Finitely generated semigroups presented by finite congruence classes II

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In this paper, we give a necessary and sufficient condition for one relator semigroups to be presented with finite congruence classes in the case of one relator of a special form. under an assumption .

1 Finitely generated monoid and their presentations

Definition Let X be a finite set of alphabets and R a finite subset of $X^* \times X^*$. Then R is *string-rewriting system*. Define the reduction relation $\Rightarrow_R$ on X^* by $\Rightarrow_R = \{((uw_1v, uw_2v)|u, v \in X^*, (w_1, w_2) \in R\}$. For $u, v \in X^*$, $(w_1, w_2) \in R$, use the denotation : $uw_1v \Rightarrow_R uw_2v$. The congruence μ_R on X^* (or X^+) generated by $\Rightarrow_R$ is called the *Thue* congruence defined by R. A monoid S has a *finite presentation* if there exists a finite set of X, there exists a surjective homomorphism ϕ of X^* to S and there exists a string-rewriting system R consisting of pairs of words over X such that the Thue congruence μ_R is the congruence $\{(w_1, w_2) \in X^* \times X^* \mid \phi(w_1) = \phi(w_2)\}$. Further, if for each $w \in X^*$, the congruence classes $\mu_R(w) = \{w' \in X^* | (w, w') \in \mu_R\}$ is finite, then the monoid $S = X^*/\mu_R$ is called to be *presented by finite congruence classes.* (Refer to [2],[3] and and see [1] for examples)

If $R = \{(u, v)\}$ then we say that R is an *one relator* and S is an *one relator monoid.*

2 The main theorems

First we have

Theorem 1. *Let u, v be word over a finite alphabet X and $R = \{(u, w)\}$ a one-relator rewriting system. Assume that u is an unbordered and the length of u is shorter than one of v . Further,* *assume that u is not a subword of v and v contain at least one letter which u does not contain. Then the relator $R = \{(u, v)\}$ does not generate the congruence such that all of the congruence classes are finite if and only if there exist non-empty words $l_{i,j}$, $r_{i,j}$ over X such that $u = l_{s,t}r_{s,t}(1 \leq s \leq 2k, 1 \leq t \leq i_s)$, $u = l_0 r_0$,*

$v \in X^+ l_{1,i_1} \cdots l_{1,1} l_0,$. $v \in r_{1,i_1-1} X^+ \cap \cdots \cap r_{1,1} X^+$,

$v \in r_{1,i_1} r_{2,1} \cdots r_{2,i_2} X^+$, $\qquad v \in X^+ l_{2,1} \cap \cdots \cap X^+ l_{2,i_2-1}$

$v \in X^+ l_{2k+1,i_{2k+1}} \cdots l_{2k+1,1} l_{2k,i_{2k}}$, $v \in r_{2k+1,i_{2k+1}-1} X^+ \cap \cdots \cap r_{2k+1,1} X^+$,

and $\quad l_{2k+1,i_{2k+1}} = l_{1,i1}$.

Then Theorem 2 follows from Theorem 1.

Theorem 2. *Under the same assumption, the problem of whether one relator monoid $S = X^*/ < (u, v) >$ are presented by finite congruence classes or not is decidable.*

References

[1] P.M. Higgins, *Techniques of semigroup theory*, Oxford Univ. press, 1992.

[2] K. Shoji, *Finitely generated semigroups which have such a presentation that all the congruence classes are regular language,* Math. Japonica, **69**(2008), 73-78.

[3] K. Shoji, *Finitely generated semigroups presented by finite congruence classes*, Suurikaisekikennkyuujo kokyuroku **1809**(2012), 160-170.