

# Infinite sequences of non-Weierstrass numerical semigroup with odd conductor<sup>1</sup>

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## Abstract

We construct infinite sequences of non-Weierstrass almost symmetric numerical semigroups with a fixed odd conductor through dividing by three.

## 1 Introduction

Let  $\mathbb{N}_0$  be the additive monoid of non-negative integers. A submonoid  $H$  of  $\mathbb{N}_0$  is called a *numerical semigroup* if the complement  $\mathbb{N}_0 \setminus H$  is finite. The cardinality of  $\mathbb{N}_0 \setminus H$  is called the *genus* of  $H$ , which is denoted by  $g(H)$ . In this article  $H$  always stands for a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteq H\},$$

which is called the *conductor* of  $H$ . It is known that  $c(H) \leq 2g(H)$ . A numerical semigroup  $H$  is said to be *symmetric* if  $c(H) = 2g(H)$ . This semigroup has the following symmetric property: For  $\gamma \in \mathbb{N}_0$  we have  $\gamma \notin H$  if and only if  $2g(H) - 1 - \gamma \in H$ . A numerical semigroup  $H$  is said to be *quasi-symmetric* if  $c(H) = 2g(H) - 1$ . We set

$$PF(H) = \{\gamma \in \mathbb{N}_0 \setminus H \mid \gamma + h \in H, \text{ all } h \in H > 0\},$$

whose elements are called *pseudo-Frobenius numbers* of  $H$ . We have  $c(H) - 1 \in PF(H)$ . We set  $t(H) = \#PF(H)$ , which is called the *type* of  $H$ .

**Remark 1.1** We have  $c(H) + t(H) \leq 2g(H) + 1$ . (For example, see [6].)

A numerical semigroup  $H$  is said to be *almost symmetric* if the equality  $c(H) + t(H) = 2g(H) + 1$  holds.

**Remark 1.2** i)  $H$  is symmetric if and only if  $t(H) = 1$ . In this case  $H$  is almost symmetric.  
ii) If  $H$  is quasi-symmetric, then  $t(H) = 2$ . The converse does not hold. In this case  $H$  is also almost symmetric.

iii) If  $c(H) = 2g(H) - 2$ , then  $t(H) = 2$  or  $3$ .

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We set  $PF^*(H) = PF(H) \setminus \{c(H) - 1\}$ .

**Remark 1.3** ([6]) *If  $H$  is almost symmetric, then it has the following symmetric property: The map sending  $\gamma$  to  $c(H) - 1 - \gamma$  induces a bijection on  $PF^*(H)$ . The converse is true.*

A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field  $k$  of characteristic 0. For a pointed curve  $(C, P)$  we denote by  $H(P)$

$$\{\alpha \in \mathbb{N}_0 \mid \exists \text{ a rational function } f \text{ on } C \text{ such that } (f)_\infty = \alpha P\}.$$

Then  $H(P)$  is a numerical semigroup of genus  $g(C)$  where  $g(C)$  is the genus of  $C$ . A numerical semigroup  $H$  is said to be *Weierstrass* if there exists a pointed curve  $(C, P)$  with  $H(P) = H$ .

For any integer  $t \geq 2$  we set

$$d_t(H) = \{h' \in \mathbb{N}_0 \mid th' \in H\},$$

which is a numerical semigroup. In this article we are interested in the case  $t = 3$ . Our main result is the following:

**Theorem 1.4** *For any  $u \geq 1$  there exist infinite sequences*

$$H_0 \xleftarrow{d_3} H_1 \xleftarrow{d_3} H_2 \xleftarrow{d_3} \dots \xleftarrow{d_3} H_{i-1} \xleftarrow{d_3} H_i \xleftarrow{d_3} \dots$$

*of non-Weierstrass numerical semigroups  $H_i$  with  $c(H_i) = 2g(H_i) - (2u - 1)$  and  $t(H_i) = 2u$ , hence,  $H_i$  is almost symmetric for any  $i$ .*

## 2 Non-Weierstrass almost symmetric numerical semigroups

In this section we find numerical semigroups in the starting points of the infinite sequences in Theorem 1.4. For a numerical semigroup  $H$  the least positive integer in  $H$  is denoted by  $m(H)$ , which is called the *multiplicity* of  $H$ . First we state the key lemmas for constructing the numerical semigroups.

**Lemma 2.1** *Let  $u$  be an integer with  $u \geq 1$  and  $H$  be a numerical semigroup. Let  $g$  be an integer with  $g > 4u - 3$ ,  $g \not\equiv u \pmod{3}$  and*

$$g > \max\{m(H) - 2u, u + \frac{3}{2}(c(H) - 1) + \frac{m(H)}{2}, 2u + 2c(H) - 3\}.$$

We set

$$\tilde{H} = 3H \cup \{g + 2u + 3\mathbb{N}_0\} \cup \{2g - 2u - 3r \mid r \in \mathbb{Z} \setminus H\}.$$

Then we have

- i)  $\tilde{H}$  is a numerical semigroup and  $g(\tilde{H}) = g$ .
- ii)  $d_3(\tilde{H}) = H$  and  $c(\tilde{H}) = 2g(\tilde{H}) - (2u - 1)$ .

See [2] for the proof. We give an example which we get by applying the above lemma.

**Example 2.1** In Lemma 2.1 let  $u = 3$ ,  $H = \langle 2, 3 \rangle$  and  $g = 10$ . Then  $\tilde{H}$  is equal to

$$\begin{aligned} & 3\langle 2, 3 \rangle \cup \{10 + 6 + 3\mathbb{N}_0\} \cup \{20 - 6 - 3r \mid r \in \mathbb{Z} \setminus \langle 2, 3 \rangle\} \\ & = \langle 6, 9 \rangle \cup \{16, 19, 22, \dots\} \cup \{11, 17, 20, 23, \dots\} = \langle 6, 9, 11, 16, 19 \rangle. \end{aligned}$$

In this case,  $c(\tilde{H}) = 15 = 2g(\tilde{H}) - 5$ .

**Lemma 2.2** Let  $u$ ,  $H$ ,  $g$  and  $\tilde{H}$  be as in Lemma 2.1. We set  $m = m(H)$ . Then we have

$$PF(\tilde{H}) \subseteq \{g + 2u + 3l - 3m \mid 0 \leq l \leq m - 1\} \cup \{2g - 2u + 3m - 3m\}.$$

Hence,  $t(\tilde{H}) \leq m + 1$ . Moreover,  $s_{\max}(\tilde{H}) = 2g - 2u + 3m$ .

See [2] for the proof.

**Theorem 2.3** Let  $u$ ,  $H$ ,  $g$  and  $\tilde{H}$  be as in Lemma 2.2. Moreover, assume that  $2u - 1 \geq m(H)$  and  $g \geq 2u + 2c(H) - 1$ . Then we have the following:

i)  $t(\tilde{H}) = m(H) + 1$ .

ii) If  $2u - 1 = m(H)$ , then  $c(\tilde{H}) + t(\tilde{H}) = 2g(\tilde{H}) + 1$ , hence  $\tilde{H}$  is almost symmetric

Using Lemmas 2.1 and 2.2 we can prove the above statement. See [2] for the details of the proof.

To construct the desired non-Weierstrass numerical semigroups we need the known facts.

**Remark 2.4** (Oliveira-Stöhr [7]) A numerical semigroup  $\tilde{H}$  satisfies that  $H = d_3(\tilde{H})$  is non-Weierstrass. If  $g(\tilde{H}) \geq 15g(H) + 11$  and  $g(\tilde{H}) \not\equiv 1 \pmod{3}$ , then  $\tilde{H}$  is non-Weierstrass.

**Remark 2.5** (Buchweitz [1] and Komeda [4]) Let  $m$  be an integer with  $m \geq 13$ . Then there exists a non-Weierstrass numerical semigroup  $H$  with  $m(H) = m$ .

**Remark 2.6** (Komeda [3]) Let  $m = 8$  or  $12$ . Then there exists a non-Weierstrass numerical semigroup  $H$  with  $m(H) = m$ .

**Corollary 2.7** Let  $u$  be an integer with  $u \geq 7$ . Then there exists a non-Weierstrass almost symmetric numerical semigroup  $\tilde{H}$  with  $c(\tilde{H}) = 2g(\tilde{H}) - (2u - 1)$ .

**Proof.** Let  $m$  be an odd integer with  $m \geq 13$ . We set  $u = \frac{m+1}{2}$ . Then  $u \geq 7$ . Using Remark 2.5, Lemma 2.1, Remark 2.4 and Theorem 2.3 in this order we get the desired non-Weierstrass numerical semigroups.  $\square$

**Corollary 2.8** Let  $u$  be an integer with  $1 \leq u \leq 6$ . Then there exists a non-Weierstrass almost symmetric numerical semigroup  $\tilde{H}$  with  $c(\tilde{H}) = 2g(\tilde{H}) - (2u - 1)$ .

**Proof.** If  $u = 2, 3, 4$  (resp. 5, resp. 6), then we take a non-Weierstrass 8-semigroup (resp. 12-semigroup, resp. 14-semigroup  $H$ ). We construct a non-Weierstrass numerical semigroup  $\tilde{H}$  with  $t(\tilde{H}) = 2u$  (see Komeda [2] for the details of the proof). In the case  $u = 1$ , the result is due to Oliveira-Stöhr [7].  $\square$

We give an example in the case  $u = 3$ , namely, a non-Weierstrass almost symmetric numerical semigroup  $\tilde{H}$  with  $c(\tilde{H}) = 2g(\tilde{H}) - 5$ .

**Example 2.2** In Corollary 2.8 let  $u = 3$ . Let  $H = \langle 8, 12, 18, 22, 45, 49 \rangle$ , which is a non-Weierstrass numerical semigroup of genus 31 from [3]. Let  $g = 15g(H) + 11 = 15 \times 31 + 11 = 476$ . We set  $\tilde{H} = 3H \cup \{476 + 2 \times 3 + 3\mathbb{N}_0\} \cup \{2 \times 476 - 2 \times 3 - 3r \mid r \in \mathbb{Z} \setminus H\}$ . Then  $PF(\tilde{H}) = \{482 + 3l - 24 \mid l = 3, \dots, 7\} \cup \{952 - 6 + 24 - 24\}$ , i.e.,  $t(\tilde{H}) = 6$ . Hence  $\tilde{H}$  is a non-Weierstrass almost symmetric numerical semigroup of genus 476 with  $c(\tilde{H}) = 2g(\tilde{H}) - 5$ .

### 3 Proof of Theorem 1.4

To prove Theorem 1.4 we need the following Lemma:

**Lemma 3.1** (Komeda [2]) *For any  $t \geq 1$  we can construct an infinite sequence of numerical semigroups  $H_i$  with  $c(H_i) = 2g(H_i) - t$  as follows:*

$$H_0 \xleftarrow{d_3} H_1 \xleftarrow{d_3} H_2 \xleftarrow{d_3} \dots \xleftarrow{d_3} H_{i-1} \xleftarrow{d_3} H_i \xleftarrow{d_3} \dots$$

Here,  $H_0$  is any almost symmetric numerical semigroup with  $c(H_0) = 2g(H_0) - t$  and  $H_i$  is also almost symmetric for any  $i \geq 1$ .

We take  $H_0$  in Lemma 3.1 as the non-Weierstrass almost numerical semigroup  $\tilde{H}$  in Corollaries 2.7 and 2.8. Using Remark 2.4 and the method of construction of Lemma 3.1 (see Komeda [2]) we get Theorem 1.4.

### References

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