数理解析研究所講究録 第2125巻 2019年 72-81

Some problems for semiclosed subspaces

Go Hirasawa[1]
Ibaraki University

1. Introduction and preliminaries

Motivated by the paper [2] which are related with ranges of operator means, we introduce 'a path' for two given semiclosed subspaces by using Uhlmann's interpolation for a symmetric operator mean. The aim of this note is to show some properties of such a path and to pose several problems that are expected to be related to the invariant subspace problem.

Let H be an infinite dimensional, separable, complex Hilbert space with an inner product $(\cdot,\cdot) = \|\cdot\|^2$ and let $\mathcal{B}(H)$ be the set of all (linear) bounded operators on H. In particular, $\mathcal{B}_+(H)$ stands for the set of all positive (semi-definite) operators on H, and

$$\mathcal{B}_+^{-1}(H) = \{A \in \mathcal{B}_+(H) : \exists A^{-1} \in \mathcal{B}(H)\}.$$

A subspace M in H is said to be semiclosed if there exists a Hilbert norm $\|\cdot\|_M$ on M such that $(M, \|\cdot\|_M) \hookrightarrow H$ (continuously embedded Hilbert space). It is easily shown that a semiclosed subspace is equivalent to an operator range, that is, a range of a bounded operator. Clearly, a closed subspace is semiclosed.

Theorem 1.1 (Douglas majorization)**.** *Let $A, B \in \mathcal{B}(H)$. The following conditions are equivalent.*

(1) $AH \subseteq BH$
(2) $AA^* \leq kBB^*$ *for some* $k > 0$
(3) $A = BX$ *for some* $X \in \mathcal{B}(H)$

[1]This work is supported by the Grant-in-Aid for Scientific Research, No.16K13760 from JSPS.

In the above cases, X in (3) uniquely determined with $\ker X^* \supseteq \ker B$ *and for such the X,*

$$\|X\|^2 = \inf\{k : AA^* \leq kBB^*\}.$$

Using Douglas majorization theorem, a parallel sum ([1]) can be defined explicitly for a general (i.e. non-invertible) case. For $A, B \in \mathcal{B}_+(H)$, since $A^{\frac{1}{2}}H \subseteq A^{\frac{1}{2}}H + B^{\frac{1}{2}}H = (A+B)^{\frac{1}{2}}H$, there uniquely exists $X \in \mathcal{B}(H)$ such that $A^{\frac{1}{2}} = (A+B)^{\frac{1}{2}}X$ with $\ker X^* \supseteq \ker(A+B)$. Similarly, there uniquely exists $Y \in \mathcal{B}(H)$ such that $B^{\frac{1}{2}} = (A+B)^{\frac{1}{2}}Y$ with $\ker Y^* \supseteq \ker(A+B)$. Then a parallel sum $A : B$ is defined by

$$A : B = A^{\frac{1}{2}}X^*YB^{\frac{1}{2}}. \tag{1.1}$$

If $A, B \in \mathcal{B}_+^{-1}(H)$, then $A : B = (A^{-1} + B^{-1})^{-1}$.

The following range equations are well known for $\mathcal{B}_+(H)$.

$$(A^2 : B^2)^{\frac{1}{2}}H = AH \cap BH, \quad (A^2 + B^2)^{\frac{1}{2}}H = AH + BH \tag{1.2}$$

Definition 1.1. *A binary operation m from $\mathcal{B}_+(H) \times \mathcal{B}_+(H)$ to $\mathcal{B}_+(H)$*

$$m : (A, B) \mapsto AmB,$$

is said to be an operator mean if the following conditions are satisfied.

(m1) $A \leq C,\ B \leq D \Longrightarrow AmB \leq CmD$. (*monotone*)
(m2) $T^*(AmB)T \leq (T^*AT)m(T^*BT)$ *for* $T \in \mathcal{B}(H)$. (*transformer*)
(m3) $A_n \downarrow A,\ B_n \downarrow B \Longrightarrow A_nmB_n \downarrow AmB$. (*upper semi-continuous*)
(m4) $ImI = I$.

Remark 1.1.

- $X_n \downarrow X$ *means* $0 \leq X_{n+1} \leq X_n,\ X_n \to X$ (*strongly*).
- m *is symmetric* $\Longleftrightarrow AmB = BmA$ *for* $A, B \in \mathcal{B}_+(H)$.
- $k(AmB) = (kA)m(kB)$ *for* $k > 0$.

According to Kubo - Ando theory ([6]), an operator mean m is one to one corresponding to a continuous operator monotone function $f \geq 0$ on $[0, \infty)$ such that $f(1) = 1$. Such a function f is called the representing function of m. An operator mean m and its representing function f are connected by the relation $f(x)I = Im(xI), x \geq 0$. When f_1 and f_2 are representing functions of m_1 and m_2 respectively, then the order relation

$m_1 \le m_2$, that is, $Am_1B \le Am_2B$ on $\mathcal{B}_+(H)$ if and only if $f_1(x) \le f_2(x)$ for $x \in [0, \infty)$.

Typical examples of operator means are power means as follows. It is known that power means m_r are symmetric.

Example 1.1. *Let $-1 \le r \le 1, r \neq 0$. Power means m_r on $\mathcal{B}_+^{-1}(H)$ is defined by*

$$Am_rB := A^{\frac{1}{2}}\left(\frac{1}{2} + \frac{1}{2}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}}.$$

For $A, B \in \mathcal{B}_+(H)$, by the definition 1.1 ($m3$),

$$Am_rB := \lim_{n\to\infty} A_n m_r B_n.$$

If $r = 1$, then $m_1 = a$ (arithmetic mean). If $r \to 0$, then $m_0(:= \lim_{r\to 0} m_r) = g$ (geometric mean). If $r = -1$, then $m_{-1} = h$ (harmonic mean). We give here the form of above three operator means for following arguments. The arithmetic mean $AaB = \frac{A+B}{2}$ on $\mathcal{B}_+(H)$. The geometric mean $AgB = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ on $\mathcal{B}_+^{-1}(H)$. Although AgB can be defined for $A \ge 0$ and $B \ge 0$ by the definition 1.1 ($m3$), we do not know the explicit form of AgB on $\mathcal{B}_+(H)$. The harmonic mean $AhB = 2(A : B)$ on $\mathcal{B}_+(H)$.

Among any symmetric mean m, it is well known that

$$h \le m \le a.$$

That is,

$$XhY \le XmY \le XaY \quad \text{for } X, Y \in \mathcal{B}_+(H). \tag{1.3}$$

Put $X = A^2$ and $Y = B^2$ in (1.3) for $A, B \in \mathcal{B}_+(H)$. Then, by Douglas majorization theorem, we have that

$$(A^2hB^2)^{\frac{1}{2}}H \subseteq (A^2mB^2)^{\frac{1}{2}}H \subseteq (A^2aB^2)^{\frac{1}{2}}H,$$

equivalently by (1.2)

$$AH \cap BH \subseteq (A^2mB^2)^{\frac{1}{2}}H \subseteq AH + BH.$$

The previous relation holds for any symmetric operator means m. However, surprisingly, the next theorem says that the expression holds for any operator means.

Theorem 1.2 ([2]). *For any (not necessarily symmetric) mean m,*

$$AH \cap BH \subseteq (A^2 m B^2)^{\frac{1}{2}} H \subseteq AH + BH.$$

2. Uhlmann's interpolation m_t $(0 \leq t \leq 1)$

Firstly, we give the definition of Uhlmann's interpolation for a symmetric operator mean.

Definition 2.1. ([5]) *A parametrized operator mean m_t $(0 \leq t \leq 1)$ on $\mathcal{B}_+(H)$ is said to be Uhlmann's interpolation for a symmetric operator mean m if the following conditions are satisfied.*

$(\mathrm{U1})_+$: *$Am_0B = A$, $Am_{\frac{1}{2}}B = AmB$ and $Am_1B = B$ on $\mathcal{B}_+(H)$.*
$(\mathrm{U2})_+$: *$(Am_pB)m(Am_qB) = Am_{\frac{p+q}{2}}B$ on $\mathcal{B}_+(H)$.*
$(\mathrm{U3})_+^{-1}$: *The mapping $t \mapsto Am_tB$ is norm continuous for each A, B. That is, for t $(0 \leq t \leq 1)$,*

$$\lim_{s \to t} \|Am_tB - Am_sB\| = 0 \text{ for each } A, B \in \underline{\mathcal{B}_+^{-1}(H)}.$$

The next theorem asserts that power means have the Uhlmann's interpolation.

Theorem 2.1. ([5]) *Let m_r $(-1 \leq r \leq 1)$ be power means on $\mathcal{B}_+(H)$. For each r, Uhlmann's interpolation $m_{r,t}$ $(0 \leq t \leq 1)$ exists :*

$$(2.1) \quad Am_{r,t}B := A^{\frac{1}{2}}\left(1 - t + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}} \quad \text{for } A, B \in \mathcal{B}_+^{-1}(H).$$

We do not know that the explicit form of $Am_{r,t}B$ for $A, B \in \mathcal{B}_+(H)$. If $r = 1$ in (2.1), then $Am_{1,t}B = Aa_tB = (1-t)A + tB$ on $\mathcal{B}_+(H)$. If $r \to 0$, then $Am_{0,t}B = Ag_tB = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ on $\mathcal{B}_+^{-1}(H)$. If $r = -1$, then $Am_{-1,t}B = Ah_tB = ((1-t)A^{-1} + tB^{-1})^{-1}$ on $\mathcal{B}_+^{-1}(H)$. Note that explicit representation of harmonic mean h on $\mathcal{B}_+(H)$ is obtained. In fact, $AhB = 2(A : B) = 2(A^{\frac{1}{2}}X^*YB^{\frac{1}{2}})$ in (1.1). For this reason, I guess that the explicit form of $h_t (= m_{-1,t})$ exists on $\mathcal{B}_+(H)$.

Theorem 2.2. ([5]) *For each t $(0 \leq t \leq 1)$, if $-1 \leq r_1 \leq r_2 \leq 1$ implies*

$$m_{r_1,t} \leq m_{r_2,t}.$$

In particular, $h_t \leq g_t \leq a_t$.

3. A path M_t between semiclosed subspaces

Motivated by a result of Theorem 1.2, we introduce a path between given two semiclosed subspaces.

Definition 3.1. *Let m_t $(0 \le t \le 1)$ on $\mathcal{B}_+(H)$ be Uhlmann's interpolation for a symmetric operator mean m. For semiclosed subspaces M_0 and M_1 in H, we define the path (with respect to m_t) between them by*

$$M_0 m_t M_1 := (A_0^2 m_t A_1^2)^{\frac{1}{2}} H,$$

where $M_0 = A_0 H$ and $M_1 = A_1 H$ such that $A_0, A_1 \in \mathcal{B}_+(H)$.

Is the definition 3.1 well defined ? Is the path determined not depending on positive operators appearing in the range representation ? For above question, we reply yes, it is well defined. Let $M_0 = A_0 H = B_0 H$ and $M_1 = A_1 H = B_1 H$, where $A_i, B_i \in \mathcal{B}_+(H)(i = 0, 1)$. Then we want to show that

$$(A_0^2 m_t A_1^2)^{\frac{1}{2}} H = (B_0^2 m_t B_1^2)^{\frac{1}{2}} H.$$

Because, there exists invertible $X_0, X_1 \in \mathcal{B}^{-1}(H)$ such that

$$A_0 = B_0 X_0,\ \ A_1 = B_1 X_1.$$

$$\begin{aligned} A_0^2 m_t A_1^2 &= (B_0 X_0 X_0^* B_0) m_t (B_1 X_1 X_1^* B_1) \\ &\le (\|X_0\|^2 B_0^2) m_t (\|X_1\|^2 B_1^2) \\ &\le \max\{\|X_0\|^2, \|X_1\|^2\} (B_0^2 m_t B_1^2) \end{aligned}$$

This means that $(A_0^2 m_t A_1^2)^{\frac{1}{2}} H \subseteq (B_0^2 m_t B_1^2)^{\frac{1}{2}} H$ by Douglas majorization theorem. Converse inclusion follows from the invertibilty of X_0 and X_1.

Remark 3.1. *From* (U1)$_+$ *in the definition of Uhlmann's interpolation, we see that*

- $t = 0 \Longrightarrow M_0 m_0 M_1 = (A_0^2 m_0 A_1^2)^{\frac{1}{2}} H = (A_0^2)^{\frac{1}{2}} H = A_0 H = M_0.$
- $t = 1 \Longrightarrow M_0 m_1 M_1 = (A_0^2 m_1 A_1^2)^{\frac{1}{2}} H = (A_1^2)^{\frac{1}{2}} H = A_1 H = M_1.$

Therefore, it is reasonable to put

$$M_t := M_0 m_t M_1. \quad (0 \le t \le 1) \tag{3.1}$$

Using the notaion (3.1), the relation

$$A_0H \cap A_1H \subseteq (A_0^2 m_t A_1^2)^{\frac{1}{2}} H \subseteq A_0H + A_1H$$

is simply represented by

$$M_0 \cap M_1 \subseteq M_t \subseteq M_0 + M_1.$$

If $M_0 \subseteq M_1$, then we see that $M_0 \subseteq M_t \subseteq M_1$.

The following examples are known facts.

Example 3.1. *Let M_0 and M_1 be semiclosed subspaces. For a_t $(0 < t < 1)$,*

$$\begin{aligned} M_t = M_0 a_t M_1 &= (A_0^2 a_t A_1^2)^{\frac{1}{2}} H \\ &= ((1-t)A_0^2 + tA_1^2)^{\frac{1}{2}} H = A_0H + A_1H \\ &= M_0 + M_1. \end{aligned}$$

Example 3.2. *Let M_0 and M_1 be closed subspaces. For g_t and h_t $(0 < t < 1)$,*

$$M_t = M_0 g_t M_1 = M_0 h_t M_1 = M_0 \cap M_1$$

Example 3.3. *Let M_0 and M_1 be semiclosed subspaces. For $h_{\frac{1}{2}} = h$,*

$$\begin{aligned} M_{\frac{1}{2}} = M_0 h_{\frac{1}{2}} M_1 &= (A_0^2 h A_1^2)^{\frac{1}{2}} H \\ &= (2(A_0^2 : A_1^2))^{\frac{1}{2}} H = A_0H \cap A_1H \\ &= M_0 \cap M_1. \end{aligned}$$

In example 3.3, we do not know a form of the path $M_t = M_0 h_t M_1$ for $0 < t < 1$.

4. M^p $(0 \leq p \leq 1)$ FOR A SEMICLOSED SUBSPACE M

We introduce a concept of p-power of a semiclosed subspace.

Definition 4.1.
For semiclosed subspace M, we define M^p by

$$M^p := A^p H, \ (0 \leq p \leq 1)$$

where $A^0 := I$ and $M = AH$ with $A \in \mathcal{B}_+(H)$. Note that $M^0 = H$.

Is the definition 4.1 well defined ? We reply yes, it is well defined. It is sufficient to show a case $0 < p < 1$. Let $M = AH = BH$ $(A, B \in \mathcal{B}_+(H))$. Then, by Douglas majorization theorem, the inequality

$$\frac{1}{k}B^2 \leq A^2 \leq kB^2$$

holds for some $k > 0$. Hence, by Löwner-Heinze inequality, we have

$$\frac{1}{k^p}B^{2p} \leq A^{2p} \leq k^p B^{2p} \quad (0 < p < 1)$$

that means $A^p H = B^p H$.

Remark 4.1. *M is closed if and only if $M = M^{\frac{1}{2}}$.*

We give the form of the path $M_t = M_0 g_t H$ between M_0 and H.

Example 4.1. *Let $M_0(= A_0 H)$ and $H(= IH)$ such that $M_0 \neq H$. Then*

$$\begin{aligned} M_t &= M_0 g_t H = (A_0^2 g_t I)^{\frac{1}{2}} H \\ &= (I g_{1-t} A_0^2)^{\frac{1}{2}} H = \left((A_0^2)^{1-t}\right)^{\frac{1}{2}} H \\ &= A_0^{1-t} H = M_0^{1-t}. \quad (0 \leq t \leq 1) \end{aligned}$$

In example 4.1, we see that $M_t(= M_0^{1-t})$ is increasing if M_0 is not closed, that is,

$$M_t \subsetneq M_s. \quad (0 \leq t < s \leq 1)$$

If M_0 is closed, then $M_t = M_0$ for $0 \leq t < 1$ and $M_1 = H$.

5. T-invariant property for a path M_t

Let $T \in \mathcal{B}(H)$. If two semiclosed subspaces are T-invariant, then each point on a path between them is also T-invariant.

Proposition 5.1. *Put $T \in \mathcal{B}(H)$. Let M_0 and M_1 be nontrivial T-invariant semiclosed subspaces in H. If m_t $(0 \leq t \leq 1)$ is Uhlmann's interpolation of a symmetric operator mean m, then a path M_t $(:= M_0 m_t M_1)$ is T-invariant for each t.*

(Proof) let $M_0 = A_0 H$ and $M_1 = A_1 H$ for $A_0, A_1 \in \mathcal{B}_+(H)$. Suppose that

$$T(A_0 H) \subseteq A_0 H, \quad T(A_1 H) \subseteq A_1 H.$$

Then, $\exists X_0$ and $\exists X_1$ in $\mathcal{B}(H)$ s.t. $TA_0 = A_0X_0$ and $TA_1 = A_1X_1$.

$$\begin{aligned} T(A_0^2 m_t A_1^2)T^* &\leq (TA_0^2T^*)m_t(TA_1^2T^*) \\ &= (A_0X_0X_0^*A_0)m_t(A_1X_1X_1^*A_1) \\ &\leq (\|X_0\|^2A_0^2)m_t(\|X_1\|^2A_1^2) \\ &\leq \max(\|X_0\|^2, \|X_1\|^2)(A_0^2m_tA_1^2) \end{aligned}$$

By Douglas's majorization theorem,

$$T(A_0^2m_tA_1^2)^{\frac{1}{2}}H \subseteq (A_0^2m_tA_1^2)^{\frac{1}{2}}H.$$

This completes the proof.

According to [7], there exists many T-invariant semiclosed subspaces. Choose non-trivial T-invariant semiclosed subspaces M_0 and M_1 $(\neq \{0\}, H)$ such that $M_0 \subsetneq M_1$. If the interval of semiclosed subspaces

$$[M_0, M_1] := \{M : M_0 \subseteq M \subseteq M_1\} \tag{5.1}$$

contains a closed subspace, then does there exists Uhlmann's interpolation m_t such that a path $M_t(= M_0m_tM_1)$ pass through the closed subspace? In particular, does the path $M_t(= M_0g_tM_1)$ run through the closed subspace? If a path M_t is closed for some t' and $M_0 \subsetneq M_{t'} \subsetneq M_1$, then $M_{t'}$ is a nontrivial T-invariant closed subspace by Proposition 5.1.

6. Some Problems

Let $\mathcal{S}$ be the set of all semiclosed subspaces in H. For $M \in \mathcal{S}$, it is known that there exists a bijective mapping $\|\cdot\|_M \to A$ from the set of Hilbert norms $\{\|\cdot\|_M : (M, \|\cdot\|_M) \hookrightarrow H\}$ to the set of positive bounded operators $\{A \geq 0 : M = AH\}$. When M is closed, the norm $\|\cdot\|$ restricted to M is corresponding to the orthogonal projection P_M onto M.

For each semiclosed subspace M, we choose a Hilbert norm $\|\cdot\|_M$ from the set of all Hilbert norms on M, and let α be its correspondence $M \to \|\cdot\|_M$, equivalently, $M \to A \geq 0$ from the above arguments. A correspondence α is a choice function to choose a positive bounded operator A from each semiclosed subspace M such that $M = AH$. We denote it $M \overset{\alpha}{=} AH$. Here we promise a rule to choose the orthogonal projection from a closed subspace. Then we define ([3]) a metric ρ_α on $\mathcal{S}$ by

$$\rho_\alpha(M,N) := \|A-B\| \quad \text{for } M \overset{\alpha}{=} AH \text{ and } N \overset{\alpha}{=} BH.$$

Since $H^\sigma(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ for $\sigma > 0$ and $d \geq 1$, Sobolev space $H^\sigma(\mathbb{R}^d)$ is a semiclosed subspace in $L^2(\mathbb{R}^d)$. Let α be the choice function that we choose the Sobolev norm $\|\cdot\|_{H^\sigma}$ from each semiclosed subspace

$$\{f \in L^2(\mathbb{R}^N) : (1+|\xi|^2)^{\frac{\sigma}{2}} \widehat{f} \in L^2(\mathbb{R}^N)\}, \quad (\sigma > 0) \tag{6.1}$$

and we suitably choose a Hilbert norm from each semiclosed subspace except for semiclosed subspaces (6.1) ($\widehat{f}$ is Fourier transform of f). Then the distance between Sobolev spaces is given as the following result.

Example 6.1 ([3]). *Let $H^{\sigma_1}(\mathbb{R}^d)$ and $H^{\sigma_2}(\mathbb{R}^d)$ be Sobolev spaces in $L^2(\mathbb{R}^d)$. For $0 < \sigma_1 < \sigma_2$,*

(1) $\rho_\alpha(H^1(\mathbb{R}^d), H^2(\mathbb{R}^d)) = 0.25$
(2) $\rho_\alpha(H^{\sigma_1}(\mathbb{R}^d), H^{\sigma_2}(\mathbb{R}^d)) = (\frac{\sigma_1}{\sigma_2})^{\frac{\sigma_1}{\sigma_2-\sigma_1}} - (\frac{\sigma_1}{\sigma_2})^{\frac{\sigma_2}{\sigma_2-\sigma_1}}$

Now we focus on the path induced from the geometric interpolation g_t $(0 \leq t \leq 1)$. As stated in previous section, we are interested in an interval case (5.1), $[M_0,\ M_1] = \{M \in \mathcal{S} : M_0 \subseteq M \subseteq M_1\}$. Concerning an interval as like this, we ask some problems.

Problem 6.1. *For non-trivial semiclosed subspaces $M_0 \subsetneq M_1$,*

$$0 \leq s < t \leq 1 \quad \overset{?}{\Longrightarrow} \quad M_s \subseteq M_t.$$

Problem 6.2. *For non-trivial semiclosed subspaces $M_0 \subsetneq M_1$, does there exist a choice function α such that the path $M_t : [0,1] \to (\mathcal{S}, \rho_\alpha)$ is continuous ?*

Problem 6.3. *For non-trivial semiclosed subspaces $M_0 \subsetneq M_1$, pick $M_{t'}$ $(0 < t' < 1)$ on the path between M_0 and M_1. Then, is the path connecting M_0 and $M_{t'}$ a part of the first path ?*

Problem 6.4. $M_s \subsetneq M_t \quad \overset{?}{\Longrightarrow} \quad \dim M_t/M_s = \infty.$

To study the invariant subspace problem, we are considering the application of method of diminishing intervals of semiclosed subspaces as described in [4]. For that purpose, the above problems are necessary.

Ibaraki University
Colleage of Engineering
Nakanarusawa 4-12-1
Hitachi 316-8511
Japan
Email address : gou.hirasawa.529@vc.ibaraki.ac.jp
茨城大学工学部　平澤　剛

References

[1] P.A.Fillmore and J.P.Williams, *On Operator Range*, Adv. in Math.7 (1971) 254-281.
[2] J.Fujii, *Operator Means and Range Inclusion*, Linear Algebra and its Applications 170 (1992) 137-146.
[3] G.Hirasawa, *A Metric for Unbounded Linear Operators in a Hilbert Space*, Integr. Equ. Oper. Theory vol.70 (2011), 363-378.
[4] ———, *A method of diminishing intervals and semiclosed subspaces*, RIMS Kôkyûroku No.1996 (2016), 113-120.
[5] E.Kamei, *Paths of Operators Parametrized by Operator Means*, Math.Japonica 39 (1994) 395-400.
[6] F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann. 246 (1980) 205-224.
[7] E.Nordgren, M.Radjabalpour, H.Radjavi and P.Rosenthal, *On Invariant Operator Ranges*, Trans. Amer. Math. Soc. vol.251 (1979), 389-398.