NOTES ON CHOW RINGS OF G/B AND BG

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1. Introduction

Let p be a prime number. Let G and T be a connected compact Lie group and its maximal torus such that $H^*(G)$ has p-torsion. Given a field k with $ch(k) = 0$, let G_k and T_k be a split reductive group and a split maximal torus over the field k, corresponding to G and T. Let us write by BG_k its classifying space defined by Totaro [To1,3]. Let B_k be the Borel subgroup containing T_k. Let $\mathbb{G}$ be a G_k-torsor. Then $\mathbb{F} = \mathbb{G}/B_k$ is a (twisted) form of the flag variety G_k/B_k.

For a smooth algebraic variety X over k, let $CH^*(X) = CH^*(X)_{(p)}$ mean the p-localized Chow ring generated by algebraic cycles modulo rational relations. The cofibering $G/T \stackrel{j}{\to} BT \stackrel{i}{\to} BG$ ([Tod1,2]) induces the maps

$$(1.1) \quad CH^*(BG_k) \stackrel{i^*}{\to} CH^*(BB_k) \stackrel{j^*}{\to} CH^*(\mathbb{G}/B_k),$$

whose composition $j^*i^* = 0$ for $* > 0$. But it is far from exact when $\mathbb{G} \cong G_k$. (Here exact means $Ker(j^+) = Ideal(Im(i^+))$.) However, we observe that it becomes near exact when $\mathbb{G}$ is sufficient twisted, while it is still not exact for most cases.

The author thanks Maski Kameko for pointing out errors in the first version of this paper.

2. $CH^*(\mathbb{G}/B_k)$

Recall that G_k and T_k are the split reductive group and split maximal torus over a field k with $ch(k) = 0$, corresponding to Lie groups G and T. Let B_k be the Borel subgroup containing T_k. Recall that $\mathbb{G}$ is a G_k-torsor, and let us write $\mathbb{F} = \mathbb{G}/B_k$ in this section.

By Petrov-Semenov-Zainoulline ([Pe-Se-Za], [Se-Zh]), it is known that the p-localized motive $M(\mathbb{F})_{(p)}$ of $\mathbb{F}$ is decomposed as

$$(2.1) \quad M(\mathbb{F})_{(p)} = M(\mathbb{G}/B_k)_{(p)} \cong R(\mathbb{G}) \otimes (\oplus_i \tilde{\mathbb{T}}^{\otimes s_i})$$

where $\tilde{\mathbb{T}}$ is the reduced Tate motive and $R(\mathbb{G})$ is some motive called generalized Rost motive. (It is the original Rost motive([Ro], [Vo1,2], [Pe-Se-Za], [Ya2]) when G is of type (I) as explained below). Hence we have maps

$$(2.2) \quad CH^*(BB_k) \to CH^*(\mathbb{F}) \stackrel{split\ surj.}{\to} CH^*(R(\mathbb{G}))$$

where BB_k is the classifying space for B_k-bundles. From Merkurjev and Karpenko [Me-Ne-Za], [Kar], we know that the first map is also surjective when $\mathbb{G}$ is a versal G_k-torsor. (For the definition of *versal torsor* see [Ga-Me-Se], [Me-Ne-Za], [Kar],

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2010 *Mathematics Subject Classification.* Primary 57T15, 20G15, 14C15.
Key words and phrases. Chow rings, classifying spaces, complete flag varieties.

[To2].) In particular, when G is of type (I), if $\mathbb{G}$ is a non-trivial G_k-torsor, then it is versal.

To explain groups of type (I), we recall arguments for $H^*(G/T)$ in algebraic topology. By Borel, its $mod(p)$ cohomology is (for p odd)

$$H^*(G;\mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1,...,x_\ell), \quad |x_i| = odd$$

where $P(y)$ is a truncated polynomial ring generated by *even* dimensional elements y_i, and $\Lambda(x_1,...,x_\ell)$ is the $\mathbb{Z}/p$-exterior algebra generated by $x_1,...,x_\ell$. When $p=2$, we consider the graded ring $grH^*(G;\mathbb{Z}/2)$ which is isomorphic to the right hand side ring above.

When G is simply connected and $P(y)$ is generated by just one generator, we say that G is of type (I). Except for $(E_7, p=2)$ and $(E_8, p=2,3)$, all exceptional (simple) Lie groups are of type (I). Note that in these cases, it is known $rank(G) = \ell \geq 2p-2$.

We consider the fibering ([Tod2], [Mi-Ni]) $G \stackrel{\pi}{\to} G/T \stackrel{i}{\to} BT$ and the induced spectral sequence

$$E_2^{*,*} = H^*(BT; H^*(G;\mathbb{Z}/p)) \Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

Here we can write $H^*(BT) \cong S(t) = \mathbb{Z}[t_1,...,t_\ell]$ with $|t_i| = 2$.

It is well known that $y_i \in P(y)$ are permanent cycles and that there is a regular sequence $(\bar{b}_1,...,\bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i) = \bar{b}_i$ ([Tod2], [Mi-Ni]).

We know that G/T is a manifold such that $H^*(G/T) = H^{even}(G/T)$ and $H^*(G/T)$ is torsion free. We also see that there is a filtration in $H^*(G/T)_{(p)}$ such that

$$grH^*(G/T)_{(p)} \cong P(y) \otimes S(t)/(b_1,...,b_\ell)$$

where $b_i \in S(t)$ with $b_i = \bar{b}_i \ mod(p)$.

For the algebraic closure $\bar{k}$ of k, let us write $\bar{X} = X|_{\bar{k}}$. Then considering (2.1) over $\bar{k}$, we see

$$CH^*(\bar{R}(\mathbb{G}))/p \cong P(y), \quad CH^*(\oplus_i \tilde{\mathbb{T}}^{\otimes s_i}) \cong S(t)/(b_1,...,b_\ell).$$

Moreover when $\mathbb{G}$ is versal, we can see ([Ya2]) that $CH^*(R(\mathbb{G}))$ is additively generated by products of $b_1,...,b_\ell$ in (2.2). Hence we have surjections $CH^*(BB_k) \to CH^*(\mathbb{F}) \stackrel{pr.}{\to} CH^*(R(\mathbb{G}))$.

By giving the filtration on $S(t)$ by b_i, we can write (additively)

$$grS(t)/p \cong A \otimes S(t)/(b_1,...,b_\ell) \quad for\ A = \mathbb{Z}/p[b_1,...,b_\ell].$$

In particular, we have maps $A \stackrel{i_A}{\to} CH^*(\mathbb{F})/p \to CH^*(R(\mathbb{G}))/p$. We also see that the above composition map is surjective.

Lemma 2.1. *([Ya2]) Suppose that there are $f_1(b),...,f_s(b) \in A$ such that $CH^*(R(\mathbb{G}))/p \cong A/(f_1(b),...,f_s(b))$. Moreover if $f_i(b) = 0$ for $1 \leq i \leq s$ also in $CH^*(\mathbb{F})/p$, we have the isomorphism*

$$CH^*(\mathbb{F})/p \cong S(t)/(p, f_1(b),...,f_s(b)).$$

For $N > 0$, let us write $A_N = \mathbb{Z}/p\{b_{i_1}...b_{i_k} | |b_{i_1}| + ... + |b_{i_k}| \leq N\}$.

Lemma 2.2. *Let $pr : A_N \to CH^*(\mathbb{F})/p \to CH^*(R(\mathbb{G}))/p$, and $b \in Ker(pr)$. Then $b = \sum b'u'$ with $b' \in A_N$, $u' \in S(t)^+/(p,b_1,...,b_\ell)$ i.e., $|u'| > 0$.*

Using these, we can prove

Theorem 2.3. ([Ya2]) *Let G be of type (I) and $rank(G)=\ell$. Let $\mathbb{G}$ be a non-trivial G_k-torsor. Then $2p-2\le \ell$, and we can take $b_i\in S(t)=CH^*(BB_k)$ for $1\le i\le \ell$ such that there are isomorphisms*

$$CH^*(R(\mathbb{G}))/p\cong \mathbb{Z}/p\{1,b_1,...,b_{2p-2}\},$$
$$CH^*(X)/p\cong S(t)/(p,b_ib_j,b_k|0\le i,j\le 2p-2<k\le \ell)$$

where $\mathbb{Z}/p\{a,b,...\}$ is the $\mathbb{Z}/p$-free module generated by $a,b,...$

3. relation $\mathbb{G}/B_k$ and BG

Let $h^*(X)=CH^*(X)/I(h)$ for some ideal $I(h)$ (e.g., $CH^*(X)/p$). We note here the following lemma for each G_k-torsor $\mathbb{G}$ (not assumed twisted).

Lemma 3.1. *For the above $h^*(X)$, the composition of the following maps is zero for $*>0$*

$$h^*(BG_k)\to h^*(BB_k)\to h^*(\mathbb{G}/B_k).$$

Proof. Take U (e.g., GL_N for a large N) such that U/G_k approximates the classifying space BG_k [To3]. Namely, we can take $\mathbb{G}=f^*U$ for the classifying map $f:\mathbb{G}/G_k\to U/G_k$. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{F}=\mathbb{G}/B_k & \longrightarrow & U/B_k \\ \downarrow & & \downarrow \\ Spec(k)\cong \mathbb{G}/G_k & \longrightarrow & U/G_k \end{array}$$

where U/B_k (resp. U/G_k) approximates BB_k (resp. BG_k). Since $h^*(Spec(k))=CH^*(Spec(k))/I(h)=0$ for $*>0$, we have the lemma. □

The above sequences of maps in the lemma is not exact, in general. However we get some informations from $h^*(\mathbb{F})$ to $h^*(BG_k)$. For example, we get much informations of $h^*(BG_k)$ from $h^*(\mathbb{F})$ than from $h^*(G_k/B_k)$ when $\mathbb{G}$ is versal.

Let us write the induced maps

$$h^+(BG_k)\stackrel{i^+}{\to} h^+(BT)\stackrel{j(\mathbb{G})^+}{\to} h^+(\mathbb{G}/B_k)$$

where $h^+(-)$ is the ideal of the positive degree parts. Let us define

$$D_h(\mathbb{G})=Ker(j^+)/(Ideal(Im(i^+)).$$

Let $\mathbb{G}$ be versal and k' is some extension of k. Then

$$D_h(\mathbb{G})\subset D_h(\mathbb{G}|_{k'})\subset D_h(G|_{\bar{k}})\cong D_h(G_k).$$

For ease of arguments we mainly consider the case $h^*(X)=CH^*(G)/p$, and write $D_h(\mathbb{G})$ simply by $D(\mathbb{G})$.

Theorem 3.2. *Let $\mathbb{G}$ be versal. Then additively*

$$D(G_k)/D(\mathbb{G})\cong CH^+(R(\mathbb{G}))/p\otimes S(t)/(b_1,...,b_\ell).$$

Proof. We consider the map $S(t)\cong CH^*(BB_k)\stackrel{j^*}{\to} CH^*(\mathbb{G}/B_k)$. Recall that

$$CH^*(G_k/B_k)/p\cong P(y)\otimes S(t)/(b)\quad (b)=Ideal(b_1,...,b_\ell).$$

So $Ker(j(G_k))=(b)$. Hence

$$D(G_k)/(D(\mathbb{G}))\cong (Ker(j(G_k)/Im(i^+))/(Ker(j(\mathbb{G}))/Im(i^+))$$

$$\cong Ker(j(G_k))/Ker(j(\mathbb{G})) \subset CH^*(\mathbb{F})/p \stackrel{pr.}{\to} CH^+(R(\mathbb{G}))/p.$$

This composition map is a surjection. Because each element

$$x \in Ker(j(G_k)) = (b_1, ..., b_\ell) \subset S(t)/p$$

can be written using $A(b)^+ = \mathbb{Z}/p[b_1, ..., b_\ell]^+$

$$x = \sum b_I t(i) \quad b_I \in A(b)^+, \quad 0 \neq t(I) \in S(t)/(b_1, ..., b_\ell).$$

This also means that the ideal $Ker(j(G)) \cong A(b)^+ \otimes S(t)/(b)$, which implies

$$Ker(j(G))/Ker(j(\mathbb{G})) \cong CH^+(R(\mathbb{G}))/p \otimes S(t)/(b).$$

□

Corollary 3.3. *There is a surjection* $D(G_k) \to CH^+(R(\mathbb{G}))/2$.

Thus we have a very weak version of the decomposition theorem by Petrov-Semenov-Zainoulline [Pe-Se-Za], without using deep motive theories.

Corollary 3.4. *Let* $\mathbb{G}$ *be versal. Then we have an additive decomposition of the* $mod(p)$ *Chow ring*

$$CH^*(\mathbb{G}/B_k)/p \cong S(t)/(p, b_1, ..., b_\ell) \oplus D(G_k)/D(\mathbb{G})$$

$$\cong (\mathbb{Z}/p\{1\} \oplus CH^+(R(\mathbb{G})/p) \otimes S(t)/(b_1, ..., b_\ell).$$

4. $SO(2\ell + 1)$

At first we consider the orthogonal groups $G = SO(m)$ and $p = 2$. The $mod(2)$-cohomology is written as (see for example [Tod-Wa], [Ni])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, ..., x_{m-1})$$

where $|x_i| = i$, and the multiplications are given by $x_s^2 = x_{2s}$.

For ease of argument, we only consider the case $m = 2\ell + 1$ so that

$$H^*(G; \mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_1, x_3, ..., x_{2\ell-1})$$

$$grP(y)/2 \cong \Lambda(y_2, ..., y_{2\ell}), \quad \textit{letting } y_{2i} = x_{2i} \ \ (\textit{hence } y_{4i} = y_{2i}^2).$$

The Steenrod operation is given as $Sq^k(x_i) = \binom{i}{k}(x_{i+k})$. The Q_i-operations are given by Nishimoto [Ni]

$$Q_n x_{2i-1} = y_{2i+2^{n+1}-2}, \qquad Q_n y_{2i} = 0.$$

In particular, $Q_0(x_{2i-1}) = y_{2i}$ in $H^*(G; \mathbb{Z}/2)$. It is well known that the transgression $b_i = d_{2i}(x_{2i-1}) = c_i$ is the i-th elementary symmetric function on $S(t)$. Hence we have

Lemma 4.1. *We have an isomorphism*

$$grH^*(G/T) \cong P(y) \otimes S(t)/(c_1, ..., c_\ell).$$

Moreover, the cohomology $H^*(G/T)$ is computed completely by Toda-Watanabe [Tod-Wa] (e.g. $2y_{2i} = c_i \ mod(4)$).

Let T be a maximal Torus of $SO(m)$ and $W = W_{SO(m)}(T)$ its Weyl group. Then $W \cong S_\ell^{\pm}$ is generated by permutations and change of signs so that $|S_k^{\pm}| = 2^k k!$. Hence we have

$$H^*(BT)^{W'} \cong \mathbb{Z}_{(2)}[p_1, ..., p_\ell] \subset H^*(BT) \cong \mathbb{Z}_{(2)}[t_1, ..., t_\ell], \ |t_i| = 2$$

where the Pontriyagin class p_i is defined by $\Pi_i(1 + t_i^2) = \sum_i p_i$.

Here we recall

$$H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, ..., w_{2\ell+1}], \quad Q_0(w_{2i}) = w_{2i+1} \ mod(w_s w_t).$$

It is known $H^*(BG)$ has no higher 2-torsion and

$$H(H^*(BG; \mathbb{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbb{Z}/2$$

where $H(A; Q_0)$ is the homology of A with the differential Q_0 and Tor is the torsion ideal in $H^*(BG)$. Hence we have

$$H^*(BG)/Tor \cong D \quad where \ \ D = \mathbb{Z}_{(2)}[c_2, c_4, ..., c_{2\ell}].$$

The isomorphism $j : H^*(BG)/Tor \to H^*(BT)^W$ is given by $c_{2i} \mapsto p_i$.

Now we consider the $mod(2)$ Chow ring and the case that $\mathbb{G}$ is the split group G_k.

Lemma 4.2. *We have additive isomorphism*

$$D(G_k) \cong \Lambda(c_1, .., c_\ell)^+ \otimes S(t, c) \quad with \ S(t, c) \cong S(t)/(c_1, ..., c_\ell),$$

namely, each element $x \in D(G_k)$ is written as $x = \sum c_I t(I)$ with $c_I \in \Lambda(c_1, ..., c_\ell)^+$ and $t(I) \neq 0 \in S(t)/(c_1, ..., c_\ell)$.

Proof. Recall that

$$CH^*(G_k/B_k)/2 \cong H^*(G/T)/2 \cong P(y)/2 \otimes S(t)/(c_1, ..., c_\ell).$$

Hence we see

$$Ker(j) \cong (c_1, ..., c_\ell) \subset CH^*(BB_k)/2 \cong H^*(BT)/2.$$

Here $j : p_i \mapsto c_i^2 \ mod(2)$ by definition of the Pontryagin class p_i.

On the other hand, we know by Totaro [To1]

$$CH^*(BG_k) \cong \mathbb{Z}[c_2, ..., c_{2\ell+1}]/(2c_{odd}).$$

Hence $CH^*(BG_k)/Tor \cong D \cong H^*(BT)^W$ by $i : c_{2i} \mapsto p_i$. Thus the ideal generated by the image is $(Im(i)) \cong (c_2, c_4, ..., c_{2\ell}) \subset S(t)$. Since $j : p_i \mapsto c_i^2$, we have

$$Ker(j)/(Im(i)) \cong (c_1, ..., c_\ell)/(c_1^2, ..., c_\ell^2) \subset S(t)/(c_1^2, ..., c_\ell^2)$$

which is additively isomorphic to $\Lambda(c_1, ..., c_\ell)^+ \otimes S(t)/(c_1, ..., c_\ell)$. □

Recall that there is a surjection $D(G_k) \to CH^+(R(\mathbb{G}))/p$ from Lemma 2.1. We can see $c_1 ... c_\ell \neq 0$ in $CH^*(R(\mathbb{G}))/2$ (for example using the torsion index $t(G) = 2^\ell$ (for the torsion index, see [To2]).

Theorem 4.3. *(Petrov* [Pe], [Ya2]*) Let $(G, p) = (SO(2\ell + 1), 2)$ and $\mathbb{F} = \mathbb{G}/B_k$ be versal. Then $CH^*(\mathbb{F})$ is torsion free, and*

$$CH^*(\mathbb{F})/2 \cong S(t)/(2, c_1^2, ..., c_\ell^2), \quad CH^*(R(\mathbb{G}))/2 \cong \Lambda(c_1, ..., c_\ell).$$

Corollary 4.4. *Let $(G, p) = (SO(2\ell + 1), 2)$ and $\mathbb{G}$ be versal.*
Then we have $D(\mathbb{G}) \cong 0$.

5. $Spin(7)$ FOR $p = 2$

Hereafter this section, we assume $G = Spin(7)$ and $p = 2$. It is well known

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8]$$

where w_i for $i \le 7$ (resp. $i = 8$) are the Stiefel-Whitney classes for the representation induced from $Spin(7) \to SO(7)$ (resp. the spin representation Δ).

Thus the integral cohomology is written as (using $Q_0w_6 = w_7$)

$$H^*(BG) \cong \mathbb{Z}_{(2)}[w_4, c_6, w_8] \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\})$$

$$\cong D \otimes \Lambda_{\mathbb{Z}}(w_4, w_8) \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\})$$

where $D = \mathbb{Z}_{(2)}[c_4, c_6, c_8]$ with $c_i = w_i^2$.

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \Longrightarrow BP^*(BG).$$

We can compute the spectral sequence

$$grBP^*(BG) \cong D \otimes (BP^*\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\}$$

$$\oplus BP^*/(2, v_1, v_2)[c_7]\{c_7\}/(v_3c_7c_8)).$$

Then $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)}$ is isomorphic to ([Ko-Ya])

$$D\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\}/(2v_1w_8) \oplus D/2[c_7]\{c_7\}.$$

On the other hand, the Chow ring of $BG_{\mathbb{C}}$ is given by Guillot ([Gu],[Ya1])

Theorem 5.1. *Let $k = \bar{k}$. Then we have isomorphisms*

$$CH^*(BG_k) \cong BP^*(BG_k) \otimes_{BP^*} \mathbb{Z}_{(2)}$$

$$\cong D \otimes (\mathbb{Z}_{(2)}\{1, c_2', c_4'.c_6'\} \oplus \mathbb{Z}/2\{\xi_3\} \oplus \mathbb{Z}/2[c_7]\{c_7\})$$

where $cl(c_i) = w_i^2$, $cl(c_2') = 2w_4$, $cl(c_4') = 2w_8$, $cl(c_6') = 2w_4w_8$, and $cl(\xi_3) = 0$, $|\xi_3| = 6$. However $cl_\Omega(\xi_3) = v_1w_8$ in $BP^*(BT)^W$, for the cycle map cl_Ω of the algebraic cobordism.

Now we consider $CH^*(\mathbb{G}/B_k)$. Let $G = Spin(7)$ and $\mathbb{G}$ be versal. The group G is of type (I) and we can take $b_1 = c_2, b_2 = c_3, b_3 = e_4$ with $|e_4| = 8$ (for details see [Ya2]). The Chow ring $CH^*(\mathbb{G}/B_k)$ is given in Theorem 2.3 (in fact, G is of type (I))

$$CH^*(\mathbb{G}/B_k) \cong S(t)/((2c_2, c_2^2, c_2c_3, c_3^2, e_4), \quad S(t) = \mathbb{Z}_{(2)}[t_1, t_2, t_3].$$

Hence we have $Ker(j(\mathbb{G})) \cong (2c_2, c_2^2, c_2c_3, c_3^2, e_4)$. Recall

$$CH^*(BG_{\bar{k}})/(Tor) \cong CH^*(BB_k)^W \cong D\{1, c_2'', c_4'', c_6''\}$$

where c_i'' is a Chern class of the (complex) spin representation. Since $i(c_2'') = 2w_4, ...,$ we see

$$D/2 \cong Im(i^*/2 : CH^*(BG_k) \to CH^*(BT)/2).$$

We can see that the map i^* is given $c_4 \mapsto c_2^2$, $c_6 \mapsto c_3^2$, $c_8'' \mapsto e_4^2$, and

$$c_2'' \mapsto 2c_2, \quad c_4'' \mapsto 2e_4, \quad c_6'' \mapsto 2c_2e_4.$$

In particular $i^*CH^*(BG_k) = i^*CH^*(BG_{\bar{k}})$. Thus we see

Proposition 5.2. *Let $G = Spin(7)$ and $\mathbb{G}$ be versal. Then we have additively*

$$D(\mathbb{G}) \cong \Lambda(c_2c_3, e_4)^+ \otimes S(t, c) \quad for\ S(t, c) \cong S(t)/(c_2, c_3, e_4).$$

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