Stabilization of bridge decompositions of knots and bridge positions of knot types

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1 Introduction

We study bridge decompositions of knots and bridge positions of knot types. A bridge decomposition of a knot in the 3-sphere is a decomposition of it into two very simple pieces. A bridge position of a knot type is a representative knot of it which lies in a good way with respect to the standard height function in the 3-sphere. (See Section 2 for precise definitions.) While they are seemingly equivalent, we will point out substantial differences between them in a separate paper [1], currently in preparation.

In this paper, we show the uniqueness of stabilization for bridge decompositions and bridge positions. A stabilization for a bridge decomposition of a knot is a certain process to obtain a new bridge decomposition of the same knot, and similarly for a bridge position. By the uniqueness of stabilization, we mean the following facts, which have been folklore without proofs in the literature as far as we are aware.

Fact 1. *Let (B_-, B_+) be a bridge decomposition of a knot, and $\left(B'_-, B'_+\right)$ be the bridge decomposition obtained from (B_-, B_+) by a stabilization. Then, the bridge isotopy class of $\left(B'_-, B'_+\right)$ depends only on the bridge isotopy class of (B_-, B_+).*

Fact 2. *Let K be a bridge position of a knot type, and K' be the bridge position obtained from K by a stabilization. Then, the bridge isotopy class of K' depends only on the bridge isotopy class of K.*

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2 Preliminaries

In this section, we review basic definitions concerning bridge decompositions and bridge positions. We work in the smooth category.

The notions of knot and knot type are defined as follows. A *knot* is a circle embedded in the 3-sphere S^3. Two knots are said to be *isotopic* if there is an ambient isotopy of S^3 which takes one to the other. A *knot type* is an isotopy class of knots.

2.1 Bridge decomposition

An *n-string trivial tangle* is the pair of a 3-ball and a collection of pairwise disjoint properly embedded n arcs in the ball simultaneously parallel to the boundary.

The notion of bridge decomposition for knots is defined as follows. Let K be a knot, and n be a positive integer. An *n-bridge decomposition* of K is the pair (B_-, B_+) of 3-balls B_- and B_+ such that $B_- \cup B_+ = S^3$ and $B_- \cap B_+ = \partial B_- = \partial B_+$, and the 2-sphere $B_- \cap B_+$ intersects K transversely, and $(B_-, K \cap B_-)$, $(B_+, K \cap B_+)$ are n-string trivial tangles. A *bridge decomposition* of K is an m-bridge decomposition of K for some positive integer m. Two bridge decompositions $(B_{1,-}, B_{1,+})$ and $(B_{2,-}, B_{2,+})$ of K are said to be *bridge isotopic* if there is an ambient isotopy $\{H_t\colon S^3 \to S^3\}_{t\in[0,1]}$ such that H_0 is the identity map, $H_1(B_{1,-}) = B_{2,-}$, and $(H_t(B_{1,-}), H_t(B_{1,+}))$ is a bridge decomposition of K for every $t \in [0, 1]$.

The notion of stabilization for bridge decompositions is defined as follows. Let K be a knot, and (B_-, B_+) be a bridge decomposition of K. Let P denote the 2-sphere $B_- \cap B_+$. Let a be an arc in S^3 such that ∂a consists of a point in $K \setminus P$ and a point in $P \setminus K$, and that the interior of a is disjoint from K and P. Suppose that there is a disk Δ in S^3 such that $\partial\Delta$ is composed of a, a subarc of K, and an arc on P, and that the interior of Δ is disjoint from K and P. Then, we call a a *stabilizing arc* for (B_-, B_+). Let ε denote the sign such that $a \subset B_\varepsilon$, and let $N(a)$ be a small closed neighborhood of a in S^3. Let $B'_{-\varepsilon}$ denote the 3-ball obtained from $B_{-\varepsilon} \cup N(a)$ by smoothing the corner $\partial B_{-\varepsilon} \cap \partial N(a)$. Let B'_ε denote the closure of $S^3 \setminus B'_{-\varepsilon}$. Then $\left(B'_-, B'_+\right)$ is a bridge decomposition of K. We say that $\left(B'_-, B'_+\right)$ is obtained from (B_-, B_+) by a *stabilization*, or, more specifically, by the *stabilization* along a.

2.2 Bridge position

We let h denote the standard height function of the 3-sphere throughout the paper. To be specific, one may regard S^3 as the unit sphere in $\mathbb{R}^4$ and $h\colon S^3 \to \mathbb{R}$ as the restriction of the projection of $\mathbb{R}^4$ to one of the $\mathbb{R}$ factors.

The notion of bridge position for knot types is defined as follows. Let $\mathcal{K}$ be a knot type, and n be a positive integer. An *n-bridge position* of $\mathcal{K}$ is a knot K in $\mathcal{K}$ such that the function $h|_K$ has exactly $2n$ critical points, they are all non-degenerate, and any locally maximal value is greater than any locally minimal value. A *bridge position* of $\mathcal{K}$ is an m-bridge position of $\mathcal{K}$ for some positive integer m. Two bridge positions K_1 and K_2 of $\mathcal{K}$ are said to be *bridge isotopic* if there is an ambient isotopy $\{H_t\colon S^3 \to S^3\}_{t\in[0,1]}$ such that H_0 is the identity map, $H_1(K_1) = K_2$, and $H_t(K_1)$ is a bridge position of $\mathcal{K}$ for every $t \in [0,1]$.

The notion of stabilization for bridge positions is defined as follows. Let $\mathcal{K}$ be a knot type, and K be a bridge position of $\mathcal{K}$. Let v_+ denote the minimum of the locally maximal values of $h|_K$, and v_- denote the maximum of the locally minimal values of $h|_K$. Let p be a point in $K \cap h^{-1}((v_-, v_+))$, which we call a *stabilizing point* for K. Let K' be the knot obtained from K by a local isotopy near p creating a canceling pair of non-degenerate critical points of $h|_K$. Then K' is a bridge position of $\mathcal{K}$. We say that K' is obtained from K by a *stabilization*, or, more specifically, by the *stabilization* at p.

3 Proofs

In this section, we give proofs of Fact 1 and Fact 2 in the following subsections, respectively.

3.1 Bridge decomposition

Note that stabilization is unique for a given bridge decomposition and a given stabilizing arc. That is to say, the bridge decomposition obtained by the stabilization is unique up to bridge isotopy, independently of the choice of small neighborhood of the arc and the way of smoothing the corner.

Note also that a proof of the uniqueness for a given bridge isotopy class of bridge decompositions can be reduced to that for one representative decomposition as follows. Let K be a knot, and $(B_{1,-}, B_{1,+})$, $(B_{2,-}, B_{2,+})$ be bridge isotopic bridge decompositions of K. Let a_1, a_2 be stabilizing arcs for $(B_{1,-}, B_{1,+})$, $(B_{2,-}, B_{2,+})$, respectively. There is an ambient isotopy $\{H_t\colon S^3 \to S^3\}_{t\in[0,1]}$ such that H_0 is the identity map, $H_1(B_{1,-}) = B_{2,-}$, and $(H_t(B_{1,-}), H_t(B_{1,+}))$ is a bridge decomposition of K for every $t \in [0,1]$. Note that $H_t(a_1)$ is a stabilizing arc for $(H_t(B_{1,-}), H_t(B_{1,+}))$. This shows that the bridge decompositions obtained from $(B_{1,-}, B_{1,+})$ and $(B_{2,-}, B_{2,+})$ by the stabilizations along a_1 and $H_1(a_1)$, respectively, are bridge isotopic. It remains to compare the bridge decompositions obtained from $(B_{2,-}, B_{2,+})$ by the stabilizations along $H_1(a_1)$ and a_2.

We show that stabilization for a bridge decomposition does not depend on the choice of stabilizing arc as follows. Let K be a knot, and n be a positive integer. Let (B_-, B_+) be an n-bridge decomposition of K, and a, b be stabilizing arcs for (B_-, B_+). Let P denote the 2-sphere $B_- \cap B_+$. Since $(B_-, K \cap B_-)$ is an n-string trivial tangle, there are pairwise disjoint disks $D_{-,1}, D_{-,2}, \ldots, D_{-,n}$ of parallelism between $K \cap B_-$ and P. Similarly for $(B_+, K \cap B_+)$, we have pairwise disjoint disks $D_{+,1}, D_{+,2}, \ldots, D_{+,n}$. Since a is a stabilizing arc for (B_-, B_+), there is a disk Δ in S^3 such that $\partial\Delta$ is composed of a, a subarc of K, and an arc on P, and that the interior of Δ is disjoint from K and P. By an isotopy along Δ, we can make a short and close to a point p_1 in $K \cap P$. Similarly for b, we have a point q in $K \cap P$. Let ε denote the sign such that $a \subset B_\varepsilon$, and i denote the index such that $D_{\varepsilon,i}$ is adjacent to p_1. By a local isotopy near p_1, we can put a on $D_{\varepsilon,i}$. Then, by an isotopy along $D_{\varepsilon,i}$, we can translate a close to another point p_2 in $K \cap P$. Then, we can convert a into a stabilizing arc in $B_{-\varepsilon}$ near p_2 preserving the bridge decomposition obtained by the stabilization up to bridge isotopy, as illustrated in Figure 1. Let j denote the index such that $D_{-\varepsilon,j}$ is adjacent to p_2. We can put a on $D_{-\varepsilon,j}$, and translate it close to another point p_3 in $K \cap P$. Since the knot K is connected, by continuing this process, a eventually comes close to q, and hence isotopic to b. This shows that the bridge decompositions obtained from (B_-, B_+) by the stabilizations along any two stabilizing arcs are bridge isotopic.

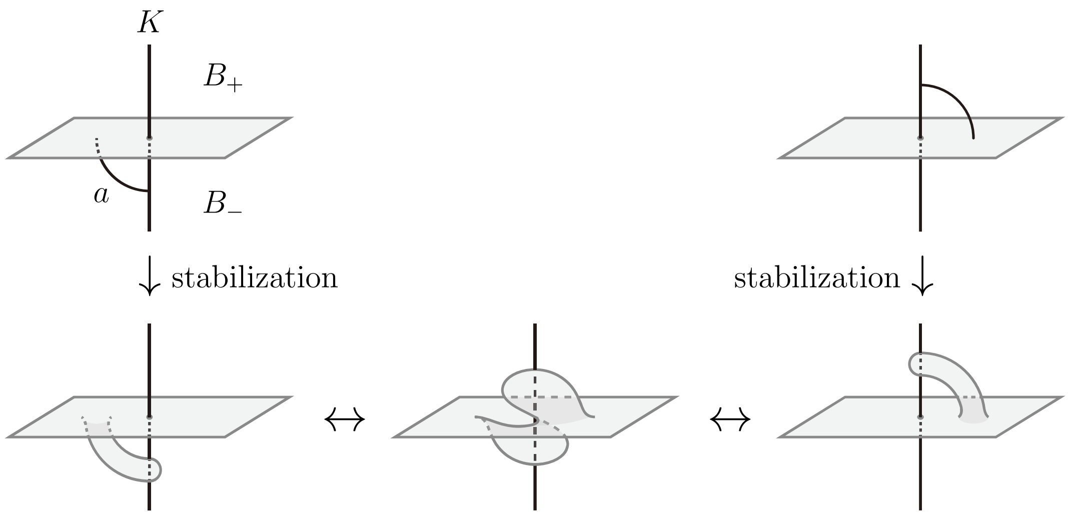

Figure 1: Stabilizing arcs for (B_-, B_+) in B_- and B_+, and the bridge decompositions obtained by the stabilizations along them, and a bridge isotopy between them.

3.2 Bridge position

Note that stabilization is unique for a given bridge position and a given stabilizing point. That is to say, the bridge position obtained by the stabilization is unique up to bridge isotopy, independently of the choice of local isotopy creating a canceling pair of critical points.

Note also that a proof of the uniqueness for a given bridge isotopy class of bridge positions can be reduced to that for one representative position as follows. Let $\mathcal{K}$ be a knot type, K be any bridge position of $\mathcal{K}$, and p be any stabilizing point for K. Note that there exists a bridge position K_0 of $\mathcal{K}$ bridge isotopic to K such that $h|_{K_0}$ has only two critical values. That is to say, all the locally minimal points of $h|_{K_0}$ have the same value, and all the locally maximal points do also. There is an ambient isotopy $\{H_t \colon S^3 \to S^3\}_{t\in[0,1]}$ such that H_0 is the identity map, $H_1(K) = K_0$, and $H_t(K)$ is a bridge position of $\mathcal{K}$ for every $t \in [0, 1]$. By composing an isotopy along K if necessary, we can arrange $\{H_t\}_{t\in[0,1]}$ so that $H_t(p)$ is a stabilizing point for $H_t(K)$. This shows that the bridge positions obtained from K and K_0 by the stabilizations at p and $H_1(p)$, respectively, are bridge isotopic. It remains to compare the bridge positions obtained from K_0 by the stabilizations at such points as $H_1(p)$.

We show that stabilization for a bridge position does not depend on the choice of stabilizing point as follows. Let $\mathcal{K}$ be a knot type, and n be a positive integer. Let K_0 be an n-bridge position of $\mathcal{K}$ such that $h|_{K_0}$ has only two critical values, and v_-, v_+ denote the minimal and maximal values, respectively. Let $k_1, k_2, \ldots, k_{2n}$ denote the component arcs of $K \cap h^{-1}((v_-, v_+))$. Let p and q be stabilizing points for K_0. Note that p lies on k_{i_1} for some index i_1 in $\{1, 2, \ldots, 2n\}$. By an isotopy along k_{i_1}, we can raise p close to a maximal point of $h|_{K_0}$. Then, we can convert p into a stabilizing point in k_{i_2} for another index i_2 preserving the bridge position obtained by the stabilization up to bridge isotopy, as illustrated in Figure 2. Then, by an isotopy along k_{i_2}, we can lower p close to a minimal point of $h|_{K_0}$, and convert it into a stabilizing point in k_{i_3} for another index i_3. Since the knot K_0 is connected, by continuing this process, p eventually meets q. This shows that the bridge positions obtained from K_0 by the stabilizations at any two stabilizing points are bridge isotopic.

Reference

[1] Y. Jang, T. Kobayashi, M. Ozawa and K. Takao, *Bridge decompositions of knots and bridge positions of knot types*, in preparation.

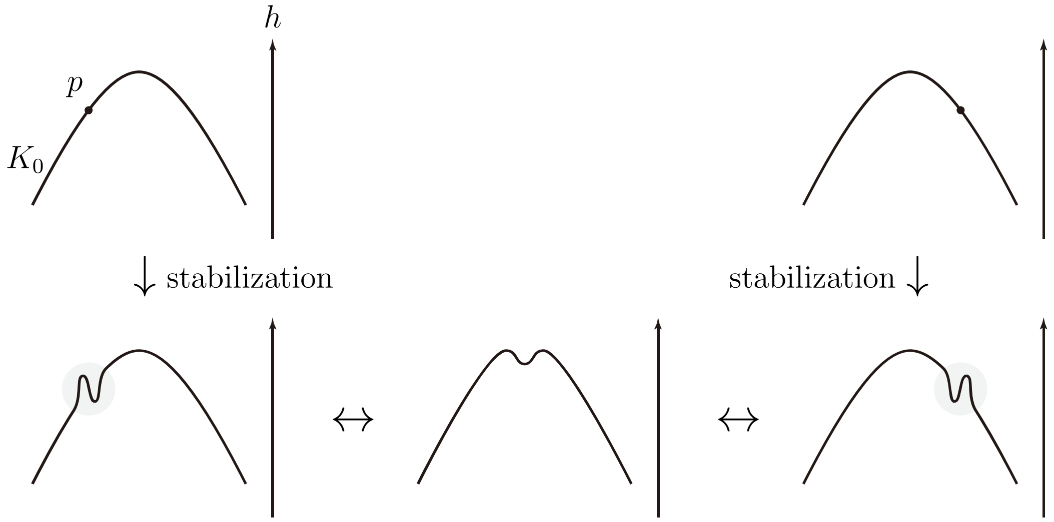

Figure 2: Stabilizing points for K_0 on different sides of a critical point of $h|_{K_0}$, and the bridge positions obtained by the stabilizations at them, and a bridge isotopy between them.

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