an extension of a Kowalski-Słodkowski's theorem [14].

Theorem 4.3 (Li, Peralta, Wang and Wang)**.** *Let B be a unital Banach algebra on $\mathbb{C}$, and let $\Delta : B \longrightarrow \mathbb{C}$ be a mapping sutisfying the following properties:*

(1) Δ : 1−homogeneous (*i.e.* $\Delta(\alpha x) = \alpha \Delta(x)$)
(2) $\Delta(x) - \Delta(y) \in \mathbb{T}\sigma(x-y) \quad (\forall x, y \in B)$

Then Δ is linear, and there exists $c \in \mathbb{T}$ such that $c\Delta$ is multiplicative.

Now we prove the main theorem by applying above theorems.

Proof of Theorem 4.1. Recall that A is $(C^1([0,1]), \|\cdot\|)$, where the norm is the C-norm or Σ-norm. Let S be a 2-local $\mathrm{Iso}_{\mathbb{R}}(A)$ map. For any pair $f, g (\in A)$, there exists $T_{f,g} \in \mathrm{Iso}_{\mathbb{R}}(A)$ such that $S(f) = T_{f,g}(f)$ and $S(g) = T_{f,g}(g)$. Applying Theorem 4.2, there exists $c_{f,g} \in \mathbb{T}$, $\varepsilon_{f,g} \in \{\pm 1\}$ and $\pi_{f,g} \in \{1, 1-id\}$ such that $T_{f,g}$ is of the following form:

$$T_{f,g}(h) = c_{f,g}[h \circ \pi_{f,g}]^{\varepsilon_{f,g}} \quad (\forall h \in A) \tag{1}$$

By this formula (1), we will show that the followings:

$$S(\lambda f) = \lambda S(f) \quad (\forall f \in A\ \forall \lambda \in \mathbb{C})\ \textit{or}\ S(\lambda f) = -\lambda S(f) \quad (\forall f \in A\ \forall \lambda \in \mathbb{C}) \tag{2}$$

$$\sigma(S(f) - S(g)) \in \mathbb{T}\sigma(f-g) \quad (\forall f, g \in A) \tag{3}$$

$$\sigma(\overline{S}(f) - \overline{S}(g)) \in \mathbb{T}\sigma(f-g) \quad (\forall f, g \in A) \tag{4}$$

First, we show that (2) holds. Take any $\lambda \in \mathbb{C} \setminus \{0\}$ and fix it. By the 2-locality of S and Theorem 4.2, for every $f (\in A)$ there exists $T_{f,\lambda f} \in \mathrm{Iso}_{\mathbb{R}}(A)$ which satisfies

$$S(\lambda f) = T_{f,\lambda f}(\lambda f) = c_f[(\lambda f) \circ \pi_f]^{\varepsilon_f} = [\lambda]^{\varepsilon_f} T_{f,\lambda f}(f) = [\lambda]^{\varepsilon_f} S(f).$$

So $S(\lambda f) = \lambda S(f)$ or $S(\lambda f) = -\lambda S(f)$ holds. To show that (2), we suppose $S(\lambda 1_A) = \lambda S(1_A)$ and show that $S(\lambda f) = \lambda S(f)$ holds for every $f (\in A)$. If there exists $F \in A \setminus \{0\}$ such that $S(\lambda F) = -\lambda S(F)$, this F is not a real-constant. We consider maps

$h_s := sF + (1-s)1_A \quad (s \in [0,1])$. Because of F is not a real-constant, h_s is also not a real-constant for all $s (\neq 0)$. Now we define

$$u = \sup\{t \in [0,1] \mid S(\lambda h_s) = \lambda S(h_s) \quad (0 \leq \forall s \leq t)\}.$$

By the definition of u, there exists two sequences $\{f_n\}, \{g_n\} \in A^{\mathbb{N}}$ such that $S(\lambda f_n) = \lambda S(f_n),\ S(\lambda g_n) = -\lambda S(g_n) \quad (\forall n \in \mathbb{N})$ and $\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n = h_u$. Since S is an isometry, the following equations hold:

$$\|S(\lambda f_n) - S(\lambda g_n)\| = \|\lambda f_n - \lambda g_n\| = |\lambda|\|f_n - g_n\| \to 0 \quad (n \to \infty),$$
$$\|S(\lambda f_n) + S(\lambda g_n)\| = \|\lambda S(f_n) - \lambda S(g_n)\| = |\lambda|\|f_n - g_n\| \to 0 \quad (n \to \infty).$$

Hence we have

$$\begin{aligned}\|2S(\lambda f_n)\| &= \|S(\lambda f_n) - S(\lambda g_n) + S(\lambda f_n) + S(\lambda g_n)\| \\ &\leq \|S(\lambda f_n) - S(\lambda g_n)\| + \|S(\lambda f_n) + S(\lambda g_n)\| \to 0 \quad (n \to \infty),\end{aligned}$$

so $0 = \lim_{n\to\infty} 2S(\lambda f_n) = 2\lambda \lim_{n\to\infty} S(f_n) = 2\lambda h_u$ holds. Hence we have $h_u = 0$ and $u \neq 0$, this is a contradiction. We have proved (2). Next, we prove that (3) holds. Take any pair $f, g \in A$ and fix them. By the 2-locality of S and (1), There exists $T_{f,g} \in \mathrm{Iso}_{\mathbb{R}}(A)$ such that

$$S(f) - S(g) = T_{f,g}(f) - T_{f,g}(g) = T_{f,g}(f-g) = c_{f,g}[(f-g) \circ \pi_{f,g}]^{\varepsilon_{f,g}}$$

Hence we have

$$\begin{aligned}\sigma(S(f) - S(g)) &= \sigma(c_{f,g}[(f-g) \circ \pi_{f,g}]^{\varepsilon_{f,g}}) \\ &= c_{f,g}[\sigma(f-g)]^{\varepsilon_{f,g}} \subset \mathbb{T}\sigma(f-g).\end{aligned}$$

Now we have proved (3) holds. Similarly, one can prove that (4) holds. Next, we define a map $U : A \longrightarrow A$ by

$$U := \begin{cases} S, & if S(i1_A) = iS(1_A) \\ \overline{S}, & if S(i1_A) = -iS(1_A) \end{cases}$$

Clearly U is also an isometry. We consider the composition mapping of U and evaluation functional $\tau_t : A \longrightarrow \mathbb{C}, \tau_t(f) := f(t) \quad (t \in [0,1])$, and define $U_t (= \tau_t \circ U) : A \longrightarrow \mathbb{C}$. By (2),(3) and (4), each U_t satisfies the assumptions of Theorem 4.4. So $\overline{U_t(1_A)}U_t$ is multiplicative. We prove that $U(1_A)$ is a constant. $\sigma(U(1_A)) \subset \mathbb{T}$ by

(3) and (4), so $\|U(1_A)\|_\infty = 1$. Hence we have $1 = \|1_A\| = \|U(1_A)\|$, and by the definition of the norm of A, we have $\|U(1_A)'\|_\infty = 0$. This yields $U(1_A)$ is a constant. Now we proved that there exists $c \in \mathbb{T}$ such that each cU_t is multiplicative. Since the maximal ideal space M_A is homeomorphic to the under lying space $[0,1]$, we can define a map $\pi : [0,1] \longrightarrow [0,1]$ satisfying $cU_t = \tau_{\pi(t)}$. Now U is represented by

$$U(f) = c(f \circ \pi) \quad (f \in A).$$

Because U equals to S or $\overline{S}$, it suffices to prove that π is id or $1 - id$ to complete the proof. By $\pi = \bar{c}U(id)$, π is differentiable and $\|\pi\| = \|U(id)\| = \|id\| = 2$ holds. Because π is continuous, π must be surjective. Suppose not, we can take a point $s \in [0,1] \backslash \mathrm{Im}(\pi)$ and its neighborhood V such that $V \cap \mathrm{Im}(\pi) = \varnothing$. We can choose $f_0 \in A \setminus 0$ as $f_0 = 0$ on V, and $0 \neq \|f_0\| = \|U(f_0)\| = \|c(f_0 \circ \pi)\| = \|0\| = 0$ holds because U is an isometry. This is a contradiction. So π is surjective. We also have $\|\pi'\|_\infty = 1$. Hence, by the mean value theorem, π is contractive. By surjectivity and contractivity of π, one can prove that π is id or $1 - id$. $\square$

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