Some approximation properties of hypergeometric Bernoulli numbers

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1 Introduction

For a positive integer N, hypergeometric Bernoulli numbers $B_{N,n}$ ([8, 9, 10, 11, 12]) are defined by

$$\frac{1}{{}_1F_1(1;N+1;x)}=\frac{x^N/N!}{e^x-\sum_{n=0}^{N-1}x^n/n!}:=\sum_{n=0}^\infty B_{N,n}\frac{x^n}{n!}\,. \tag{1}$$

where

$${}_1F_1(a;b;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!}$$

is the confluent hypergeometric function and $(x)^{(n)}=x(x+1)\dots(x+n-1)$ $(n\ge 1)$ with $(x)^{(0)}=1$ is the rising factorial. $B_{1,n}=B_n$ are the classical Bernoulli numbers with $B_1=-1/2$.

Many kinds of generalizations of the Bernoulli numbers have been considered by many authors. For example, Poly-Bernoulli number, multiple Bernoulli numbers, Apostol Bernoulli numbers, multi-poly-Bernoulli numbers, degenerated Bernoulli numbers, various types of q-Bernoulli numbers, Bernoulli Carlitz numbers. However, hypergeometric Bernoulli numbers have several advantages. In this paper, we show some relations, which are yielded from continued fraction expansions.

2 Preliminaries

First, we show some basic properties of hypergeometric Bernoulli numbers. From the definition (1),

Proposition 1.

$$\sum_{m=0}^{n}\binom{N+n}{m}B_{N,m}=0\,.$$

Remark. When $N=1$, we have a famous identity for Bernoulli numbers.

$$\sum_{m=0}^{n}\binom{n+1}{m}B_m=0\quad (n\ge 1)\,.$$

By using Proposition 1 or

$$B_{N,n}=-\sum_{k=0}^{n-1}\frac{\binom{N+n}{k}}{\binom{N+n}{n}}B_{N,k} \tag{2}$$

with $B_{N,0}=1$ $(N\ge 1)$, some values of $B_{N,n}$ $(0\le n\le 9)$ are explicitly given by the following.

$$\begin{aligned}
B_{N,0}=&1\,,\\
B_{N,1}=&-\frac{1}{N+1}\,,\\
B_{N,2}=&\frac{2}{(N+1)^2(N+2)}\,,\\
B_{N,3}=&\frac{3!(N-1)}{(N+1)^3(N+2)(N+3)}\,,\\
B_{N,4}=&\frac{4!(N^3-N^2-6N+2)}{(N+1)^4(N+2)^2(N+3)(N+4)}\,,\\
B_{N,5}=&\frac{5!(N-1)(N^3-3N^2-14N+2)}{(N+1)^5(N+2)^2(N+3)(N+4)(N+5)}\,,\\
B_{N,6}=&\frac{6!(N^7-3N^6-49N^5-57N^4+222N^3+264N^2-198N+12)}{(N+5)(N+4)(N+6)(N+3)^2(N+2)^3(N+1)^6}\,,\\
B_{N,7}=&\frac{7!(N-1))(N^7-7N^6-81N^5-37N^4+766N^3+1048N^2-390N+12)}{(N+6)(N+5)(N+4)(N+3)^2(N+2)^3(N+7)(N+1)^7}\,,\\
B_{N,8}=&\frac{8!}{(N+7)(N+6)(N+5)(N+3)^2(N+8)(N+4)^2(N+2)^4(N+1)^8}\\
&\times(N^{11}-8N^{10}-172N^9-354N^8+3265N^7+13498N^6+1164N^5\\
&-46836N^4-23650N^3+38356N^2-6096N+96)\,.
\end{aligned}$$

In general, we have an explicit expression of $B_{N,n}$.

Proposition 2. *For $N, n \ge 1$, we have*

$$B_{N,n} = n! \sum_{k=1}^{n} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} \frac{(-N!)^k}{(N+i_1)!\cdots(N+i_k)!}\,.$$

Hypergeometric Bernoulli numbers can be expressed in terms of determinant.

Theorem 1. *For $N, n \ge 1$, we have*

$$B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & 0 & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix}\,.$$

Remark. When $N = 1$, we have a determinant expression of Bernoulli numbers ([7, p.53]).

This type of continued fractions has many advantages in combinatorial interpretations ([4]), Trudi's formula ([3, 15]), inversion formulas and so on. For example, by inversion formula (see, e.g., [14]), we have the following.

Corollary 1. *For $n \ge 1$*

$$\frac{N!}{(N+n)!} = \begin{vmatrix} -\frac{B_{N,1}}{1!} & 1 & & \\ \frac{B_{N,2}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n B_{N,n}}{n!} & \cdots & \frac{B_{N,2}}{2!} & -\frac{B_{N,1}}{1!} \end{vmatrix}\,.$$

3 Continued fractions of hypergeometric Bernoulli numbers

In [2, 13] by studying the convergents of the continued fraction of

$$\frac{x/2}{\tanh x/2} = \sum_{n=0}^{\infty} B_{2n} \frac{x^{2n}}{(2n)!}\,,$$

some identities of Bernoulli numbers are obtained. In this section, the n-th convergent of the generating function of hypergeometric Bernoulli numbers is explicitly given. As an application, we give some identities of hypergeometric Bernoulli numbers in terms of binomial coefficients.

The generating function on the left-hand side of (1) can be expanded as a continued fraction

$$\frac{1}{{}_1F_1(1;N+1;x)} = 1 - \cfrac{x}{N+1+\cfrac{x}{N+2-\cfrac{(N+1)x}{N+3+\cfrac{2x}{N+4-\cfrac{(N+2)x}{N+5+\ddots}}}}} \tag{3}$$

(*Cf.* [16, (91.2)]). Its n-th convergent $P_n(x)/Q_n(x)$ $(n \geq 0)$ is given by the recurrence relation

$$P_n(x) = a_n(x)P_{n-1}(x) + b_n(x)P_{n-2}(x) \ (n \geq 2), \tag{4}$$
$$Q_n(x) = a_n(x)Q_{n-1}(x) + b_n(x)Q_{n-2}(x) \ (n \geq 2), \tag{5}$$

with initial values

$$P_0(x) = 1, \quad P_1(x) = (N+1) - x;$$
$$Q_0(x) = 1, \quad Q_1(x) = N+1\,,$$

where for $n \geq 1$, $a_n(x) = N+n$, $b_{2n}(x) = nx$ and $b_{2n+1}(x) = -(N+n)x$.

We have explicit expressions of both the numerator and the denominator of the n-th convergent of (3).

Theorem 2. *For $n \geq 1$, we have*

$$P_{2n-1}(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\prod_{l=1}^{2n-j-1}(N+l)\cdot x^j\,,$$
$$P_{2n}(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\prod_{l=1}^{2n-j}(N+l)\cdot x^j$$

and

$$Q_{2n-1}(x)=\sum_{j=0}^{n-1}\sum_{k=0}^{j}(-1)^{j-k}(2n-j-1)_k\binom{n-k-1}{j-k}\prod_{l=k+1}^{2n-j-1}(N+l)\cdot x^j\,,$$

$$Q_{2n}(x)=\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{j-k}(2n-j)_k\binom{n-k-1}{j-k}\prod_{l=k+1}^{2n-j}(N+l)\cdot x^j\,.$$

Remark. Here we use the convenient values

$$\binom{n}{k}=0\ (0\le n<k),\quad \binom{-1}{0}=1$$

and recognize the empty product as 1. Otherwise, we should write $Q_{2n}(x)$ as

$$Q_{2n}(x)=\sum_{j=0}^{n-1}\sum_{k=0}^{j}(-1)^{j-k}(2n-j)_k\binom{n-k-1}{j-k}\prod_{l=k+1}^{2n-j}(N+l)\cdot x^j+n!x^n\,.$$

If we use the unsigned Stirling numbers of the first kind $\left[{n \atop k}\right]$, which generating function is given by

$$\sum_{n=k}^{\infty}(-1)^{n-k}\left[{n \atop k}\right]\frac{z^n}{n!}=\frac{\big(\log(1+z)\big)^k}{k!}\,,$$

we can express the products as

$$\prod_{l=1}^{2n-j-1}(N+l)=\sum_{i=1}^{2n-j}\left[{2n-j \atop i}\right]N^{i-1}$$

or

$$\prod_{l=k+1}^{2n-j-1}(N+l)=\sum_{i=1}^{2n-j-k}\left[{2n-j-k \atop i}\right](N+k)^{i-1}\,.$$

3.1 Some more identities of hypergeometric Bernoulli numbers

Since $P_{2n-1}(x)$, $P_{2n}(x)$ and $Q_{2n}(x)$ are the polynomials with degree n and $Q_{2n-1}(x)$ is the polynomial with degree $n-1$, by the approximation property of the continued fraction, we have the following.

Lemma 1. *Let $P_n(x)/Q_n(x)$ denote the n-th convergent of the continued fraction expansion of (3). Then we have for $n \ge 0$*

$$Q_n(x)\sum_{\kappa=0}^\infty B_{N,\kappa}\frac{x^\kappa}{\kappa!} \equiv P_n(x) \pmod{x^{n+1}}\,.$$

This approximation property implies that the coefficients x^j $(0 \le j \le n)$ of

$$Q_n(x)\sum_{\kappa=0}^\infty B_{N,\kappa}\frac{x^\kappa}{\kappa!} - P_n(x)$$

are nullified. By using the results in Theorem 2, we can show the following identities.

Theorem 3. *We have*

$$\sum_{j=0}^{\min\{h,n\}}\sum_{k=0}^{j}(-1)^{j-k}(2n-j)_k\binom{n-k-1}{j-k}\prod_{l=k+1}^{2n-j}(N+l)\cdot\frac{B_{N,h-j}}{(h-j)!}$$
$$=\begin{cases}(-1)^h\binom{n}{h}\prod_{l=1}^{2n-h}(N+l) & (0\le h\le n);\\ 0 & (h>n)\end{cases} \tag{6}$$

and

$$\sum_{j=0}^{\min\{h,n\}}\sum_{k=0}^{j}(-1)^{j-k}(2n-j-1)_k\binom{n-k-1}{j-k}\prod_{l=k+1}^{2n-j-1}(N+l)\cdot\frac{B_{N,h-j}}{(h-j)!}$$
$$=\begin{cases}(-1)^h\binom{n}{h}\prod_{l=1}^{2n-h-1}(N+l) & (0\le h\le n);\\ 0 & (h>n)\,.\end{cases} \tag{7}$$

When $N=1$, we have the identities for the classical Bernoulli numbers.

Corollary 2. *We have*

$$\sum_{j=0}^{\min\{h,n\}}\sum_{k=0}^{j}(-1)^{j-k}(2n-j)_k\binom{n-k-1}{j-k}\frac{(2n-j+1)!}{(k+1)!(2n-h+1)!}\cdot\frac{B_{h-j}}{(h-j)!}$$

$$= \begin{cases} (-1)^h \binom{n}{h} & (0 \le h \le n); \\ 0 & (h > n) \end{cases} \tag{8}$$

and

$$\sum_{j=0}^{\min\{h,n\}} \sum_{k=0}^{j} (-1)^{j-k} (2n-j-1)_k \binom{n-k-1}{j-k} \frac{(2n-j)!}{(k+1)!(2n-h)!} \cdot \frac{B_{h-j}}{(h-j)!}$$
$$= \begin{cases} (-1)^h \binom{n}{h} & (0 \le h \le n); \\ 0 & (h > n)\,. \end{cases} \tag{9}$$

Remark. Since

$$\sum_{k=0}^{j} (-1)^{j-k} \frac{(2n-j)_k}{(k+1)!} \binom{n-k-1}{j-k}$$
$$= \begin{cases} \dfrac{1}{j+1} \dbinom{n}{j} & \text{if } j \text{ is even;} \\ \dfrac{1}{4j} \dbinom{j-2}{(j-1)/2}^{-1} \dbinom{n-(j+1)/2}{(j-1)/2} \dbinom{n}{(j-1)/2} & \text{if } j \text{ is odd} \ge 3; \\ \dfrac{1}{2} & \text{if } j = 1\,, \end{cases}$$

we can write (8) as

$$\sum_{j=0}^{\lfloor \frac{h}{2} \rfloor} \frac{(2n-2j+1)!}{2j+1} \binom{n}{2j} \frac{B_{h-2j}}{(h-2j)!} + \frac{(2n)!}{2} \frac{B_{h-1}}{(h-1)!}$$
$$+ \sum_{j=1}^{\lfloor \frac{h-1}{2} \rfloor} \frac{(2n-2j)!}{4(2j+1)} \binom{2j-1}{j}^{-1} \binom{n-j-1}{j} \binom{n}{j} \frac{B_{h-2j-1}}{(h-2j-1)!}$$
$$= \begin{cases} (-1)^h \binom{n}{h} (2n-h+1)! & \text{if } 1 \le h \le n; \\ 0 & \text{if } n < h \le 2n+1\,. \end{cases}$$

Since

$$\sum_{k=0}^{j} (-1)^{j-k} \frac{(2n-j-1)_k}{(k+1)!} \binom{n-k-1}{j-k}$$

$$= \begin{cases} \dfrac{\big((j/2)!\big)^2}{(j+1)!}\dbinom{n}{j/2}\dbinom{n-j/2-1}{j/2} & \text{if } j \text{ is even;} \\ 0 & \text{if } j \text{ is odd}\,, \end{cases}$$

we can write (9) as

$$\sum_{j=0}^{\lfloor \frac{h}{2} \rfloor} \frac{(j!)^2(2n-2j)!}{(2j+1)!}\binom{n}{j}\binom{n-j-1}{j}\frac{B_{h-2j}}{(h-2j)!}$$
$$= \begin{cases} (-1)^h\binom{n}{h}(2n-h)! & \text{if } 1 \le h \le n; \\ 0 & \text{if } n < h \le 2n\,. \end{cases}$$

Here the empty summation is recognized as 0, as usual.

3.2 Final comments

In a similar method by continued fractions, we can obtain some identities for Cauchy numbers ([5]).

It is not so difficult to get the identities after finding explicit forms of convergents $P_n(x)/Q_n(x)$. However, it is often fairly hard to find any explicit form of the convergents $P_n(x)/Q_n(x)$ for general n.

For example, in [2, 6] the following continued fraction expansion for non-exponential Bernoulli numbers is given.

$$\sum_{n=1}^{\infty} B_{2n}(4x)^n = \cfrac{x}{1+\frac{1}{2}+\cfrac{x}{\frac{1}{2}+\frac{1}{3}+\cfrac{x}{\frac{1}{3}+\frac{1}{4}+\cfrac{x}{\ddots}}}}\,. \tag{10}$$

Its convergents can be calculated as

$$\frac{P_0(x)}{Q_0(x)} = \frac{0}{1}, \quad \frac{P_1(x)}{Q_1(x)} = \frac{x}{3/2}, \quad \frac{P_2(x)}{Q_2(x)} = \frac{(5x)/6}{(5+4x)/4}\,,$$
$$\frac{P_3(x)}{Q_3(x)} = \frac{x(35+72x)/72}{5(7+20x)/48}, \quad \frac{P_4(x)}{Q_4(x)} = \frac{7x(15+88x)/480}{(21+140x+64x^2)/64}\,,$$
$$\frac{P_5(x)}{Q_5(x)} = \frac{x(385+4592x+4800x^2)/4800}{7(11+140x+224x^2)/640}\,,$$

$$\frac{P_6(x)}{Q_6(x)}=\frac{x(5005+103796x+321120x^2)/201600}{(143+3080x+11312x^2+3840x^3)/3840},$$
$$\frac{P_7(x)}{Q_7(x)}=\frac{x(25025+820820x+5205728x^2+3763200x^3)/3763200}{(715+24024x+166320x^2+194816x^3)/71680},$$
$$\frac{P_8(x)}{Q_8(x)}=\frac{11x(7735+376012x+4145440x^2+9010176x^3)/54190080}{(2431+120120x+1393392x^2+3703040x^3+1032192x^4)/1032192}.$$

But no explicit form of $P_n(x)$ or $Q_n(x)$ has been known yet.

Nevertheless, we can see several relations. For example, for $n=7$, we see that all the coefficients up to x^7 are coincided in

$$\frac{P_7(x)}{Q_7(x)}=\frac{2}{3}x-\frac{8}{15}x^2+\frac{32}{21}x^3-\frac{128}{15}x^4+\frac{2560}{33}x^5-\frac{1415168}{1365}x^6+\frac{57344}{3}x^7$$
$$-\frac{129511227392}{306735}x^8+\frac{21520583360512}{2147145}x^9-\cdots$$

and

$$\sum_{n=1}^\infty B_{2n}(4x)^n=\frac{2}{3}x-\frac{8}{15}x^2+\frac{32}{21}x^3-\frac{128}{15}x^4+\frac{2560}{33}x^5-\frac{1415168}{1365}x^6+\frac{57344}{3}x^7$$
$$-\frac{118521856}{255}x^8+\frac{5749735424}{399}x^9-\cdots.$$

Then,

$$\frac{4\cdot 715}{71680}B_2=\frac{143}{21504}=\frac{25025}{3763200},$$
$$\frac{4\cdot 24024}{71680}B_2+\frac{4^2\cdot 715}{71680}B_4=\frac{5863}{26880}=\frac{820820}{3763200},$$
$$\frac{4\cdot 166320}{71680}B_2+\frac{4^2\cdot 24024}{71680}B_4+\frac{4^3\cdot 715}{71680}B_6=\frac{162679}{117600}=\frac{5205728}{3763200},$$
$$\frac{4^n\cdot 194816B_{2n}+4^{n+1}\cdot 166320B_{2n+2}+4^{n+2}\cdot 24024B_{2n+4}+4^{n+3}\cdot 715B_{2n+6}}{71680}$$
$$=\begin{cases}1 & (n=1);\\ 0 & (n=2,3,4).\end{cases}$$

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