Algebraic independence of the values of a certain map defined on the set of orbits of the action of Klein four-group

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1 Introduction

Let \( \{ R_k \}_{k \geq 1} \) be a linear recurrence of positive integers satisfying

\[
R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 1),
\]

where \( n \geq 2 \) and \( c_1, \ldots, c_n \) are nonnegative integers with \( c_n \neq 0 \). The author [9] studied the two-variable function \( E(x, q) \) defined by

\[
E(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x q^{R_l}}{1 - q^{R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{R_1})(1-q^{R_2}) \cdots (1-q^{R_k})},
\]

which may be regarded as an analogue of \( q \)-exponential function

\[
E_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k q^{1+2+\cdots+k}}{(1-q)(1-q^2) \cdots (1-q^k)}
\]

(cf. Gasper and Rahman [2]), if we replace \( k \) in the exponent of \( q \) in \( E_q(x) \) with \( \{ R_k \}_{k \geq 1} \) defined above.

Let

\[
\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n
\]

(2)

and let \( \overline{\mathbb{Q}}^x \) be the set of nonzero algebraic numbers. The author proved the following

**Theorem 0** (Corollary 4 of [9]). Let \( \{ R_k \}_{k \geq 1} \) be a linear recurrence satisfying (1). Suppose that \( \Phi(\pm 1) \neq 0 \) and the ratio of any pair of distinct roots of \( \Phi(X) \) is not a root of unity. Assume that \( \{ R_k \}_{k \geq 1} \) is not a geometric progression. Then the values

\[
E(x, q) \quad (x, q \in \overline{\mathbb{Q}}^x, \ |q| < 1)
\]

are algebraically dependent if and only if there exist some distinct pairs \( (x_1, q_1) \) and \( (x_2, q_2) \) of nonzero algebraic numbers \( x_1, x_2, q_1, \) and \( q_2 \) with \( |q_1|, |q_2| < 1 \) such that \( x_1 = x_2 \) and \( q_1^{N_k} = q_2^{N_k} \) for some \( k \geq 1 \), where \( N_k = \text{g.c.d.}(R_k, R_{k+1}, \ldots, R_{k+n-1}) \).

In particular, if \( N_k = 1 \) for any \( k \geq 1 \), then the values \( E(x, q) \) are algebraically independent for any distinct pairs \( (x, q) \) of nonzero algebraic numbers \( x \) and \( q \) with \( |q| < 1 \).
Example 0. Let \( \{F_k\}_{k \geq 1} \) be the sequence of Fibonacci numbers defined by \( F_1 = 1, F_2 = 1, \) and \( F_{k+2} = F_{k+1} + F_k \) \( (k \geq 1) \). Since \( \{F_k\}_{k \geq 1} \) satisfies the conditions in Theorem 0, the infinite set of the values
\[
\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{F_1 + F_2 + \cdots + F_k}}{(1 - q^{F_1})(1 - q^{F_2}) \cdots (1 - q^{F_k})} \bigg| x, q \in \overline{\mathbb{Q}}^\times, \ |q| < 1 \right\}
\]
is algebraically independent.

The two-variable function \( E(x, q) \) converges on the domain
\[
(\mathbb{C} \times \{ |q| < 1 \}) \cup \left( \{ |x| < 1 \} \times \{ |q| > 1 \} \right) := \{(x, q) \in \mathbb{C}^2 \mid |q| < 1 \lor (|x| < 1 \land |q| > 1)\},
\]
whereas a ‘balanced’ analogue
\[
\Theta(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x q^{R_l}}{1 - q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1 + R_2 + \cdots + R_k}}{(1 - q^{2R_1})(1 - q^{2R_2}) \cdots (1 - q^{2R_k})}
\]
converges on the wider domain
\[
\mathbb{C} \times \{|q| \neq 1\} := \{(x, q) \in \mathbb{C}^2 \mid |q| \neq 1\}.
\]
Indeed, if \( q \neq 0, \Theta(x, q) \) is invariant under the map
\[
\sigma_1 : (x, q) \mapsto (-x, q^{-1}),
\]
namely
\[
\Theta(\sigma_1(x, q)) = \sum_{k=1}^{\infty} \frac{(-x)^k q^{-R_1 - R_2 - \cdots - R_k}}{(1 - q^{-2R_1})(1 - q^{-2R_2}) \cdots (1 - q^{-2R_k})} = \Theta(x, q)
\]
and so \( \Theta(x, q) \) converges on \( \mathbb{C} \times \{|q| \neq 1\} \) by the similar reason to the convergence of \( E(x, q) \).

Moreover, if \( \{R_k\}_{k \geq 1} \) is a sequence of odd integers, then \( \Theta(x, q) \) is invariant also under the maps
\[
\sigma_2 : (x, q) \mapsto (-x, -q),
\]
\[
\sigma_3 : (x, q) \mapsto (x, -q^{-1}).
\]
Since \( \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \text{id} \) and \( \sigma_1 \circ \sigma_3 = \sigma_3 \circ \sigma_1 = \sigma_3 \), we see that \( G_4 = \{\text{id}, \sigma_1, \sigma_2, \sigma_3\} \) is Klein four-group. Therefore, \( \Theta(x, q) \) can be regarded as a map defined on the set of orbits \( (\mathbb{C} \times \{|q| \neq 0, 1\})/G_4 \), where \( \mathbb{C} \times \{|q| \neq 0, 1\} = \{(x, q) \in \mathbb{C}^2 \mid |q| \neq 0, 1\} \), namely the map
\[
\overline{\Theta} : (\mathbb{C} \times \{|q| \neq 0, 1\})/G_4 \longrightarrow \Theta(\mathbb{C} \times \{|q| \neq 0, 1\})
\]
given by
\[
\text{the orbit of } (x, q) \mapsto \Theta(x, q)
\]
is well-defined. Hence the restriction to algebraic points
\[
\overline{\Theta} : \left( (\mathbb{C} \times \{|q| \neq 0, 1\}) \cap \left( \overline{\mathbb{Q}}^\times \right)^2 \right)/G_4 \longrightarrow \Theta \left( (\mathbb{C} \times \{|q| \neq 0, 1\}) \cap \left( \overline{\mathbb{Q}}^\times \right)^2 \right),
\]
or equivalently
\[ \overline{\Theta} : \left( \overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{ |q| = 1 \}) \right) / G_4 \rightarrow \Theta \left( \overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{ |q| = 1 \}) \right) \]

is also well-defined, where the second $\overline{\mathbb{Q}}^\times$ denotes the multiplicative group of nonzero algebraic numbers while the first $\overline{\mathbb{Q}}^\times$ simply denotes the set of nonzero algebraic numbers. In this paper we prove the following

**Theorem 1.** Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1). Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\text{g.c.d.}(R_k, R_{k+1}, \ldots, R_{k+n-1}) = 1$ for any $k \geq 1$. Assume further that $\Phi(2) < 0$ and that $\{R_k\}_{k \geq 1}$ is a sequence of odd integers. Then the infinite set of the values
\[ \overline{\Theta} \left( \left( \overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{ |q| = 1 \}) \right) / G_4 \right) \]
is algebraically independent.

**Remark 1.** The condition that $\text{g.c.d.}(R_k, R_{k+1}, \ldots, R_{k+n-1}) = 1$ for any $k \geq 1$ implies that the sequence $\{R_k\}_{k \geq 1}$ is not a geometric progression.

**Corollary 1.** Let $\{R_k\}_{k \geq 1}$ be as in Theorem 1. Then the infinite set consisting of the distinct values of
\[ \left\{ \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\cdots(1-q^{2R_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\} \]
is algebraically independent.

**Example 1.** Let $\{P_k\}_{k \geq 1}$ be the sequence defined either by $P_1 = P_2 = 1$ and $P_{k+2} = 2P_{k+1} + P_k$ ($k \geq 1$) or by $P_1 = P_2 = P_3 = 1$ and $P_{k+3} = P_{k+2} + P_{k+1} + 3P_k$ ($k \geq 1$). Since $\{P_k\}_{k \geq 1}$ satisfies all the conditions of Theorem 1, the infinite set consisting of the distinct values of
\[ \left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1+P_2+\cdots+P_k}}{(1-q^{2P_1})(1-q^{2P_2})\cdots(1-q^{2P_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\} \]
is algebraically independent.

If $\{R_k\}_{k \geq 1}$ is a sequence of odd integers, then
\[ \Theta_4(x, q) := \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x q^{R_l}}{1 + q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\cdots+R_k}}{(1+q^{2R_1})(1+q^{2R_2})\cdots(1+q^{2R_k})} \]
is invariant under the maps
\[ \tau_1 : (x, q) \mapsto (x, q^{-1}), \]
\[ \tau_2 : (x, q) \mapsto (-x, -q), \]
\[ \tau_3 : (x, q) \mapsto (-x, -q^{-1}). \]
Since \( \tau_1 \circ \tau_1 = \tau_2 \circ \tau_2 = \text{id} \) and \( \tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \tau_3 \), we see that \( G'_4 = \{ \text{id}, \tau_1, \tau_2, \tau_3 \} \) is also Klein four-group. Hence the map

\[
\Theta_+ : (\mathbb{C} \times \{ |q| \neq 0, 1 \})/G'_4 \rightarrow \Theta_+ (\mathbb{C} \times \{ |q| \neq 0, 1 \})
\]

given by

\[
\text{the orbit of } (x, q) \mapsto \Theta_+ (x, q)
\]
is well-defined. We also have the following

**Theorem 2.** Let \( \{ R_k \}_{k \geq 1} \) be as in Theorem 1. Then the infinite set of the values

\[
\tilde{\Theta}_+ \left( \left( \mathbb{Q}^\times \times (\mathbb{Q}^\times \setminus \{ |q| = 1 \}) \right) / G'_4 \right)
\]
is algebraically independent.

**Example 2.** Let \( \{ P_k \}_{k \geq 1} \) be one of the linear recurrences defined in Example 1. Since \( \{ P_k \}_{k \geq 1} \) satisfies all the conditions of Theorem 1, the infinite set consisting of the distinct values of

\[
\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1 + P_2 + \ldots + P_k}}{(1 + q^{2P_1})(1 + q^{2P_2}) \ldots (1 + q^{2P_k})} \right| x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \}
\]
is algebraically independent.

## 2 Lemmas

Let \( F(z_1, \ldots, z_n) \) and \( F[[z_1, \ldots, z_n]] \) denote the field of rational functions and the ring of formal power series in variables \( z_1, \ldots, z_n \) with coefficients in a field \( F \), respectively, and \( F^\times \) the multiplicative group of nonzero elements of \( F \). Let \( \Omega = (\omega_{ij}) \) be an \( n \times n \) matrix with nonnegative integer entries. Then the maximum \( \rho \) of the absolute values of the eigenvalues of \( \Omega \) is itself an eigenvalue (cf. Gantmacher [1, p. 66, Theorem 3]). If \( z = (z_1, \ldots, z_n) \) is a point of \( \mathbb{C}^n \), we define a transformation \( \Omega : \mathbb{C}^n \rightarrow \mathbb{C}^n \) by

\[
\Omega z = \left( \prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right).
\]

We suppose that \( \Omega \) and an algebraic point \( \alpha = (\alpha_1, \ldots, \alpha_n) \), where \( \alpha_i \) are nonzero algebraic numbers, have the following four properties:

(I) \( \Omega \) is nonsingular and none of its eigenvalues is a root of unity, so that in particular \( \rho > 1 \).

(II) Every entry of the matrix \( \Omega^k \) is \( O(\rho^k) \) as \( k \) tends to infinity.

(III) If we put \( \Omega^k \alpha = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}) \), then

\[
\log |\alpha_i^{(k)}| \leq -c \rho^k \quad (1 \leq i \leq n)
\]

for all sufficiently large \( k \), where \( c \) is a positive constant.
(IV) For any nonzero \( f(z) \in \mathbb{C}[[z_1, \ldots, z_n]] \) which converges in some neighborhood of the origin, there are infinitely many positive integers \( k \) such that \( f(\Omega^k \alpha) \neq 0 \).

**Lemma 1** (Lemma 4 and Proof of Theorem 2 in [6]). Suppose that \( \Phi(\pm1) \neq 0 \) and the ratio of any pair of distinct roots of \( \Phi(X) \) is not a root of unity, where \( \Phi(X) \) is the polynomial defined by (2). Let

\[
\Omega = \begin{pmatrix}
  c_1 & 1 & 0 & \ldots & 0 \\
  c_2 & 0 & 1 & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & 0 & \vdots \\
  c_n & 0 & \ldots & \ldots & 0
\end{pmatrix}
\]

(4)

and let \( \beta_1, \ldots, \beta_s \) be multiplicatively independent algebraic numbers with \( 0 < |\beta_j| < 1 \) (1 \( \leq \) \( j \leq \) \( s \)). Let \( p \) be a positive integer and put \( \Omega' = \text{diag}(\Omega^p, \ldots, \Omega^p) \). Then the matrix \( \Omega' \) and the point \( \beta = (1, \ldots, 1, \beta_1, \ldots, \ldots, 1, \beta_s) \) have the properties (I)–(IV).

**Lemma 2** (Kubota [3], see also Nishioka [5]). Let \( K \) be an algebraic number field. Suppose that \( f_1(z), \ldots, f_m(z) \in K[[z_1, \ldots, z_n]] \) converge in an \( n \)-polydisc \( U \) around the origin and satisfy the functional equations

\[
f_i(z) = a_i(z)f_i(\Omega z) + b_i(z) \quad (1 \leq i \leq m),
\]

where \( a_i(z), b_i(z) \in K(z_1, \ldots, z_n) \) and \( a_i(z) \) (1 \( \leq \) \( i \leq \) \( m \)) are defined and nonzero at the origin. Assume that the \( n \times n \) matrix \( \Omega \) and a point \( \alpha \in U \) whose components are nonzero algebraic numbers have the properties (I)–(IV) and that \( a_i(z) \) (1 \( \leq \) \( i \leq \) \( m \)) are defined and nonzero at \( \Omega^k \alpha \) for any \( k \geq 1 \). If \( f_1(z), \ldots, f_m(z) \) are algebraically independent over \( K(z_1, \ldots, z_n) \), then the values \( f_1(\alpha), \ldots, f_m(\alpha) \) are algebraically independent.

In what follows, \( C \) denotes a field of characteristic 0. Let \( L = C(z_1, \ldots, z_n) \) and let \( M \) be the quotient field of \( C[[z_1, \ldots, z_n]] \). Let \( \Omega \) be an \( n \times n \) matrix with nonnegative integer entries having the property (I). We define an endomorphism \( \tau : M \to M \) by \( f^\tau(z) = f(\Omega z) \) (\( f(z) \in M \)) and a subgroup \( H \) of \( L^\times \) by

\[
H = \{ g^\tau g^{-1} \mid g \in L^\times \}.
\]

**Lemma 3** (Kubota [3], see also Nishioka [5]). Let \( f_{ij} \in M \) (\( i = 1, \ldots, h; \ j = 1, \ldots, m(i) \)) satisfy

\[
f_{ij} = a_i f_{ij} + b_{ij},
\]

where \( a_i \in L^\times, \ b_{ij} \in L \) (1 \( \leq \) \( i \leq \) \( h, \ 1 \leq \) \( j \leq m(i) \)), and \( a_i a_j^{-1} \notin H \) for any distinct \( i, i' \) (1 \( \leq \) \( i, i' \leq \) \( h \)). Suppose for any \( i \) (1 \( \leq \) \( i \leq \) \( h \)) there is no element \( g \) of \( L \) satisfying

\[
g = a_i g^\tau + \sum_{j=1}^{m(i)} c_j b_{ij},
\]

where \( c_1, \ldots, c_{m(i)} \in C \) are not all zero. Then the functions \( f_{ij} \) (\( i = 1, \ldots, h; \ j = 1, \ldots, m(i) \)) are algebraically independent over \( L \).
Let \( \{R_k\}_{k \geq 1} \) be a linear recurrence satisfying (1) and define a monomial
\[
M(z) = z_1^{R_1} \cdots z_n^{R_n},
\]
which is denoted similarly to (3) by
\[
M(z) = (R_n, \ldots, R_1)z.
\]
Let \( \Omega \) be the matrix defined by (4). It follows from (1), (3), and (6) that
\[
M(\Omega^k z) = z_1^{R_{k+n}} \cdots z_n^{R_{k+1}} \quad (k \geq 0).
\]

**Lemma 4** (Theorem 2 of [7]). Suppose that \( \{R_k\}_{k \geq 1} \) is not a geometric progression. Assume that \( \Phi(\pm 1) \neq 0 \) and the ratio of any pair of distinct roots of \( \Phi(X) \) is not a root of unity. Let \( \overline{C} \) be an algebraically closed field of characteristic 0. Suppose that \( F(z) \) is an element of the quotient field of \( \overline{C}[[z_1, \ldots, z_n]] \) satisfying the functional equation of the form
\[
F(z) = \left( \prod_{k=u}^{p+u-1} Q_k(M(\Omega^k z)) \right) F(\Omega^p z),
\]
where \( \Omega \) is defined by (4), \( p > 0, u \geq 0 \) are integers, and \( Q_k(X) \in \overline{C}(X) \) \( u \leq k \leq p + u - 1 \) are defined and nonzero at \( X = 0 \). If \( F(z) \in \overline{C}(z_1, \ldots, z_n) \), then \( F(z) \in \overline{C}^\times \) and \( Q_k(X) \in \overline{C}^\times \) \( u \leq k \leq p + u - 1 \).

We adopt the usual vector notation, that is, if \( I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n \) with \( \mathbb{Z}_{\geq 0} \) the set of nonnegative integers, we write \( z^I = z_1^{i_1} \cdots z_n^{i_n} \). We denote by \( C[z_1, \ldots, z_n] \) the ring of polynomials in variables \( z_1, \ldots, z_n \) with coefficients in \( C \).

**Lemma 5** (Lemma 3.2.3 in Nishioka [5]). If \( A, B \in C[z_1, \ldots, z_n] \) are coprime, then \( \text{g.c.d.}(A^r, B^r) = z^I \), where \( I \in \mathbb{Z}_{\geq 0}^n \).

**Lemma 6** (Lemma 12 of [7]). Let \( \Omega \) be an \( n \times n \) matrix with nonnegative integer entries which has the property (I). Let \( R(z) \) be a nonzero polynomial in \( C[z_1, \ldots, z_n] \). If \( R(\Omega z) \) divides \( R(z)z^I \), where \( I \in \mathbb{Z}_{\geq 0}^n \), then \( R(z) \) is a monomial in \( z_1, \ldots, z_n \).

**Lemma 7** (Lemma 6 of [8]). Let \( P(z) \) be a nonconstant polynomial in \( z = (z_1, \ldots, z_n) \) with \( n \geq 2 \). Let \( \Omega \) be an \( n \times n \) matrix with positive integer entries which has the property (I). Then
\[
\deg_z P(\Omega z) > \deg_z P(z).
\]

## 3 Proof of the main theorem

We prove only Theorem 1, since Theorem 2 is proved in the same way.

**Proof of Theorem 1.** A complete set of representatives of the orbits \( \left( \overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{ |q| = 1 \}) \right) / G_4 \) is given by
\[
\left\{ (x, q) \in (\overline{\mathbb{Q}}^\times)^2 \left| |q| < 1, \ 0 \leq \text{Arg} q < \pi \right. \right\} =: \Lambda
\]
since, under the action of the Klein four-group $G_4$, the second component $q$ is transformed either to $q$, $q^{-1}$, $-q$, or $-q^{-1}$. Hence it is enough to prove that the values

$$
\eta_i := \Theta(x_i, q) = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}} \quad (i = 1, \ldots, r)
$$

are algebraically independent for any finite number of distinct pairs $(x_1, q_1), (x_2, q_2), \ldots, (x_r, q_r)$ belonging to $\Lambda$.

Assume that the values $\eta_1, \ldots, \eta_r$ are algebraically dependent. There exist multiplicatively independent algebraic numbers $\beta_1, \ldots, \beta_s$ with $0 < |\beta_j| < 1$ ($1 \leq j \leq s$) and a primitive $N$-th root of unity $\zeta$ such that

$$q_i = \zeta^{m_i} \prod_{j=1}^{s} \beta_j^{e_{ij}} \quad (1 \leq i \leq r), \quad (8)$$

where $m_1, \ldots, m_s$ are integers with $0 \leq m_i \leq N - 1$ and $e_{ij}$ ($1 \leq i \leq r$, $1 \leq j \leq s$) are nonnegative integers (cf. Loxton and van der Poorten [4], Nishioka [5]). We can choose a positive integer $p$ and a sufficiently large integer $u$, which will be determined later, such that $R_{k+p} = R_k \pmod{N}$ for any $k \geq u + 1$. Let $y_{j,m}$ ($1 \leq j \leq s$, $1 \leq m \leq n$) be variables and let $y_j = (y_{j,1}, \ldots, y_{j,n})$ ($1 \leq j \leq s$), $y = (y_1, \ldots, y_s)$. Define

$$f_i(y) = \sum_{k=u}^{\infty} \prod_{l=1}^{k} \frac{x_i^{m_i R_{l+1}} \prod_{j=1}^{n} M(\Omega^l y_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^{s} M(\Omega^l y_j)^{e_{ij}}\right)^2} \quad (1 \leq i \leq r),$$

where $M(z)$ and $\Omega$ are defined by (5) and (4), respectively. Letting

$$\beta = (\beta_1, \ldots, \beta_s),$$

we see by (7) and (8) that

$$f_i(\beta) = \sum_{k=u}^{\infty} \prod_{l=1}^{k} \frac{x_i q_i^{R_{l+1}}}{1 - q_i^{2R_{l+1}}} = \sum_{k=u+1}^{\infty} \prod_{l=u+1}^{k} \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}}$$

and so

$$\eta_i = \left(\prod_{k=1}^{u} \frac{x_i q_i^{R_k}}{1 - q_i^{2R_k}}\right) f_i(\beta) + \sum_{k=1}^{u} \prod_{l=1}^{k} \frac{x_i q_i^{R_l}}{1 - q_i^{2R_l}}.$$

Since $\eta_1, \ldots, \eta_r$ are algebraically dependent, so are $f_i(\beta)$ ($1 \leq i \leq r$). Let

$$\Omega' = \operatorname{diag}(\Omega^p, \ldots, \Omega^p).$$

Then each $f_i(y)$ satisfies the functional equation

$$f_i(y) = \left(\prod_{k=u}^{p+u-1} \frac{x_i^{m_i R_{k+1}} \prod_{j=1}^{n} M(\Omega^k y_j)^{e_{ij}}}{1 - \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k y_j)^{e_{ij}}\right)^2}\right) f_i(\Omega' y)$$
\[ + \sum_{k=u}^{p+u-1} \prod_{l=u}^{k-1} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{e_{ij}}}{1 - \left( \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{e_{ij}} \right)^2}, \]

where \( \Omega' y = (\Omega^p y_1, \ldots, \Omega^p y_s) \). Let \( D = |\det(\Omega - E)| = |\Phi(1)| \), where \( E \) is the identity matrix. Then \( D \) is a positive integer, since \( \Phi(1) \neq 0 \). Let \( y_{j_1}^{1/D} \) \((1 \leq j < s, 1 \leq \lambda < n)\), \( y_{j_1}^{1/D} = (y_{j_1}, \ldots, y_{j_1}^{s}) \) \((1 \leq j \leq s)\), and \( y' = (y_{1}', \ldots, y_{s}') \). Noting that \( \prod_{j=1}^{s} M(\Omega - E)^{-1} \Omega^u y_j)^{e_{ij}} \in \mathcal{Q}(y') \), we define

\[
g_i(y') = \left( \prod_{j=1}^{s} M(\Omega - E)^{-1} \Omega^u y_j)^{e_{ij}} \right) f_i(y) - T_i(y') \quad (1 \leq i \leq r),
\]

where

\[
f_i(y') = \sum_{k=u}^{\infty} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}}}{1 - \left( \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}} \right)^2} \in \mathcal{Q}(y'),
\]

\[
T_i(y') = \left( \prod_{j=1}^{s} M(\Omega - E)^{-1} \Omega^u y_j)^{e_{ij}} \right) \sum_{k=u}^{k_1} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}}}{1 - \left( \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}} \right)^2} \in \mathcal{Q}(y'),
\]

and \( k_1 \) is such a large integer that \( g_i(y') \in \mathcal{Q}(y') \) \((1 \leq i \leq r)\). Since \( M(\Omega - E)^{-1} \Omega^u y_j)^{D} = M(\Omega - E)^{-1} \Omega^{u+p} y_j)^{D} \), each \( g_i(y') \) satisfies the functional equation

\[
g_i(y') = \left( \prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}}}{1 - \left( \zeta^{m_i R_{k+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}} \right)^2} \right) g_i(\Omega' y')
\]

\[
+ \left( \prod_{j=1}^{s} M(\Omega - E)^{-1} \Omega^u y_j)^{e_{ij}} \right) \sum_{k=u}^{p+u-1} \prod_{l=u}^{k} \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}}}{1 - \left( \zeta^{m_i R_{l+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}} \right)^2} \]

\[
+ \left( \prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}}}{1 - \left( \zeta^{m_i R_{k+1}} \prod_{j=1}^{l} M(\Omega^i y_j)^{D e_{ij}} \right)^2} \right) T_i(\Omega' y') - T_i(y'),
\]

where \( \Omega' y' = (\Omega^p y_1', \ldots, \Omega^p y_s') \). Since \( f_i(\beta) \) \((1 \leq i \leq r)\) are algebraically dependent, so are \( g_i(\beta') \) \((1 \leq i \leq r)\), where

\[ \beta' = (1, \ldots, 1, \beta_1^{1/D}, \ldots, 1, \ldots, 1, \beta_s^{1/D}). \]

By Lemma 1, the matrix \( \Omega' \) and \( \beta' \) have the properties (I)–(IV). By Lemma 2, the functions \( g_i(y') \) \((1 \leq i \leq r)\) are algebraically dependent over \( \mathcal{Q}(y') \).
In order to apply Lemma 3, we assert that

\[
Q_{uv}(y') = \prod_{k=u}^{p+u-1} x_i \zeta^{m_i R_{k+1}} \left( 1 - \left( \zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k y_j)^{D_{\nu_j}} \right)^2 \right)
\]

\[
\in H = \left\{ \frac{h(\Omega^k y')}{h(y')} \mid h(y') \in \mathcal{Q}(y') \setminus \{0\} \right\}
\]

if and only if \( m_i = m_{\nu}, (e_{i1}, \ldots, e_{i8}) = (e_{\nu1}, \ldots, e_{\nu8}), \) and \( x_{\nu} = x_{\nu} \). It is clear that, if \( m_i = m_{\nu}, (e_{i1}, \ldots, e_{i8}) = (e_{\nu1}, \ldots, e_{\nu8}), \) and \( x_{\nu} = x_{\nu} \), then \( Q_{uv}(y') = 1 \in H \). Conversely, suppose that \( Q_{uv}(y') \in H \). Then there exists an \( F(y') \in \mathcal{Q}(y') \setminus \{0\} \) satisfying

\[
F(y') = \left( \prod_{k=u}^{p+u-1} x_i \zeta^{m_i R_{k+1}} \left( 1 - \left( \zeta^{m_i R_{k+1}} \prod_{j=1}^{s} M(\Omega^k y_j)^{D_{\nu_j}} \right)^2 \right) \right)^{-1} \quad (9)
\]

Let \( P \) be a positive integer divisible by \( D \) and let

\[
y_j = (y_{j1}, \ldots, y_{jn}) = (z_{j1}^{p_j/D}, \ldots, z_{jn}^{p_j/D}) \quad (1 \leq j \leq s).
\]

We choose a sufficiently large \( P \) such that the following two properties are both satisfied:

(a) If \( (e_{i1}, \ldots, e_{i8}) \neq (e_{\nu1}, \ldots, e_{\nu8}), \) then \( \sum_{j=1}^{s} e_{ij} p_j \neq \sum_{j=1}^{s} e_{\nu j} p_j \).

(b) \( F^*(z) = F(z_{1}^{p_1/D}, \ldots, z_{n}^{p_1/D}, \ldots, z_{1}^{p_s/D}, \ldots, z_{n}^{p_s/D}) \in \mathcal{Q}(z_1, \ldots, z_n) \setminus \{0\} \).

Then by (9), \( F^*(z) \) satisfies the functional equation

\[
F^*(z) = \left( \prod_{k=u}^{p+u-1} x_i \zeta^{m_i R_{k+1}} \left( 1 - \left( \zeta^{m_i R_{k+1}} M(\Omega^k z)^{D_{\nu_j}} \right)^2 \right) \right)^{-1} \quad (10)
\]

where \( \ell_i = \sum_{j=1}^{s} e_{ij} p_j \) (\( 1 \leq i \leq r \)). Therefore by Lemma 4 we see that

\[
x_i \zeta^{m_i R_{k+1}} \left( 1 - \zeta^{2m_i R_{k+1}} X^2 \zeta^{2\ell_i} \right) \in \mathcal{Q}^X
\]

for any \( k \) (\( u \leq k \leq p + u - 1 \)), where \( X \) is a variable, and \( F^*(z) \in \mathcal{Q}^X \). Hence \( \ell_i = \ell_{\nu} \) and \( \zeta^{2m_i R_{k+1}} = \zeta^{2m_{\nu} R_{k+1}} \) (\( u \leq k \leq p + u - 1 \)). Thus \( (e_{i1}, \ldots, e_{i8}) = (e_{\nu1}, \ldots, e_{\nu8}) \) by the property (a), and \( \zeta^{2m_i} = \zeta^{2m_{\nu}} \) since \( g.c.d.(R_k, R_{k+1}, \ldots, R_{k+p-1}) = 1 \) for any \( k \geq 1 \). Hence \( q_{\nu} = q_{\nu} \) by (8) and so \( q_i = q_{\nu} \) since \( 0 \leq \text{Arg } q_i < \pi \) (\( 1 \leq i \leq r \)). Then \( m_i = m_{\nu} \), and the functional equation (10) becomes \( x_i F^*(z) = x_{\nu} F^*(\Omega^p z) \). Since \( F^*(z) \in \mathcal{Q}^X \), we have \( x_i = x_{\nu} \), and the assertion is proved.

Now let \( S \) be a maximal subset of \( \{1, \ldots, r\} \) such that \( (x_i, q_i) = (x_{\nu}, q_{\nu}) \) for any \( i, \nu \in S \), which is equivalent to \( x_i = x_{\nu}, m_i = m_{\nu}, \) and \( (e_{i1}, \ldots, e_{i8}) = (e_{\nu1}, \ldots, e_{\nu8}) \). Fix a \( \lambda \in S \) and let \( \xi = x_{\lambda}, m = m_{\lambda}, \) and \( e_j = e_{\lambda j} \) (\( 1 \leq j \leq s \)). Then \( x_i = \xi, m_i = m, \) and \( (e_{i1}, \ldots, e_{i8}) = (e_{1}, \ldots, e_{s}) \) for any \( i \in S \) and by Lemma 3 there exits a \( G(y') \in \mathcal{Q}(y') \) satisfying
$$G(y') = \xi \left( \prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}}}{1 - \left( \zeta^{mR_{k+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{ej}} \right)^2} \right) G(\Omega^p y')$$

$$+ \left( \prod_{j=1}^{s} M(D(\Omega - E)^{-1} \Omega^u y'_j)^{e_j} \right)$$

$$\times \sum_{k=u}^{p+u-1} \left( \sum_{i \in S} c_i x_i^{k-u+1} \right) \prod_{l=u}^{k} \frac{\zeta^{mR_{l+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{el}}}{1 - \left( \zeta^{mR_{l+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{ej}} \right)^2}$$

$$+ \xi \left( \prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}}}{1 - \left( \zeta^{mR_{k+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{ej}} \right)^2} \right) \sum_{i \in S} c_i T_i(\Omega^p y') - \sum_{i \in S} c_i T_i(y'),$$

where $c_i (i \in S)$ are algebraic numbers not all zero. Then

$$G^*(y') = \left( \prod_{j=1}^{s} M(D(\Omega - E)^{-1} \Omega^u y'_j)^{e_j} \right)^{-2} \left( G(y') + \sum_{i \in S} c_i T_i(y') \right) \in \overline{Q}(y')$$

satisfies the functional equation

$$G^*(y') = \xi \left( \prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{2D_{ej}}}{1 - \left( \zeta^{mR_{k+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{ej}} \right)^2} \right) G^*(\Omega^p y')$$

$$+ \frac{1}{\prod_{j=1}^{s} M(D(\Omega - E)^{-1} \Omega^u y'_j)^{e_j}}$$

$$\times \sum_{k=u}^{p+u-1} \left( \sum_{i \in S} c_i x_i^{k-u+1} \right) \prod_{l=u}^{k} \frac{\zeta^{mR_{l+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{el}}}{1 - \left( \zeta^{mR_{l+1}} \prod_{j=1}^{s} M(\Omega^k y'_j)^{D_{ej}} \right)^2}. \tag{11}$$

Let $P$ be a positive integer and let $y'_j = (y'_j, \ldots, y'_n) = (z_1^{P_j}, \ldots, z_n^{P_j})$ ($1 \leq j \leq s$). We choose $P$ sufficiently large such that

$$H(z) = G^*(z_1^{P_1}, \ldots, z_1^{P_n}, \ldots, z_n^{P_1}, \ldots, z_n^{P_n}) \in \overline{Q}(z_1, \ldots, z_n).$$

Then by (11), $H(z)$ satisfies the functional equation

$$H(z) = \xi \left( \prod_{k=u}^{p+u-1} \frac{\zeta^{mR_{k+1}} M(\Omega^k z)^{2D_{ej}}}{1 - \left( \zeta^{mR_{k+1}} M(\Omega^k z)^{D_{ej}} \right)^2} \right) H(\Omega^p z)$$

$$+ \frac{1}{M(D(\Omega - E)^{-1} \Omega^{u} z)\ell} \sum_{k=u}^{p+u-1} \left( \sum_{i \in S} c_i x_i^{k-u+1} \right) \prod_{l=u}^{k} \frac{\zeta^{mR_{l+1}} M(\Omega^k z)^{D_{el}}}{1 - \left( \zeta^{mR_{l+1}} M(\Omega^k z)^{D_{ej}} \right)^2},$$

where $\ell = \sum_{j=1}^{s} e_j P_j$. Letting $H(z) = A(z)/B(z)$, where $A(z)$ and $B(z)$ are coprime polynomials in $\overline{Q}[z_1, \ldots, z_n]$ with $B \neq 0$, and letting $M(D(\Omega - E)^{-1} \Omega^{u} z) = M_1(z)/M_2(z)$, where $M_1(z)$ and $M_2(z)$ are coprime monomials in $\overline{Q}[z_1, \ldots, z_n]$, we have

$$A(z)B(\Omega^{p} z)M_1(z)^{\ell} \prod_{k=u}^{p+u-1} \left( 1 - \left( \zeta^{mR_{k+1}} M(\Omega^k z)^{D_{ej}} \right)^2 \right)$$
\[
\begin{align*}
\xi A(\Omega^p z)B(z)M_1(z)^\ell \prod_{k=u}^{p+u-1} \zeta^{mR_{k+1}} M(\Omega^k z)^{2D\ell} \\
+ \sum_{k=u}^{p+u-1} \left( \sum_{i \in S} c_i z_i^{k-u+1} \right) B(z)B(\Omega^p z)M_2(z)^\ell \prod_{l=u}^{k} \zeta^{mR_{l+1}} M(\Omega^l z)^{2D\ell} \\
\times \prod_{l'=u}^{p+u-1} \left( 1 - \left( \zeta^{mR_{l'+1}} M(\Omega^{l'} z)^{2D\ell} \right)^2 \right).
\end{align*}
\]

In what follows, let \( u \) be sufficiently large. By the condition \( \Phi(2) < 0 \), the root \( \rho \) of \( \Phi(X) \) such that \( R_k = b \rho^k + o(\rho^k) \) with \( b > 0 \) (cf. Remark 4 in [6]) satisfies \( \rho > 2 \) and hence \( R_{k+1} > 2R_k \) for all sufficiently large \( k \). Then the componentwise inequality \( (R_{u+1}, \ldots, R_1)D(\Omega - E)^{-1} \Omega = (R_{u+1}, \ldots, R_1)\Omega^u D(\Omega - E)^{-1} = (R_{u+1}, \ldots, R_{u+1})D(\Omega - E)^{-1} < D(R_{u+1}, \ldots, R_{u+1}) \) holds and so \( z_1 \cdots z_n M_1(z)^\ell \) divides \( M(\Omega^p z)^{2D\ell} = M(\Omega^u z)^{\ell} \).

In what follows, \( p \) is replaced with its multiple if necessary. We can put the greatest common divisor of \( A(\Omega^p z) \) and \( B(\Omega^p z) \) as \( z^{I(p)} \), where \( I(p) \) is an \( n \)-dimensional vector with nonnegative integer components, by Lemma 5. Then \( B(\Omega^p z) \) divides \( B(z)M_1(z)^\ell z^{I(p)} \prod_{k=u}^{p+u-1} M(\Omega^k z)^{2D\ell} \) by (12). Therefore \( B(z) \) is a monomial in \( z_1, \ldots, z_n \) by Lemmas 1 and 6. Since \( p \) and \( u \) are independent, the right-hand side of (12) is divisible by \( z_1 \cdots z_n M_1(z)^\ell B(\Omega^p z) \) and thus \( A(z) \) is divisible by \( z_1 \cdots z_n \). Since \( A(z) \) and \( B(z) \) are coprime, \( B(z) \in Q[z_1, \ldots, z_n] \). If \( A(z) \notin Q \) and if \( p \) is sufficiently large, then by Lemma 7, \( \deg_A(\Omega^p z) > \max \{ \deg_A A(z), \deg_A M_2(z)^{\ell} \} \), which is a contradiction by comparing the total degrees of both sides of (12). Hence \( A(z) \in Q \). Then by (12), we see that \( \sum_{i \in S} c_i x_i^{k-u+1} = 0 \) (\( u \leq k \leq p + u - 1 \)) and so \( \sum_{i \in S} c_i x_i^k = 0 \) (\( 1 \leq k \leq p \)). Hence \( x_i = x_{i'} \) for some distinct \( i, i' \in S \) since \( c_i (i \in S) \) are not all zero. Then \( (x_i, q_i) = (x_{i'}, q_{i'}) \), which is a contradiction, and the proof of the theorem is completed. \( \square \)

References