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Positive flow-spines and contact structures — a short summary

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1 Introduction

In this report, we discuss a relationship between positive flow-spines and contact structures of 3-manifolds.

Let M be a closed oriented 3-manifold. A *contact structure* on M is a totally non-integrable plane field in TM. There is a well-known relationship between open book decompositions of M and contact structures on M, called the *Giroux correspondence* [3]. On the other hand, a *flow-spine*, defined by the first author [6], of M is a special kind of spine, which defines a non-singular flow on M in such a way that the flow is transverse to the spine, and the flow in the complement of the spine is diffeomorphic to a constant flow in an open ball. We say that a contact structure is *supported* by a flow-spine if the flow of a Reeb vector field for the contact structure is defined by the flow-spine. The following is the main theorem of this report.

Theorem. (1) *Every positive flow-spine of a* 3*-manifold supports a unique contact structure up to isotopy; and*

(2) *Every contact structure on a* 3*-manifold is supported by a positive flow-spine.*

This report is adapted from the talk at 2019 Intelligence of Low-dimensional Topology held in Research Institute for Mathematical Sciences, Kyoto University. The details will be given in the forthcoming paper [7].

2 Preliminaries

Let M be an oriented, smooth 3-manifold. A *positive contact structure* on M is a transversely orientable 2-plane field ξ on M, given as the kernel of a 1-form (called a *contact form*) α on M, where α satisfies $\alpha \wedge d\alpha > 0$. In this paper we will omit the word "positive" for simplicity. The pair (M, ξ) is called a *contact* 3*-manifold*. We denote by $\mathrm{Cont}(M)$ the set of contact structures on M. Two contact structures $\xi_0, \xi_1 \in \mathrm{Cont}(M)$ are said to be *isotopic* if there exists a 1-parameter family of contact structures connecting them. Two contact 3-manifolds $(M; \xi)$ and $(M'; \xi')$ are said to be *contactomorphic* if there exists a diffeomorphism $f : M \to M'$ such that $f_*(\xi) = \xi'$. Contact geometry has no local invariants due to the following theorem.

Theorem 2.1 (Darboux's theorem)**.** *Let α be a contact form on an oriented* 3*-manifold M, and let p be a point in M. Then there exists a chart* $(U; x, y, z)$ (*called a Darboux chart*) *around p such that* $p = (0, 0, 0)$ *and* $\alpha|_U = dz + xdy$.

The next theorem claims that there are no non-trivial deformation of contact structures on M: it is especially useful when we prove that two contact structures are isotopic.

Theorem 2.2 (Gray's stability [4])**.** *Let $\{\xi_t\}_{t \in [0,1]}$ be a smooth family of contact structures on a closed oriented* 3*-manifold M. Then there exists an isotopy $\{\psi_t\}_{t \in [0,1]}$ of M such that $(\psi_t)_*(\xi_0) = \xi_t$ for each $t \in$* $[0, 1]$.

For a contact form α on an oriented 3-manifold M, the *Reeb vector field R_α* on M is defined by $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. We also call R_α a Reeb vector field of the contact structure $\xi = \ker \alpha$. The flow generated by R_α is called the *Reeb flow* of α (or a Reeb flow of ξ). A contact structure ξ on M is said to be *overtwisted* if there exists a disk D embedded in M such that ∂D is everywhere tangent to ξ and the framing of D along ∂D coincides with that of ξ. Otherwise ξ is said to be *tight*. For a discussion of the basic theory of contact 3-manifolds, we refer the reader to [9] and [2].

A 2-dimensional polyhedron P in M is called a *flow-spine* if there exists a non-singular flow $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ on M such that

1. for each point of P, there exists a positive chart $(U; x, y, z)$ of M around the point such that $(U, U \cap P)$ is diffeomorphic (by an orientation-preserving diffeomorphism) to one of the four models shown in Figure 1, where the flow Φ on U is generated by the vector field $\partial/\partial z$; and

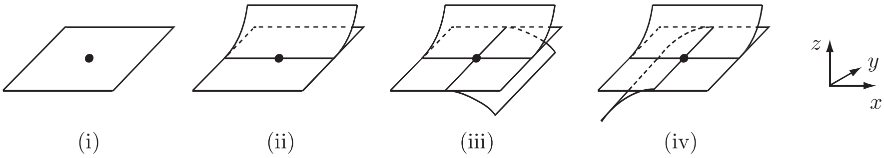

Figure 1: The local models of a flow-spine.

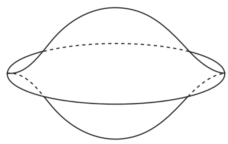

Figure 2: The complement of a flow-spine.

2. P is a *spine*, that is, $M \setminus P$ is an open 3-ball, and the flow Φ in $M \setminus P$ is diffeomorphic to a constant flow in an open ball, see Figure 2.

A point of P whose neighborhood is shaped on the model (iii) (resp. (iv)) in Figure 1 is called a *vertex of ℓ-type* (resp. *r-type*), and we denote the set of vertices of P by $V(P)$. The set of points whose neighborhoods are shaped on the models (ii), (iii) or (v) in Figure 1 is called the *singular set* of P, and we denote it by $S(P)$. Each component of $S(P) \setminus V(P)$ is called an *edge*. Each component of $P \setminus S(P)$ is called a *region*. We remark that if a flow-spine P contains an edge e diffeomorphic to a circle, then P has no vertices. Moreover, it is known that in that case, the ambient 3-manifold M is diffeomorphic to $S^2 \times S^1$, and the set of regions of P consists of a single disk and a single annulus. We also remark that if P contains at least one vertex, then every edge of P is diffeomorphic to an open interval. See [6] and [1] for the details.

A flow-spine P is said to be *positive* if $V(P)$ is non-empty and P has no point of the model (iv) in Figure 1. In the above setting, we say that the flow Φ is *carried* by P. A contact structure ξ on M is said to be *supported* by a flow-spine P if a Reeb flow of ξ is carried by P. We note that when a contact structure ξ on M is supported by a flow-spine P, $M \setminus P$ is an ultimately large Darboux chart.

The following is our main theorem.

Theorem 2.3. *Let M be a closed oriented 3-manifold. Then the following holds:*

(1) *For any positive flow-spine $P \subset M$, there exists a unique contact structure on M supported by P up to isotopy.*

(2) *For any contact structure ξ on M, there exists a flow-spine of M that supports ξ.*

The above theorem implies that the map

$$\{\text{positive flow-spines of } M\}/\text{isotopy} \;\to\; \mathrm{Cont}(M)/\text{isotopy}$$

that takes a positive flow-spine P (up to isotopy) to a contact structure ξ (up to isotopy) whose Reeb flow is carried by P is a well-defined surjective map. We remark that we cannot remove the positivity condition for flow-spines from Theorem 2.3. In fact, the following holds.

Theorem 2.4. *Suppose that M admits a tight contact structure. Then at least one of the following holds:*

(1) *There exists a flow-spine of M that does not support any contact structure; or*

(2) *There exists a flow-spine of M supporting two contact structures that are not contactomorphic.*

3 Summary of the proof of Theorem 2.3

First we briefly explain the following two notions, which play key roles in the proof of Theorem 2.3 (1).

- the *admissibility* condition for flow-spines, and
- a *reference* 1*-form* associated with a flow-spine.

Let P be a flow-spine of a closed oriented 3-manifold M. Let Φ be a non-singular flow on M carried by P. Let $R_1, \ldots, R_n$ be the regions of P. Equip each region R_i of P with the orientation compatible with the orientation of M and the direction of Φ. Let $\bar{R}_i$ be the metric completion of R_i with the path metric inherited from a Riemannian metric on R_i. Let $\kappa_i : \bar{R}_i \to M$ be the natural extension of the inclusion $R_i \hookrightarrow M$. Assign an orientation to each edge of P in an arbitrary way.

Definition. P is said to be *admissible* if there exists an assignment of real numbers $x_1, \ldots, x_m$ to the edges $e_1, \ldots e_m$, respectively, of P such that for any $i \in \{1, \ldots, n\}$

$$\sum_{\tilde{e}_j \subset \partial \bar{R}_i} \varepsilon_{ij} x_j > 0,$$

where $\tilde{e}_j$ is an open interval or a circle on $\partial \bar{R}_i$ such that $\kappa_i|_{\tilde{e}_j} : \tilde{e}_j \to e_j$ is a homeomorphism, and $\varepsilon_{ij} = 1$ if the orientation of e_j coincides with that of $\kappa_i(\tilde{e}_j)$ induced from the orientation of R_i and is $\varepsilon_{ij} = -1$ otherwise.

The proof of the following proposition is given by the combinatorics of the *DS-diagram* (see for instance [5], [6] and [8]) corresponding to a positive flow-spine.

Proposition 3.1. *Every positive flow-spine satisfies the admissibility condition.*

Let P be a positive flow-spine of a closed oriented 3-manifold M. A *reference* 1*-form* η on M associated with P is roughly defined as follows. We consider a compact neighborhood of each of vertices, edges and regions of P. On a compact neighborhood $R_i \times [0,1]$ of each region R_i of P, the 1-form is defined as $\eta \fallingdotseq dt_i$, where t_i is the parameter on $[0,1]$, see Figure 3. The 1-form η is exactly dt_i outside of a neighborhood of $\partial R_i \times [0,1]$. We define η on the neighborhoods of vertices and edges in a similar way. Figure 3 shows the 1-form η on the neighborhood $\mathrm{Nbd}(v_j)$ of a vertex v_j, and the gluing map from $R_i \times [0,1]$ to $\mathrm{Nbd}(v_j)$. Finally, we extend η from $\mathrm{Nbd}(P)$ to the whole M using the product structure $D^2 \times [0,1]$ defined by a non-singular flow carried by P. Since each vertex of P is of ℓ-type, η turns out to be a *confoliation*, i.e. $\eta \wedge d\eta \geq 0$.

Proof sketch of Theorem 2.3 (1). Let M be a closed oriented 3-manifold and $P \subset M$ be a positive flow-spine. Since P is admissible by Proposition 3.1, we can assign real numbers to the edges of P satisfying the condition in the definition of the admissibility. Then we can find a 1-form β on a neighborhood of $S(P)$ in P such that $d\beta > 0$ and $\int_{\partial R_i} \beta$ is

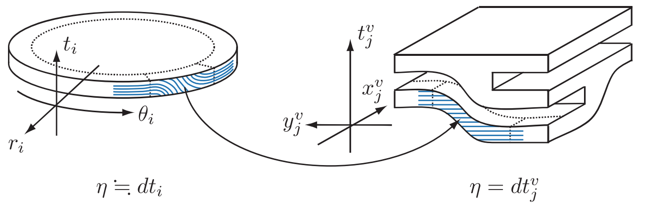

Figure 3: The 1-form η.

the real number given by the assignment. We extend β to the whole P by using Stokes' Theorem in a similar way as in Thurston-Winkelnkemper's construction [10], and then extend that to a neighborhood Nbd(P) of P in a natural way. Set $\alpha := \alpha_R = \beta + R\eta$, where $R > 0$. Then we have

$$\alpha \wedge d\alpha = \beta \wedge d\beta + R(\eta \wedge d\beta + \beta \wedge d\eta) + R^2\eta \wedge d\eta.$$

We can show that $\beta \wedge d\beta = 0$, $\beta \wedge d\eta = 0$ and $\eta \wedge d\beta > 0$, thus α_R is a contact form on Nbd(P). The Reeb vector field R_α is positively transverse to P provided that R is sufficiently large. We extend such a contact form α to the whole M in a natural way. The contact structure $\xi = \ker \alpha$ is then supported by P. The uniqueness of the contact structures supported by P up to isotopy is due to Gray's stability. □

Proof sketch of Theorem 2.3 (2). Let M be a closed oriented 3-manifold and ξ be a contact structure on M. By Giroux [3], there exists an open book decomposition of M whose pages are transverse to a Reeb flow of ξ. We then construct a positive flow-spine P from a finite number of pages and adding more regions according to its monodromy vector field. By Theorem 2.3 (1), we know that there exists a contact form whose Reeb vector field is carried by P. We may choose such a contact form so that it is supported by the open book decomposition. The contact structure thus obtained is isotopic to ξ by the Giroux correspondence. Consequently, ξ is carried by P. □

4 Complexity of contact 3-manifolds

For a contact 3-manifold (M, ξ), we define the *complexity* $c(M, \xi)$ of (M, ξ) to be the minimum number of vertices of any positive flow-spine that supports (M, ξ). By Theorem 2.3 (2), $c(M, \xi)$ is well-defined for any contact 3-manifold (M, ξ). Since there exist only finitely may flow-spines, which are actually simple polyhedra equipped with some additional structures, of a given number of vertices, the complexity c is a finite-to-one invariant.

There exists exactly one positive flow-spine P, called a *positive abalone*, with a single vertex. The left-hand side in Figure 4 shows a neighborhood of $S(P)$ in P. This is a spine of S^3. The right-hand side in Figure 4 depicts the metric completion of $M \setminus P$ with the path metric inherited from a Riemannian metric on M. The pattern, which is a 3-regular graph, on the bounadry of the 3-ball comes from the singular set $S(P)$ of P. We can show that after moving the boundary-pattern of the 3-ball by an isotopy, the constant

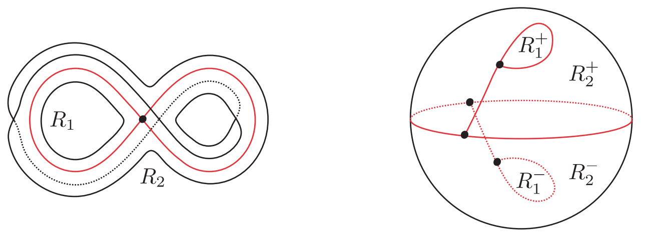

Figure 4: The positive abalone P and the metric completion of $M \setminus P$.

vertical flow on the 3-ball defines a non-singular flow on S^3 whose orbits form the Seifert fibration of S^3 with a regular fiber a trefoil. This is the Reeb flow of the contact form $(2(x_1dy_1 - y_1dx_1) + 3(x_2dy_2 - y_2dx_2))|_{S^3}$, where x_1, y_1, x_2, y_2 are the standard coordinates of $\mathbb{R}^4$ and S^3 is the unit sphere in $\mathbb{R}^4$. The kernel of this form is contactomorphic to the standard contact structure ξ_{std} on S^3. Consequently, P supports the standard contact structure on S^3. In other words, $c(M, \xi) = 1$ if and only if (M, ξ) is contactomorphic to $(S^3, \xi_{\mathrm{std}})$.

There exists exactly three positive flow-spines with two vertices, and we can check that they respectively support $(S^3, \xi_{\mathrm{std}})$, $(\mathbb{RP}^3, \xi_{\mathrm{tight}})$ and $(L(3,2), \xi_{\mathrm{tight}})$, where ξ_{tight} is the unique tight contact structure on $\mathbb{RP}^3$ or $L(3,2)$. Thus $c(M, \xi) = 2$ if and only if (M, ξ) is contactomorphic to $(\mathbb{RP}^3, \xi_{\mathrm{tight}})$ or $(L(3,2), \xi_{\mathrm{tight}})$.

It seems that any positive flow-spine with at most 3 vertices supports a tight contact structure. On the other hand, there exists a positive flow-spine of S^3 with 5 vertices supporting an overtwisted contact structure. It is interesting to determine whether there exists a positive flow-spine with 4 vertices supporting an overtwisted contact structure. It is also an interesting problem to give a criterion for the tightness of contact structures in terms of supporting positive flow-spines.

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