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Optimal Selection of the Penultimate Candidate

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Abstract

A known number n of objects appear one at a time. Let $X_k, 1 \leq k \leq n$, denote the value of the kth object and suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed continuous random variables with a known distribution function. Let $L_k = \max(X_1, \ldots, X_k)$, and call the kth object a *candidate* if it is a relative maximum, i.e. $X_k = L_k$. We denote by C_j the jth to last candidate, $j \geq 1$. Hence C_1 is the last candidate and C_2 the penultimate candidate, etc. The problem we consider here seeks a stopping rule that maximizes the probability of choosing C_2. We give the optimal rule and the corresponding success probability. It can be shown that this success probability tends to 0.416002 as $n \to \infty$. Some comparisons with other related problems are also made.

1 Introduction

We first review the 'full-information' best-choice problem originally studied by Gilbert and Mosteller (1966, Sec.3) as a variation of the secretary problem. A known number n of objects appear one at a time. Let $X_k, 1 \leq k \leq n$, denote the value of the kth object and suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed continuous random variables with a known distribution function F. As each object appears, we observe its value and decide either to select or reject it based on the values observed so far. The objective is to find a stopping rule that maximizes the probability of choosing the best, i.e. stopping with the largest of $X_1, X_2, \ldots, X_n$, and compute the probability of choosing the best. We can assume without loss of generality that $X_1, X_2, \ldots, X_n$ are uniformly distributed on the interval (0, 1), because order relationship is preserved under transformation $F(X_k)$.

Let $L_k = \max(X_1, \ldots, X_k), 1 \leq k \leq n$, and call the kth object (or X_k) a *candidate* if it is a relative maximum, that is, $X_k = L_k$. A candidate is sometimes referred to as a record in the literature. Denote by C_j the jth to last candidate, $j \geq 1$. Hence C_1 is the last candidate (i.e. best overall) and C_2 the penultimate candidate, etc. Then we are tempted to consider the problem of choosing C_j if it exists. Since the problem of choosing C_1 is just the best-choice problem, we here try to solve the problem of choosing C_2 as a first step toward these problems. Needless to say, this problem can be viewed as a two-choice problem of choosing both C_1 and C_2, because the identification of C_2 also identifies C_1 as the candidate next to C_2 with certainty. The main results will be summarized in Section 2. In the 'no-information' analogue, where we can only observe the relative rank of the current object with respect to its predecessors, Bruss and Paindaveine (2000) solved the problem of choosing C_j for all $j(\geq 1)$. In Section 3, we compare our problem with other related problems.

2 Main results

The objective of the problem we consider here is to find a stopping rule that maximizes the probability of choosing C_2, if any, and derive the corresponding success probability (if C_1 appears at the first stage, we do not have C_2, so, in such a case, our trial is unsuccessful). Our problem makes sense for $n \geq 2$ and the decision of selection takes place only when a candidate appears.

For $n = 2$, the optimal rule obviously stops with X_1 and yields the success probability $P_2^* = P\{X_1 < X_2\} = 1/2$. For $n = 3$, the optimal rule also stops with X_1 and yields the success probability $P_3^* = 1/2$. This can be seen as follows. Suppose that we stop with X_1. We then have three cases with respect to the size of X_1 relative to X_2 and X_3. Since the respective success probabilities are 1/2, 1, 0, depending on whether X_1 is the smallest, middle or largest, and these three cases are equally likely, the success probability is $(1/2)(1/3) + (1)(1/3) + (0)(1/3) = 1/2$. On the other hand, if we pass over X_1, we succeed only when $X_1 < X_2 < X_3$, implying that the success probability is 1/6. Thus the maximal success probability is $P_3^* = \max\{1/2,\ 1/6\} = 1/2$.

Before considering our problem for $n \geq 4$, we need some preparations. We review the distribution of the number of relative maxima. Let $r_m(k)$ be the probability that the total number of relative maxima is k when we observe m independent and identically distributed continuous random variables (or, equivalently, we observe the random permutation of m rankable objects). Then it is well known (see, e.g. Section 6.2 of Blom et al.(1994)) that $r_m(k)$ satisfies the recursion

$$r_m(k) = \frac{1}{m} r_{m-1}(k-1) + \left(1 - \frac{1}{m}\right) r_{m-1}(k), \quad 1 \leq k \leq m,\ 2 \leq m,$$

with $r_1(1) = 1$ and $r_m(k) = 0$ for $k = 0$ or $k > m$. In particular, we have by induction

$$r_m(1) = \frac{1}{m}, \quad r_m(2) = \frac{h_{m-1}}{m} \tag{1}$$

where $h_j = \sum_{i=1}^{j} 1/i$ for $j \geq 1$.

We denote by (k, x) a state of the process where we have just observed the kth object to be a candidate having value x, i.e. $X_k = L_k = x$. Let $p_k(x)$ be the success probability by stopping immediately with the current candidate in state (k, x) and $q_k(x)$ be the one by stopping with the next candidate (after leaving (k, x)), if any. These two quantities are expressed as

$$p_k(x) = P\{M_k(x) = 1\} \tag{2}$$

$$q_k(x) = P\{M_k(x) = 2\}, \tag{3}$$

if we denote by $M_k(x)$ the number of future candidates. For further computation, we proceed by conditioning on $N_k(x)$ which denotes the number of future observations that are bigger than x. Since $N_k(x)$ is a binomial random variable with parameters $(n-k, 1-x)$ and the conditional probability $P\{M_k(x) = 1 \mid N_k(x) = m\}$ is given as $r_m(1)$ from the former half of (1) regardless of x, we have from (2)

$$\begin{aligned} p_k(x) &= \sum_{m=1}^{n-k} P\{M_k(x) = 1 \mid N_k(x) = m\} P\{N_k(x) = m\} \\ &= \sum_{m=1}^{n-k} \frac{1}{m} \binom{n-k}{m} (1-x)^m x^{n-k-m} \end{aligned}$$

for $k \leq n-1$ with $p_n(x) = 0$. In a similar manner, we have from (3) combined with the latter half of (1)

$$q_k(x) = \sum_{m=2}^{n-k} \frac{h_{m-1}}{m} \binom{n-k}{m} (1-x)^m x^{n-k-m}$$

for $k \leq n-2$ with $q_{n-1}(x) = q_n(x) = 0$.

We are now ready to state the main results for $n \geq 4$.

Theorem 1. *(a) Optimal stopping rule: Let, for $0 < x < 1$,*

$$\phi_r(x) = \sum_{m=2}^{r} \frac{h_m - 1}{(m+1)^2} \binom{r}{m} \left(\frac{1-x}{x}\right)^m, \quad r \geq 2.$$

Then there exists an increasing sequence of the thresholds $\{b_r, r \geq 2\}$ with b_r defined as a unique solution $x \in (0, 1)$ to the equation

$$\phi_r(x) = 1 \tag{4}$$

such that the optimal rule is to choose the first candidate $X_k (= L_k)$ that exceeds the threshold $b_{(n-1)-k}$ (note that $(n-1)-k$ denotes the remaining *number of observations to the penultimate stage $n-1$).*

(b) Maximal success probability P_n^*: *We have that*

$$P_n^* = \int_{a_1}^{1} p_1(x)dx + \sum_{k=2}^{n-1}\sum_{j=1}^{k-1}\int_{a_k}^{1} p_k(x)\frac{[\min(x,a_j)]^{k-1}}{k-1}dx, \tag{5}$$

where $a_k = b_{(n-1)-k}$ for $1 \le k \le n-3$ and $a_k = 0$, otherwise.

(c) Asymptotics: *Define $c^*(\approx 3.27201)$ as a unique solution c to the equation*

$$\sum_{m=2}^{\infty} \frac{h_m - 1}{(m+1)^2}\cdot\frac{c^m}{m!} = 1. \tag{6}$$

Then, using the exponential-integral functions

$$\begin{aligned} I(c) &= \int_1^{\infty} \frac{e^{-cx}}{x}dx \\ J(c) &= \int_0^1 \frac{e^{cx}-1}{x}dx = \sum_{j=1}^{\infty}\frac{c^j}{j!j}, \end{aligned}$$

we have that, as $n \to \infty$,

$$\begin{aligned} P_n^* \to P^* &= e^{-c^*}J(c^*) - \left\{(1+c^*)J(c^*) + e^{c^*}J(-c^*)\right\} I(c^*) \\ &\approx 0.416002. \end{aligned}$$

Proof. Omitted.

We conclude this section with some comments concerning the approximation for b_r and the simplification of P_n^*. It is easy to see from (4) that $b_2 = (3\sqrt{2}-1)/17 \approx 0.1907$ and $b_3 \approx 0.3409$ is a unique root $x(>0)$ of the equation $85x^3 + 17x^2 - x - 5 = 0$. For large r, we can give an asymptotic approximation for the b_r. Write b_r as $1 - c_r/r$ and write (4) in the form

$$\sum_{m=2}^{r} \frac{h_m - 1}{(m+1)^2 m!}\prod_{k=0}^{m-1}\left[\left(1-\frac{k}{r}\right)\left(\frac{c_r}{1-c_r/r}\right)\right] = 1.$$

In order for this to stay equal to 1 as $r \to \infty$, the c_r must converge to a constant $c_r \to c^*$, where c^* satisfies the equation (6). Thus $b_r \approx 1 - 3.272/r$.

Performing integrations in (5) for $n = 4$, we have

$$P_4^* = \frac{11}{24} + \frac{1}{6}a_1 - \frac{1}{4}a_1^2 - \frac{5}{6}a_1^3 + \frac{17}{24}a_1^4 \approx 0.47618$$

because of $a_1 = b_2$. In a similar manner, we obtain another expression of P_n^* for general n, as given in Corollary 1 below. Consider a binomial random variable with parameters (r, p) and denote its cumulative distribution function by

$$B_{in}(m; r, p) = \sum_{k=0}^{m} \binom{r}{k} p^k (1-p)^{r-k}, \quad 0 \le m \le r.$$

Then, applying to (5) the well-known formula

$$\int_0^a x^{s-1}(1-x)^{t-1} dx = B(s,t) B_{in}(t-1; s+t-1, 1-a),$$

where $0 \le a \le 1$ and $B(s,t)$ is the *beta function* defined as

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1} dx = \frac{(s-1)!(t-1)!}{(s+t-1)!}$$

for positive integers s and t, we obtain the following result.

Corollary 1. *We have*

$$P_n^* = A_n + \sum_{k=2}^{n-1} \sum_{j=1}^{k-1} \left[B_n(j,k) + C_n(j,k) \right],$$

where

$$\begin{aligned}
A_n &= \sum_{m=1}^{n-1} \frac{1}{m} \binom{n-1}{m} B(m+1, n-m) \left[1 - B_{in}(m; n, 1-a_1) \right] \\
B_n(j,k) &= \frac{1}{k-1} \sum_{m=1}^{n-k} \frac{1}{m} \binom{n-k}{m} B(m+1, n-m) \\
&\quad \times \left[B_{in}(m; n, 1-a_j) - B_{in}(m; n, 1-a_k) \right] \\
C_n(j,k) &= \frac{a_j^{k-1}}{k-1} \sum_{m=1}^{n-k} \frac{1}{m} \binom{n-k}{m} B(m+1, n-m-k+1) \\
&\quad \times \left[1 - B_{in}(m; n-k+1, 1-a_j) \right].
\end{aligned}$$

3 Some comparisons

(a). In the no-information problem of choosing C_2, Bruss and Paindaveine (2000) showed that there exists a threshold r_n such that the optimal rule stops with the first candidate that appears after time r_n, if any. Moreover, as $n \to \infty$, r_n/n tends to e^{-2} and the corresponding success probability tends to $2e^{-2} \approx 0.2707$ (compare this with our value $P^* \approx 0.4160$). It is also interesting to observe that the same limiting values appear in two

quite different optimal stopping problems; one is the best-choice problem with an unknown random number of objects having a uniform distribution on $\{1, \ldots, n\}$ considered by Presman and Sonin (1972) and the other the *duration problem* considered in Section 2.2 of Ferguson et al. (1992), where the objective is to maximize the time of possession of a candidate.
(b). In (a), we pointed out the coincidences among three problems in the no-information case. How about the corresponding triple coincidences in the full-information case? The answer is "No". Samuels (2004) showed that the best-choice problem with uniform number of objects and the duration problem have the same asymptotic optimal payoff

$$\hat{P} = e^{-\hat{c}} J(\hat{c}) + \left\{ e^{\hat{c}} - 1 - \hat{c} J(\hat{c}) \right\} I(\hat{c}) \approx 0.43517,$$

where $\hat{c} \approx 2.1198$ is a unique solution c to the equation

$$-J(-c)e^{c} - J(c) = e^{c} - 1.$$

Our payoff P^* is obviously different from $\hat{P}$. See also Porosinski (1987, 2002), Petruccelli (1980), Gnedin (2004) and Mazalov and Tamaki (2006) for $\hat{P}$.
(c). Finally we compare our problem with two related problems. One is the best-choice problem. Its asymptotic optimal payoff is

$$P_1 = e^{-c_1} + (e^{c_1} - c_1 - 1) I(c_1) \approx 0.58016,$$

where $c_1 \approx 0.80435$ is a unique solution c to the equation $J(c) = 1$. For these, see Gilbert and Mosteller (1966), Samuels (1982), (1991), (2004), Gnedin (2004) and Berezovsky and Gnedin (1984). The other is a one-choice problem of choosing either C_1 or C_2, considered by Tamaki (2010) (see Theorem 4.1 and Table 2 for $m = 2$). The asymptotic optimal payoff is given by

$$\begin{aligned} P_2 &= \left[e^{c_2} \{1 - J(-c_2)\} - (1 + c_2) \{1 + J(c_2)\} \right] I(c_2) \\ &\quad + e^{-c_2} \{1 + J(c_2)\} \\ &\approx 0.8424, \end{aligned}$$

where $c_2 \approx 1.5151$ is a unique solution c to the equation

$$\int_0^c \frac{-J(-x)e^{x} - J(x)}{x} dx = 1.$$

Note that P_2 here is denoted by P_2^* in Tamaki (2010) and that (7) follows since the functions $J_2(t)$ and $K_2(t)$ in Tamaki (2010) can be expressed as

$$\begin{aligned} J_2(t) &= 1 + J(t) \\ K_2(t) &= e^{t} - 1 + \left\{ -J(-t)e^{t} - J(t) \right\} \end{aligned}$$

in terms of $J(\cdot)$. To derive (8) and (9), we have used Gnedin's identity (2004, p.322). Incidentally the asymptotic optimal payoff of the corresponding no-information problem is $(1+\sqrt{2})e^{-\sqrt{2}} \approx 0.5869$ (see Tamaki (2010)).
(d). Let T_j be the random arrival time of C_j and Y_j be its random value on the PPP defined in the proof (c) of Theorem 1. Then random atom of C_j is identified as (T_j, Y_j), $j \geq 1$. From the property of PPP, we easily find

$$\begin{aligned} T_j &= U_1 U_2 \cdots U_j \\ Y_j &= E_1 + \frac{E_2}{U_1} + \cdots + \frac{E_j}{U_1 U_2 \cdots U_{j-1}}, \end{aligned}$$

where $U_1, U_2, \ldots, U_j$ and $E_1, E_2, \ldots, E_j$ are all independent and U_k are uniform on $(0,1)$ and E_k are exponential with parameter 1. For the best-choice problem, Samuels (2004) showed that, in his Section 10.3, P_1 has another expression

$$P_1 = P\left\{ Y_1 < \frac{c_1}{1-T_1} \text{ and } Y_2 > \frac{c_1}{1-T_2} \right\},$$

because we succeed if C_1 is below the optimal threshold $y = c_1/(1-t)$ and C_2 is above. Similarly we have, as another expression for P^*,

$$P^* = P\left\{ Y_2 < \frac{c^*}{1-T_2} \text{ and } Y_3 > \frac{c^*}{1-T_3} \right\},$$

because we succeed if C_2 is below the optimal threshold $y = c^*/(1-t)$ and C_3 is above. This is equivalently written as

$$P^* = P\left\{ \left(E_1 + \frac{E_2}{U_1}\right)(1 - U_1 U_2) < c^* < \left(E_1 + \frac{E_2}{U_1} + \frac{E_3}{U_1 U_2}\right)(1 - U_1 U_2 U_3) \right\}.$$

See also Gnedin (2004) for EU-representation.

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