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The Heegaard Floer complexes of a trivalent graph defined on two Heegaard diagrams

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1 Introduction

In the report, let G denote an embedded oriented connected trivalent graph in S^3 without source or sink. Namely each vertex of G is one of the following two cases.

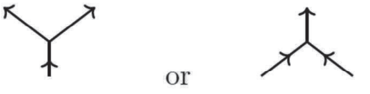

Figure 1: Vertices of G.

Choose a generic point δ in G and assign a coloring c on G. Then we consider the Heegaard Floer complex $\mathrm{CFG}(G,\delta,c)$. The homology of $\mathrm{CFG}(G,\delta,c)$, which is denoted by $\mathrm{HFG}(G,\delta,c)$, is a topological invariant of G with the choice of δ and c.

In this report, we consider two types of Heegaard diagrams, which have both been studied in [5] for singular knots. We extend the constructions to the case of trivalent graphs and study the Heegaard Floer complexes $\mathrm{CFG}(G,\delta,c)$ defined on them.

For the Heegaard diagram of type one, which is given in Example 2.2, we generalize the results in [5]. This part is a joint work with Zhongtao Wu.

Let $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$. Let $C(G,\delta,c)$ be the $\mathbb{F}$-vector space

$$\bigoplus_{\mathbf{x}\in S(G,\delta)} \bigotimes_{e\in E_{\mathbf{x}}} \left(\frac{\mathbb{F}[U_e]}{U_e^{c(e)}=0}\right),$$

where $S(G,\delta)$ is the set of Kauffman states, $E_{\mathbf{x}}$ is a subset of edges of G associated with $\mathbf{x}$ and U_e is the variable corresponding to an edge e. We define the $\mathcal{A}$-grading and $\mathcal{M}$-grading of a Kauffman state by using the local contributions in Figures 6 and 7, and extend them to gradings of $C(G,\delta,c)$ by adding the rules $\mathcal{A}(U_e)=-c(e)$ and $\mathcal{M}(U_e)=0$.

Theorem 1.1 (B. and Wu). *Let $C_d(G,\delta,c;s)$ be the $\mathbb{F}$-vector space generated by the generators of $C(G,\delta,c)$ with $\mathcal{A}$-grading s and $\mathcal{M}$-grading d. Then there is a differential*

$$\partial: C(G,\delta,c;s)\to C(G,\delta,c;s)$$

that carries $C_d(G,\delta,c;s)$ to $C_{d-1}(G,\delta,c;s)$ satisfying

$$H_d(G,\delta,c;s)\cong \mathrm{HFG}_d(G,\delta,c;s).$$

Note that $\mathcal{A}$-grading and $\mathcal{M}$-grading for a Kauffman states correspond to the Alexander grading and Maslov grading of the original chain complex.

As a corollary of Theorem 1.1, we get the following formula for the Euler characteristic. Let $\Delta_{(G,c)}(t)$ be the Alexander polynomial defined in [3], which is defined up to a factor of $\pm t^k$ for $k \in \mathbb{Z}$. We say $f(t) \doteq g(t)$ if $f(t) = \pm t^k g(t)$ for some $k \in \mathbb{Z}$.

Corollary 1.2 (B. and Wu)**.**

$$\chi(\mathrm{HFG}(G, \delta, s)) \doteq [\delta]\Delta_{(G,c)}(t), \tag{1}$$

where $[\delta]$ is a factor related to the position of δ.

For a plane trivalent graph, we have the following result.

Corollary 1.3 (B. and Wu)**.** *Suppose G is a plane trivalent graph. Then the group* $\mathrm{HFG}(G, \delta, c)$ *is determined by the Alexander polynomial $\Delta_{(G,c)}(t)$; indeed, the homology* $\mathrm{HFG}(G, \delta, c)$ *is supported on the zero Maslov grading level.*

The Heegaard diagram of type two is given in Example 2.4. The results on this Heegaard diagram were written in [2], which has been submitted for publication, so we briefly summarize the results in Section 4.

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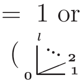
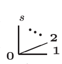
2 Preliminaries

2.1 Heegaard diagram

Suppose G is a connected trivalent graph. Let V be the set of vertices of G and E be the set of edges of G. Suppose $|V| = n$ and $|E| = m$. Choose a generic point on G and call it δ. It is convenient to regard δ as a new bivalent vertex of G. The Heegaard diagram of (G, δ) is defined as follows, which was proposed in [1].

Definition 2.1. A quintet $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is called a Heegaard diagram for (G, δ) if it satisfies the following conditions.

1. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ is an $(n+1)$-pointed Heegaard diagram for S^3, and $\boldsymbol{z}$ is a set of $(m+1)$ points in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{w})$.
2. For each vertex $v \in V \cup \{\delta\}$ whose indegree is $l = 1$ or 2 (resp. outdegree is $s = 1$ or 2), there exists a smooth embedding $\varphi_v : ($  $, \{0\}, \{1, 2, \cdots, l\}) \hookrightarrow (\Sigma \backslash \boldsymbol{\alpha}, \boldsymbol{w}, \boldsymbol{z})$ (resp. $\psi_v : ($  $, \{0\}, \{1, 2, \cdots, s\}) \hookrightarrow (\Sigma \backslash \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$) so that the images of φ_v (resp. ψ_v) are pairwisely disjoint and $\bigcup_{v \in V \cup \{\delta\}} (\mathrm{Im}(\varphi_v) \cup \mathrm{Im}(\psi_v))$ recovers G, when we push the interior of $\mathrm{Im}(\varphi_v)$ (resp. $\mathrm{Im}(\psi_v)$) slightly into $U_{\boldsymbol{\alpha}}$ (resp. $U_{\boldsymbol{\beta}}$). Here $U_{\boldsymbol{\alpha}}$ (resp. $U_{\boldsymbol{\beta}}$) is obtained from Σ by attaching 2-handles along $\boldsymbol{\alpha}$-curves (resp. $\boldsymbol{\beta}$-curves).

From the definition it is easy to see that on the Heegaard diagram Σ, we assign a base point w_v around a vertex and a basepoint z_e at an edge e of G, and the base points w_δ and z_δ around δ. In this report, we focus on two types of Heegaard diagrams. Here we introduce them as examples.

Example 2.2 (Heegaard diagram of type one)**.** This Heegaard diagram was introduced in [1], which extended the construction in [5].

Consider a graph diagram $D \subset \mathbb{R}^2$ for a given graph $G \subset S^3$.

1. Regard D as a 1-complex in S^3 and take a tubular neighbourhood of it in S^3. It is a handlebody and its boundary is the Heegaard surface Σ.
2. The diagram D divides $\mathbb{R}^2$ into several regions. For each bounded region, introduce an $\boldsymbol{\alpha}$-curve on Σ which encloses the region.
3. For each crossing of D, introduce a $\boldsymbol{\beta}$-curve as shown in Figure 2.
4. Place the base point w_v on each vertex $v \in V$.
5. For a vertex $v \in V$ with indegree $l = 1$ or 2, introduce l $\boldsymbol{\beta}$-curves which are meridians of the edges pointing to v and l base points of type $\boldsymbol{z}$ on the edges pointing to v. Introduce an $\boldsymbol{\alpha}$-curve α_v which bounds a disk in Σ that contains w_v and all the base points of type $\boldsymbol{z}$ on the edges pointing to v.
6. For δ, introduce a $\boldsymbol{\beta}$-curve which is the meridian of the edge containing δ and introduce two base points w_δ and z_δ around δ.

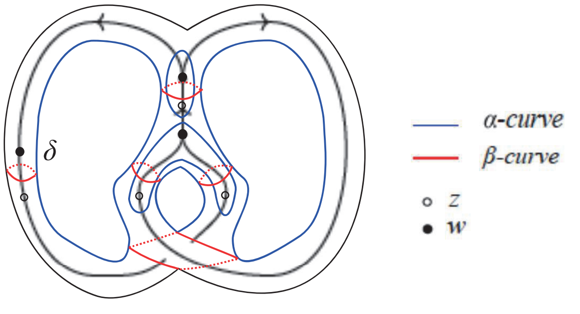


Figure 2: The Heegaard diagram of type one associated with a graph diagram.

Remark 2.3. *Example 2.2 was first proposed in [1], where we studied bipartite graphs. A trivalent graph without source or sink can be regarded as a perticular bipartite graph.*

Example 2.4 (Heegaard diagram of type two)**.** We consider the second type of Heegaard diagram only for plane graphs. It extends the construction in [5] for a singular knot.

For a plane oriented graph G without source or sink, choose a point δ on an edge of G. We introduce a Heegaard diagram for (G, δ) as follows.

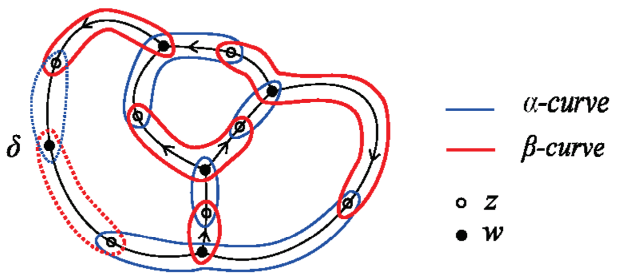


Figure 3: A Heegaard diagram for G with a point δ.

1. The Heegaard surface is S^2 where G is embedded as a plane graph.
2. At each vertex $v \in V$, introduce a base point w_v, and on each edge e of G introduce a base point z_e.
3. For δ, introduce two base points w_δ and z_δ around δ.
4. Around each vertex $v \in V$, introduce a curve α_v which encloses the base point w_v at v and the base point(s) z('s) on the edge(s) pointing to v. Introduce a curve β_v which encloses the base point w_v at v and the base point(s) z('s) on the edge(s) pointing out of v.

As a result, we get the following data $H = (S^2, \{\alpha_v\}_{v\in V}, \{\beta_v\}_{v\in V}, \boldsymbol{w}, \boldsymbol{z})$, which is a Heegaard diagram for (G, δ). See Fig. 3 for an example.

2.2 The Heegaard Floer complex

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ be a Heegaard diagram for a graph (G, δ) whose number of $\boldsymbol{\alpha}$-curves (and also $\boldsymbol{\beta}$-curves) is d. Let

$$\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \alpha_1 \times \cdots \times \alpha_d \text{ and } \mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \beta_1 \times \cdots \times \beta_d$$

be the tori in the symmetric product $\mathrm{Sym}^d(\Sigma)$. Given $x, y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, let $\pi_2(x, y)$ be the set of relative homology classes of Whitney disks from x to y with boundary in $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. For $\phi \in \pi_2(x, y)$, let $\mu(\phi)$ be its Maslov index and $\widehat{\mathcal{M}}(\phi)$ be the moduli space of pseudo-holomorphic disks in the class ϕ modulo $\mathbb{R}$.

Let $D_1, D_2, \cdots, D_h$ denote the closures of the components of $\Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. A *domain* is a 2-chain on Σ of the form $D = \sum_{i=1}^h a_i D_i$, where $a_i \in \mathbb{Z}$ is called the local multiplicity of D at D_i. For a point p in the interior of D_i, let $n_p(D)$ denote the local multiplicity of D at the point p, which equals a_i. A domain D is a *positive domain* if $a_i \geq 0$ for $1 \leq i \leq h$. A domain $P = \sum_{i=1}^h a_i D_i$ is called a *periodic domain* if ∂P is a $\mathbb{Z}$-linear combination of $\boldsymbol{\alpha}$-curves and $\boldsymbol{\beta}$-curves and $P \cap \boldsymbol{w} = P \cap \boldsymbol{z} = \emptyset$.

To each base point z_e, we assign a variable U_e. We define the chain complex $\mathrm{CFG}(G, \delta)$ to be the free $\mathbb{F}[\{U_e\}_{e\in E}]$-module generated by $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Note that the generating set $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ has a one-to-one correspondence with the set

$$S = \{(x_1, x_2, \cdots, x_d) | x_i \in \alpha_i \cap \beta_{\sigma(i)}, 1 \leq i \leq d, \sigma \in S_d\}.$$

Therefore S is also regarded as the generating set.

Then $\mathrm{CFG}(G,\delta)$ is endowed with the differential

$$\partial(x)=\sum_{y\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}}\ \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1,n_{\boldsymbol{w}}(\phi)=\{0\},n_{z_\delta}(\phi)=0\}}\#\widehat{\mathcal{M}}(\phi)\cdot(\prod_{e\in E}U_e^{n_{z_e}(\phi)})\cdot y, \tag{2}$$

where $n_{\boldsymbol{w}}(\phi)=\{n_w(\phi)|w\in\boldsymbol{w}\}$.

To simplify the discussion, we further endow a coloring to G. A *balanced coloring* is a map $c:E\to\mathbb{Z}_{\geq 0}$ such that for each vertex v of G,

$$\sum_{e\text{: pointing into } v}c(e)=\sum_{e\text{: pointing out of } v}c(e).$$

Let $c(v)$ be the value above.

Definition 2.5. The relative *Alexander grading* A and *Maslov grading* M are as follows:

$$A(x)-A(y)=\sum_{v\in V}c(v)\cdot n_{w_v}(\phi)-\sum_{e\in E}c(e)\cdot n_{z_e}(\phi), \tag{3}$$

$$M(x)-M(y)=\mu(\phi)-2\sum_{v\in V\cup\{\delta\}}n_{w_v}(\phi), \tag{4}$$

for $x,y\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$, where $\phi\in\pi_2(x,y)$.

The functions A and N defined above induce on $\mathrm{CFG}(G,\delta)$, the *Alexander* and *Maslov* gradings, with the convention that

$$A(U_e)=-c(e),\ M(U_e)=0.$$

The differential ∂ drops the Maslov grading by one, and preserves the Alexander grading.

Proposition 2.6. *The homology of* $\mathrm{CFG}(G,\delta,c)$ *is a topological invariant of* G *together with the choice of* δ *and* c.

3 Heegaard Floer complex: type one

The purpose of this section is to prove Theorem 1.1 and Corollaries 1.2 and 1.3.

3.1 Kauffman states

We recall the definition of Kauffman states, which was described in detail in [3].

Definition 3.1. Starting from a trivalent graph diagram (D,δ), we can obtain a *decorated diagram* (D,δ) by drawing a circle around each vertex of D.

1. $\mathrm{Cr}(D)$: denotes the set of crossings, including the types  and  which are the double points of the diagram and the type  which are the intersection points around each vertex between the incoming edges with the circle.

2. Re(D): denotes the set of regions, including the *regular regions* of $\mathbb{R}^2$ separated by D and the *circle regions* around the vertices. *Marked regions* are the regions adjacent to the base point δ, and the others are called *unmarked regions*.

3. Corners: For a crossing of type  or , there are four corners around it, and we call them the *north*, *south*, *west*, and *east corners* of the crossing. Around a crossing of type  there are three corners, and we call the one inside the circle region the *north* corner, the one on the left of the crossing the *west* corner and the one on the right the *east* corner. Note also that every corner belongs to a unique region in Re(D).

Calculating the Euler characteristic of $\mathbb{R}^2$ using D gives

$$|\operatorname{Re}(D)| = |\operatorname{Cr}(D)| + 2.$$

The point δ is chosen so that it is adjacent to two distinct regions, which will be denoted by R_u and R_v.

Definition 3.2. A *Kauffman state*, or simply, a *state* for a decorated diagram (D, δ) is a bijective map

$$\mathbf{x} : \operatorname{Cr}(D) \to \operatorname{Re}(D) \backslash \{R_u, R_v\},$$

which sends a crossing in Cr(D) to one of its corners. Denote $S(D, \delta)$ the set of all states.

For each Kauffman state $\mathbf{x}$, around each vertex v there is a unique edge the crossing on which is sent to the circle region. Let $E_{\mathbf{x}}$ be the set of such edges.

Consider the Heegaard diagram in Example 2.2. We see that each Kauffman state is associated to 2^n different generators in $\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$, where n is the number of vertices. More precisely, each intersection point in $\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ is represented by a pair $(\mathbf{x}, \epsilon_{\mathbf{x}})$, where $\mathbf{x}$ is a Kauffman state and $\epsilon_{\mathbf{x}} : V \to \{\pm 1\}$.

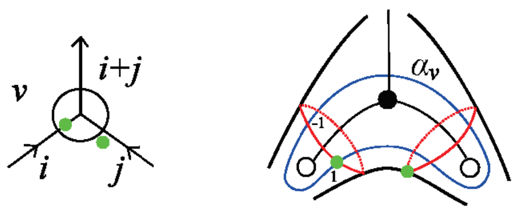


Figure 4: The internal curve α_v intersects the same β-curve at two points, which we label them as 1 and -1.

Suppose x_+ and x_- are two generators which are identical except that around the vertex v, x_+ is labeled by 1 and x_- is labeled by -1. There is a bigon connecting x_- to x_+ which contains a base point z_e. Then we have

$$A(x_-) - A(x_+) = -c(e) \leq 0.$$

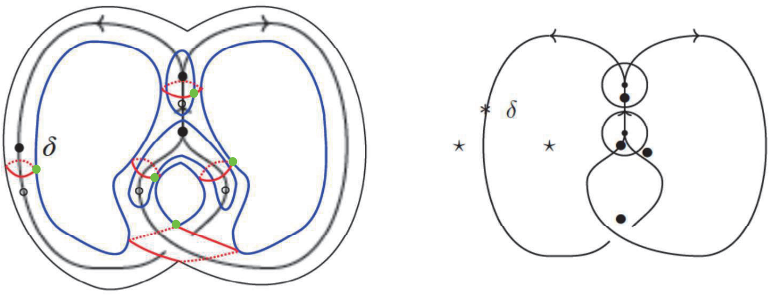
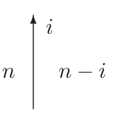


Figure 5: A kauffman state and its corresponding Alexander grading maximizer.

Therefore among all the intersection points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ which corresponds to the same Kauffman state $\mathbf{x}$, the one with $\epsilon_{\mathbf{x}}(v) = +1$ is the unique intersection point that has maximal Alexander grading. We call it *the Alexander grading maximizer* of the state $\mathbf{x}$.

For each regular region in $\mathrm{Re}(D)$, we recall the definition of its index $\mathrm{Ind}(\cdot)$ in [3], which was defined as follows.

1. The index of the unbounded region is set to be 0.
2. The indices of the other regular regions are inductively determined by the rule as exhibited below: when an edge with color i points upward, let the difference of the index of its left-hand side region and that of its right-hand side region be i.



The definition of a balanced coloring guarantees that the above rules give rise to a well-defined index for each regular region.

Definition 3.3. Suppose δ is on an edge with color i, and the indices of the regions adjacent to δ are n and $n-i$. Define

$$|\delta| = t^{n-i} - t^{n} \text{ and } [\delta] = (t^{-1/2} - t^{1/2})^{-1}|\delta|.$$

3.2 Alexander grading

Lemma 3.4. *Let $\mathbf{x}, \mathbf{y}$ be two Kauffman states and let x, y be the Alexander grading maximizers that correspond to them. Then we have*

$$\mathcal{A}(\mathbf{x}) - \mathcal{A}(\mathbf{y}) = A(x) - A(y),$$

where $\mathcal{A}(\mathbf{x}) = \sum_{c \in \mathrm{Cr}(D)} \mathcal{A}_c^{\mathbf{x}(c)}$ *is calculated using the local contribution in Figure 6.*

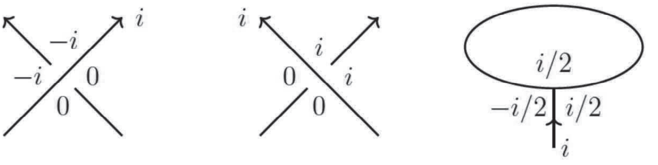


Figure 6: The local contribution $\mathcal{A}_c^{\Delta}$.

Proof. The proof is similar with the proof of Lemma 4.2 in [5]. For x and y, It suffices to show that any homology class $\phi \in \pi_2(x, y)$ satisfies

$$\mathcal{A}(\mathbf{x}) - \mathcal{A}(\mathbf{y}) = \sum_{v \in V} c(v) \cdot n_{w_v}(\phi) - \sum_{e \in E} c(e) \cdot n_{z_e}(\phi).$$

Around each vertex v, let $\overrightarrow{ve}$ be the oriented simple arc from w_v to z_e on the Heegaard surface for each edge e adjecent to v, and $\overrightarrow{ev}$ be the inverse of $\overrightarrow{ve}$. Suppose each arc $\overrightarrow{ve}$ intersects transversely with the boundary of ϕ. For an intersection point $p \in \overrightarrow{ve} \cap \partial\phi$ (or $\overrightarrow{ev} \cap \partial\phi$), define

$$\sharp p = (-1)^{\text{sign}(p)} c(e)$$

to be the algebraic intersection number of $\overrightarrow{ve}$ (or $\overrightarrow{ev}$) and $\partial\phi$ at p, where sign(p) is $+1$ if the orientation of the intersection agrees with the orientation of the Heegaard surface and -1 otherwise.

Around a vertex v of G, let η_v be the union of $\overrightarrow{ve}$'s for those edges e pointing to v. Let θ_v be the union of $\overrightarrow{ev}$'s for all the edges pointing to v and $\overrightarrow{ve}$'s for all the edges leaving v. Then we have

$$\begin{aligned} A(x) - A(y) &= \sum_{v \in V} \sharp(\eta_v \cap \partial\phi) \\ &= \sum_{v \in V} \sharp((\eta_v + \theta_v) \cap \partial\phi). \end{aligned}$$

The first equality follows from the definition of $A(x) - A(y)$ and the fact that ϕ is a 2-complex on the Heegaard surface. The second equality follows from the facts that $\cup_v \theta_v$ replicates G and c is a balanced coloring defined on G. Namely $\sum_{v \in V} \sharp(\theta_v \cap \partial\phi) = 0$.

Let β_c be the $\boldsymbol{\beta}$-curve around a double crossing c, and α_v be the internal $\boldsymbol{\alpha}$-curve around a vertex v. For $\boldsymbol{\alpha}$-curves, the only ones that intersect $(\eta_v + \theta_v)$ are the internal α_v's around vertices, and the only $\boldsymbol{\beta}$-curves that algebraically intersect $(\eta_v + \theta_v)$ are those β_c's. Note that $\partial\phi$ are union of arcs on $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$. Therefore we have

$$\begin{aligned} &\sum_{v \in V} \sharp((\eta_v + \theta_v) \cap \partial\phi) \\ = &\sum_{v \in V} \sharp((\eta_v + \theta_v) \cap \partial\phi|_{\alpha_v}) + \sum_{c:\text{ double crossing}} \sharp[(\cup_v \theta_v) \cap \partial\phi|_{\beta_c}]. \end{aligned}$$

We claim that

$$\sharp((\eta_v+\theta_v)\cap\partial\phi|_{\alpha_v}) = \mathcal{A}_v^{\mathbf{x}(v)}-\mathcal{A}_v^{\mathbf{y}(v)}+[\mathrm{Ind}(\mathbf{x}(v))-\mathrm{Ind}(\mathbf{y}(v))], \tag{5}$$

$$\sharp[(\cup_v\theta_v)\cap\partial\phi|_{\beta_c}] = \mathcal{A}_c^{\mathbf{x}(c)}-\mathcal{A}_c^{\mathbf{y}(c)}+[\mathrm{Ind}(\mathbf{x}(c))-\mathrm{Ind}(\mathbf{x}(c))], \tag{6}$$

where $\mathcal{A}_v^{\mathbf{x}(v)}$ is the sum of the local contributions around v for the state $\mathbf{x}$, and $\mathrm{Ind}(\mathbf{x}(v))$ is the sum of the indices of the regular regions occupied by $\mathbf{x}$ around v. For each vertex v, the intersection number of $(\eta_v+\theta_v)$ with the internal α_v is zero. For each β_c, the intersection number of $(\cup_v\theta_v)$ with it is also zero. Therefore it suffices to verify (5) and (6) by choosing any arc connecting x to y on the internal α_v's and β_c's.

We consider all the possible locations of x and y around a vertex and around a crossing to verify (5) and (6). It is easy to see that all the possible locations can be obtained as compositions of the six cases in Tables 1 and 2.

Now summing equations (5) and (6) we have

$$A(x)-A(y)=[\mathcal{A}(\mathbf{x})+\sum_{c\in\mathrm{Cr}(D)}\mathrm{Ind}(\mathbf{x}(c))]-[\mathcal{A}(\mathbf{y})+\sum_{c\in\mathrm{Cr}(D)}\mathrm{Ind}(\mathbf{y}(c))].$$

On the other hand, we see that

$$\sum_{c\in\mathrm{Cr}(D)}\mathrm{Ind}(\mathbf{x}(c))=\sum_{c\in\mathrm{Cr}(D)}\mathrm{Ind}(\mathbf{y}(c)),$$

which is the sum of indices over all unmarked regular regions of D. □

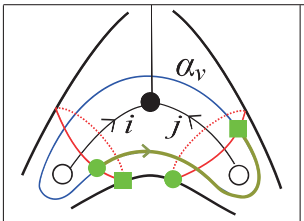
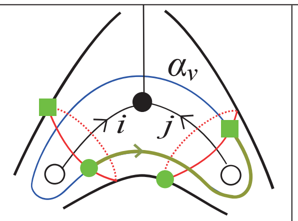
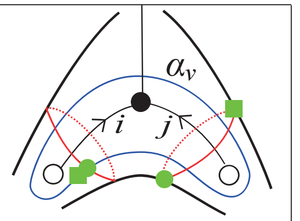
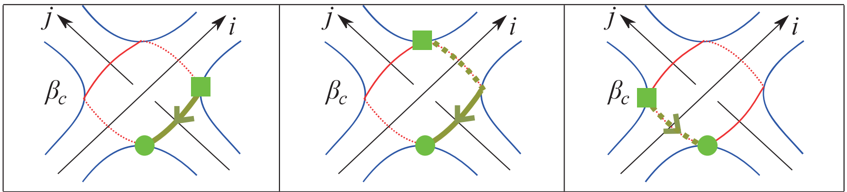
		
$\sharp((\eta_v+\theta_v)\cap\partial\phi|_{\alpha_v})=-j$ $\mathcal{A}_v^{\mathbf{x}(v)}=i/2-j/2$ $\mathcal{A}_v^{\mathbf{y}(v)}=i/2+j/2$ $\mathrm{Ind}(\mathbf{x}(v))=\mathrm{Ind}(\mathbf{y}(v))$	$\sharp((\eta_v+\theta_v)\cap\partial\phi|_{\alpha_v})=-j$ $\mathcal{A}_v^{\mathbf{x}(v)}=i/2-j/2$ $\mathcal{A}_v^{\mathbf{y}(v)}=-i/2+j/2$ $\mathrm{Ind}(\mathbf{x}(v))=\mathrm{Ind}(\mathbf{y}(v))-i$	$\sharp((\eta_v+\theta_v)\cap\partial\phi|_{\alpha_v})=0$ $\mathcal{A}_v^{\mathbf{x}(v)}=i/2-j/2$ $\mathcal{A}_v^{\mathbf{y}(v)}=i/2+j/2$ $\mathrm{Ind}(\mathbf{x}(v))=\mathrm{Ind}(\mathbf{y}(v))+j$

Table 1: Verify Eq. (5) around a vertex.

3.3 Maslov grading

For the Maslov grading, we can also calculate it locally as below.



$\sharp((\eta_v+\theta_v)\cap\partial\phi\vert_{\beta_c}) = j$ $\mathcal{A}_c^{\mathbf{x}(c)} = 0$ $\mathcal{A}_v^{\mathbf{y}(v)} = 0$ $\mathrm{Ind}_{\mathbf{x}(v)} = \mathrm{Ind}_{\mathbf{y}(v)} - j$	$\sharp((\eta_v+\theta_v)\cap\partial\phi\vert_{\alpha_v}) = j$ $\mathcal{A}_v^{\mathbf{x}(v)} = 0$ $\mathcal{A}_v^{\mathbf{y}(v)} = -i$ $\mathrm{Ind}_{\mathbf{x}(v)} = \mathrm{Ind}_{\mathbf{y}(v)} + i - j$	$\sharp((\eta_v+\theta_v)\cap\partial\phi\vert_{\alpha_v}) = 0$ $\mathcal{A}_v^{\mathbf{x}(v)} = 0$ $\mathcal{A}_v^{\mathbf{y}(v)} = -i$ $\mathrm{Ind}_{\mathbf{x}(v)} = \mathrm{Ind}_{\mathbf{y}(v)} + i$

Table 2: Verify Eq. (6) around a double crossing.

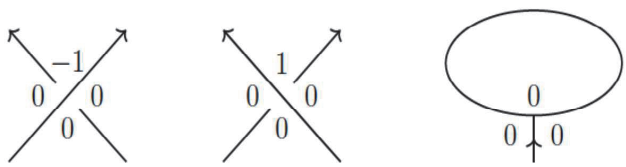


Figure 7: The local contribution $\mathcal{M}_c^{\Delta}$.

Lemma 3.5. *Let* $\mathbf{x}$ *be a Kauffman state and let* x *be the Alexander grading maximizer that corresponds to it. Then we have*

$$\mathcal{M}(\mathbf{x}) = M(x),$$

where $\mathcal{M}(\mathbf{x}) = \sum_{c\in \mathrm{Cr}(D)} \mathcal{M}_c^{\mathbf{x}(c)}$ *is calculated using the local contribution in Figure 7.*

The proof in [5] can be generalized trivially to our case. In the proof of [5], only four-valent vertices (singular crossings) are considered. However, the proof works for graphs which contains both trivalent and four-valent vertices.

For the readers' convenience, we summarize the sketch of the proof and point out the notational differences. Note that the definition of $M(x)$ does not depend on the $\boldsymbol{z}$ base points and the coloring c. Suppressing the $\boldsymbol{z}$ base points, we get an $(n+1)$-pointed Heegaard diagram for S^3.

Let G be a plane trivalent graph. Two Kauffman states are said to be *equivalent* if they coincide with each other inside the circle regions. This definition is an adaption of Definition 4.3 [5] to our situation. By using exactly the same argument, we could prove that two Kauffman states are equivalent if and only if they are the same state.

Following the discussion in Proposition 4.6 of [5], we can show the following fact. Let (G,δ) be a plane trivalent graph. Suppose $\mathbf{x}$ is a Kauffman state and x is the corresponding Alexander maximizer. Then we have $M(x) = 0$.

Now we consider a general trivalent graph (G,δ). Consider the Heegaard diagram of type one $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ for (G, δ), where $\boldsymbol{z}$ is suppressed. Consider a Kauffman state $\mathbf{x}$ for (G, δ). Then following the recipe in the proof of Theorem 4.1 of [5], one can construction a new Heegaard diagram $H_{\mathbf{x}} = (\Sigma, \boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{w})$, which deeply depends on $\mathbf{x}$. It is an $(n+1)$-pointed Heegaard diagram for S^3. Then we can show that the generators induced from $\mathbf{x}$ are the only generators for $H_{\mathbf{x}}$, the number of which is 2^n. Namely the differential of the chain complex defined on $H_{\mathbf{x}}$ is trivial. Let x be the Alexander maximizer of $\mathbf{x}$ in $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$, and let x' be the induced generator on $H_{\mathbf{x}}$. Then we have $M(x') = 0$.

On the other hand, $H_{\boldsymbol{\alpha},\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{w})$ is a Heegaard diagram for $(S^1 \times S^2)^{\sharp d}$, where d is the cardinality of $\boldsymbol{\alpha}$. The differential defined on $H_{\boldsymbol{\alpha},\gamma}$ is also trivial. Suppose θ is the generator with maximal Maslov grading, which is $M(\theta) = 0$.

Consider the homological triangle connecting x, x' and θ. One can finally show that $M(x) = n - p$, where n is the number of negative crossings whose images under $\mathbf{x}$ are north corners and p is the number of positive crossings whose images under $\mathbf{x}$ are north corners. It is indeed the conclusion we want to show in Lemma 3.5.

3.4 Proof of Theorem 1.1.

The proof follows the idea in the proofs of Proposition 4.6 and Theorem 4.1 in [5].

Since G is connected, we have the following lemma.

Lemma 3.6. *Let P be a periodic domain in the Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ of type one. If $n_{\boldsymbol{z}}(P) = n_{\boldsymbol{w}}(P) = \{0\}$, P is a trivial domain.*

Using the lemma above, we can define a filtration on $\mathrm{CFG}(G, \delta, c)$. Consider all the regions of $\Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, and place a point in each region disjoint from z_e's. Let Q be the set of such points.

Definition 3.7. We define a partial order $\mathcal{K}$ on $\mathbb{F}[\{U_e\}_{e\in E}]\langle \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \rangle$. For two elements $a, b \in \mathbb{F}[\{U_e\}_{e\in E}]\langle \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \rangle$, if there is a domain ϕ connecting a to b with $n_{\boldsymbol{w}}(\phi) = \{0\}$ and $n_{z_\delta}(\phi) = 0$, let

$$\mathcal{K}(a) - \mathcal{K}(b) = \sharp_{alg}(\phi \cap Q),$$

where the right hand side denotes the algebraic intersection number of ϕ and Q.

The partial order K is well-defined since by Lemma 3.6 if the domain ϕ exists, it is unique.

Lemma 3.8. *For $a, b \in \mathbb{F}[\{U_e\}_{e\in E}]\langle \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \rangle$, suppose b is a summand in $\partial(a)$. Then we have*

$$\mathcal{K}(b) \leq \mathcal{K}(a).$$

Namely $\mathcal{K}$ induces a filtration on $(\mathrm{CFG}(G, \delta, c), \partial)$.

Proof. The pseudo-holomorphic disk ϕ connecting a to b always has non-negative intersection number with Q. ∎

Proof of Theorem 1.1. For a Kauffman state $\mathbf{x}$, let x_+ and x_- are two generators corresponding to $\mathbf{x}$ which are identical except around a base point z_e. Then the bigon containing z_e connects x_- to x_+. Its homology class ϕ admits a single holomorphic representative up to reparameterization. The differential on the E_0 page of $\mathrm{CFG}(G, \delta, c)$ under the filtration $\mathcal{K}$ counts only those bigons described above. Namely it only connects generators which have common underlying Kauffman state. Therefore the chain complex on the E_0 page splits along the set $S(G, \delta)$.

For each Kauffman state $\mathbf{x}$, the chain complex on the E_0 page is the tensor product of the chain complexes of the form

$$\mathbb{F}[\{U_e\}_{e\in E}] \xrightarrow{U_e^{c(e)}} \mathbb{F}[\{U_e\}_{e\in E}],$$

for $e \in E_{\mathbf{x}}$. The cokernal of each such short differential is generated as a $\mathbb{F}$-vector space by $\{x_+, U_e \cdot x_+, \cdots, U_e^{c(e)-1} \cdot x_+\}$.

The E_1 page is now a free module generated by Kauffman states. More precisely, as a $\mathbb{F}$-vector space, it is

$$\bigoplus_{\mathbf{x}\in S(G,\delta)} \bigotimes_{e\in E_{\mathbf{x}}} \left(\frac{\mathbb{F}[U_e]}{U_e^{c(e)} = 0} \right).$$

□

Now we prove Corollaries 1.2 and 1.3.

Proof of Corollary 1.2. We elaborate the proof here by looking back at the differential ∂_0 in E_0-page in the proof of Theorem 1.1. The differential given by the bigon that passes z_e is of the form

$$\partial_0 x_- = U_e^{c(e)} \cdot x_+,$$

from which we see that the cokernal is generated by $\{x_+, U_e \cdot x_+, \cdots, U_e^{c(e)-1} \cdot x_+\}$. There are a total of $c(e)$ number of generators with the same Maslov grading and different Alexander gradings, which contribute a factor

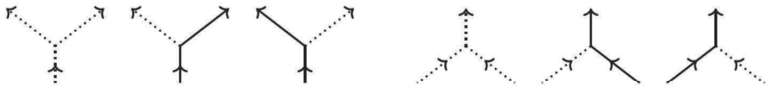
$$1 + t + \cdots t^{c(e)-1} = [c(e)] \cdot t^{\frac{c(e)-1}{2}}$$

around z_e for $e \in E_{\mathbf{x}}$, where $[k] := \frac{t^{k/2} - t^{-k/2}}{t^{1/2} - t^{-1/2}}$. Finally by comparing the local contributions in Figures 6 and 7 and the local contributions in [3], we obtain the identification. □

Proof of Corollary 1.2. When G is a plane trivalent graph, all the generators in Theorem 1.1 have zero $\mathcal{M}$-grading. Hence, the differential on the chain complex is trivial. □

4 Heegaard Floer complex: type two

We consider a plane trivalent graph G. Viro in [6] proposed a vertex state sum formula for the $gl(1|1)$-Alexander polynomial. Choose a point δ on an outermost edge of G. A *Viro state* is a map $s : E \to \{0, 1\}$ which sends the edge containing δ to zero and satisfies the condition that at each vertex, the sum of $s(e)$ for all the edges e pointing toward the vertex equals that for the edges pointing out of the vertex. Let $\mathcal{S}$ be the set of states. By definition, around each vertex we have the following six possibilities under a state, where the dotted edges are those which are sent to zero and the solid edges are those which are sent to one.



Consider the intersection points of $\boldsymbol{\alpha}$-curves and $\boldsymbol{\beta}$-curves. Let

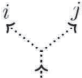
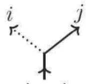
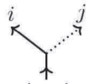
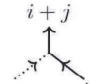
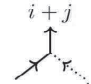
$$\mathcal{T} := \{\{x_v\}_{v\in V} | x_v \in \alpha_{\sigma(v)} \cap \beta_v \text{ for a bijection } \sigma \text{ of } V\}.$$

Let σ_x be the bijection that defines $x \in \mathcal{T}$. It is easy to see that the intersection points always appear in pairs around a base point. Two elements $x = \{x_v\}_{v\in V}$ and $y = \{y_v\}_{v\in V}$ are said to be equivalent ($x \sim y$) if x_v and y_v are around the same base point for any $v \in V$. Let $[x]$ be the equivalence class of x. It is obvious that σ_x keeps invariant within the equivalence class of x.

Our first result is:

Proposition 4.1. *There is an identification between $\mathcal{T}/\!\sim$ and $\mathcal{S}$.*

The $gl(1|1)$-Alexander polynomial is defined by assigning Boltzmann weights to each vertex under each state as follows. The Boltzmann weights were originally obtained by scaling the Clebsch-Gordan morphisms for irreducible $U_q(gl(1|1))$-modules of dimension $(1|1)$.

State	 i, j, $i+j$	 i, j, $i+j$	 i, j, $i+j$	 $i+j$, i, j	 $i+j$, i, j	 $i+j$, i, j
$Wt(v;s)$	$\{2i+2j\}_q$	$\{2j\}_q$	$\{2i\}_q q^{-2j}$	1	q^{2i}	1

Our second result is:

Proposition 4.2. *The Boltzmann weights around trivalent vertices, which determine the $gl(1|1)$-Alexander polynomial, can be obtained from the Fox calculus on the Heegaard diagram of type two.*

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