

# FINITE GROUPS NOT HAVING THE BORSUK-ULAM PROPERTY

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ABSTRACT. Determination of finite groups with the Borsuk-Ulam property is one of classical problems in the study of Borsuk-Ulam type theorems. Many relevant results have been shown so far, and recently, we succeeded in providing a complete answer to this problem. In this article, we shall provide an outline of the proof and some remarks on the Borsuk-Ulam constant.

## 1. BORSUK-ULAM PROPERTIES AND THE MAIN STATEMENT

Let  $G$  be a finite group and all maps between spaces are assumed to be continuous. For a unitary or orthogonal  $G$ -representation  $V$ , we denote the unit sphere by  $S(V)$ , called the representation sphere. Moreover we assume that representations are fixed-point-free, i.e.,  $V^G = 0$  unless otherwise stated.

K. Borsuk [3] proved the so-called Borsuk-Ulam theorem, which is described in various way. For example, the following is known.

**Proposition 1.1.** *The following statements hold.*

- (1) *For any map  $f : S^n \rightarrow \mathbb{R}^n$ , there exists a point  $x \in S^n$  such that  $f(x) = f(-x)$ .*
- (2) *There is no odd map  $f : S^n \rightarrow S^{n-1}$ .*
- (3) *If  $h : S^{n-1} \rightarrow S^{n-1}$  is an odd map, then  $\deg h \neq 0$ , i.e.,  $f$  is not null-homotopic.*

*Remark.* Furthermore,  $\deg h$  is odd for any odd map  $h : S^{n-1} \rightarrow S^{n-1}$ .

Many generalizations of the Borsuk-Ulam theorem have been studied; for example, the following result was shown in 1980's.

**Proposition 1.2.** *Let  $G = C_p^k$  or  $(S^1)^k$  ( $p$ : prime,  $k \geq 0$ ).*

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- (1) *If there exists a  $G$ -map  $f : S(V) \rightarrow S(W)$ , then  $\dim V \leq \dim W$ .*
- (2) *If there exists a  $G$ -map  $h : S(V) \rightarrow S(W)$  with  $\dim V = \dim W$ , then  $\deg h \neq 0$ .*

On the other hand, T. Bartsch [1] and W. Marzantowicz [7] have shown that Borsuk-Ulam type results as in Proposition 1.2 do not hold for many finite groups. This leads us to the following definition and problem.

*Definition.* Let  $G$  be a finite group.

- (1) We say that  $G$  has the *Borsuk-Ulam property of type (I)* (BUP (I) for short) if  $\dim V \leq \dim W$  holds whenever there exists a  $G$ -map  $f : S(V) \rightarrow S(W)$ .
- (2) We say that  $G$  has the *Borsuk-Ulam property of type (II)* (BUP (II) for short) if  $\deg h \neq 0$  holds whenever there exists a  $G$ -map  $h : S(V) \rightarrow S(W)$  with  $\dim V = \dim W$ .

**Problem.** *Determine finite groups with BUP (I) [resp. (II)].*

A complete answer to this problem is given as follows.

**Theorem 1.3** ([10]). *The following statements are equivalent.*

- (1)  *$G$  has BUP (I).*
- (2)  *$G$  has BUP (II).*
- (3)  *$G$  is an elementary abelian  $p$ -group  $C_p^k$ ,  $k \geq 0$ .*

In the following sections, we give an outline of the proof of this theorem. The details are described in [10].

## 2. REDUCTION TO THE CASE OF $M_p$

The Borsuk-Ulam theorem does not hold for almost finite groups, more precisely, the following is known.

**Proposition 2.1** ([1], [7]). (1) *Non  $p$ -groups have neither BUP (I) nor BUP (II).*  
 (2) *Finite groups with an element of order  $p^2$  have neither BUP (I) nor BUP (II).*

Therefore the remaining groups are finite  $p$ -groups of exponent  $p$ . If  $G$  is abelian, then such  $G$  is isomorphic to an elementary abelian  $p$ -group. If  $|G| \leq p^2$  or  $p = 2$ , then  $G$  is abelian and hence an elementary abelian  $p$ -group. Therefore we may consider only non-abelian finite  $p$ -groups of exponent  $p$ , where  $p$  is an odd prime. For any odd prime  $p$ , if  $|G| = p^3$ , then there exists only one non-abelian  $p$ -group  $M_p$  of exponent  $p$ :

$$M_p = \langle a, b, c \mid a^p = b^p = c^p = 1, ac = ca, bc = cb, a^{-1}ba = bc \rangle,$$

see [6]. We would like to show that non-abelian finite  $p$ -groups of exponent  $p$  have neither BUP (I) nor BUP (II). The following proposition shows that the problem is reduced to the case of  $M_p$ .

**Proposition 2.2** ([10]). *If  $M_p$  does not have BUP (I) [resp. (II)], then every non-abelian  $p$ -group of exponent  $p$  does not have BUP (I) [resp. (II)].*

This will be proved by using the following basic property.

**Proposition 2.3** (Basic property [10]). *If  $G$  has BUP (I) [resp. (II)], then*

- (1) *any subgroup  $H$  has BUP (I) [resp. (II)], and*
- (2) *any quotient group  $Q = G/K$  has BUP (I) [resp. (II)].*

*Outline of the proof of Proposition 2.2.* It is shown by induction on  $k \geq 3$  ( $|G| = p^k$ ). If  $k = 3$ , then  $G \cong M_p$ , hence it is true by assumption.

Since  $G$  is non-abelian, there exists a unitary irreducible representation  $U$  such that  $\dim_{\mathbb{C}} U \geq 2$ . If  $K := \text{Ker } U \neq 1$ , then  $U = U^K$  is a unitary irreducible  $G/K$ -representation of  $\dim_{\mathbb{C}} U^K \geq 2$ . Therefore  $G/K$  is non-abelian and of exponent  $p$ . By inductive assumption,  $G/K$  does not have BUP (I) [resp. (II)], hence  $G$  does not have BUP (I) [resp. (II)] by the basic property.

If  $K = 1$ , then  $U$  is faithful, and hence the center  $Z(G)$  is cyclic by representation theory. Therefore we have  $Z(G) \cong C_p$ . If  $G/Z(G)$  is non-abelian, then  $G$  does not have BUP (I) [resp. (II)] by the basic property. If  $G/Z(G)$  is abelian, then we have  $G/Z(G) \cong C_p^{k-1}$ . Thus  $G$  is an extra-special  $p$ -group. By [6], an extra-special  $p$ -group  $G$  of exponent  $p$  has a subgroup isomorphic to  $M_p$ . This shows that  $G$  does not have BUP (I) [resp. (II)] by the basic property.  $\square$

### 3. CONSTRUCTION OF COUNTEREXAMPLES

In this section, we would like to construct counterexamples to BUP (I) and (II) for  $M_p$ .

To do that, we consider a compact Lie group  $\widetilde{M}_p$  including  $M_p$  as a subgroup:

$$\widetilde{M}_p = \langle a, b, \zeta \mid a^p = b^p = 1, \zeta a = a\zeta, \zeta b = b\zeta \ (\forall \zeta \in S^1), a^{-1}ba = b\xi_p \rangle,$$

where  $\xi_p = \exp(2\pi\sqrt{-1}/p)$ . We first construct a  $\widetilde{M}_p$ -map  $h : S(V) \rightarrow S(W)$  of degree 0 for some fixed-point-free  $\widetilde{M}_p$ -representations  $V$  and  $W$ . By restricting to  $M_p$ , we obtain a counterexample to BUP (II). We next construct a  $\widetilde{M}_p$ -map  $f : S(V) \rightarrow S(W)$  for some  $V$  and  $W$  with  $\dim V > \dim W$  using a degree 0  $\widetilde{M}_p$ -map  $h$ . By restricting to  $M_p$ , we will obtain a counterexample to BUP (I).

We consider the following irreducible (unitary)  $\widetilde{M}_p$ -representations  $V_{k,l}$ ,  $(k, l) \in \mathbb{F}_p^2 \setminus \{(0, 0)\}$  and  $U_m$ ,  $m \in \mathbb{F}_p^*$ :

(1)  $V_{k,l}$  is 1-dimensional and its character is given by  $V_{k,l}(a) = \xi_p^k$ ,  $V_{k,l}(b) = \xi_p^l$  and  $V_{k,l}(\xi) = 1$ .

(2)  $U_m$  is  $p$ -dimensional and its character is given by  $U_m(g) = \begin{cases} \xi_p^{mu} & \text{if } g = \xi_p^u \\ 0 & \text{otherwise.} \end{cases}$

Restricting  $V_{k,l}$  and  $U_m$  to  $M_p$ , we obtain fixed-point-free  $M_p$ -representations.

Set

$$V = 2U_1 \quad \text{and} \quad W = V_{0,p-1} \oplus 2V_{1,p-1} \oplus \cdots \oplus 2V_{p-1,p-1} \oplus V_{1,0}.$$

Note that  $\dim V = \dim W$ . By equivariant obstruction theory [4], [5], there exists an  $\widetilde{M}_p$ -map  $h : S(V) \rightarrow S(W)$ . Then we have

**Proposition 3.1.**  $\deg h = 0$ . Hence  $M_p$  does not have BUP (II).

*Proof (Sketch).* We use the Euler class

$$e(V) := e(E\widetilde{M}_p \times_{\widetilde{M}_p} V) \in H^{2n}(B\widetilde{M}_p; \mathbb{Q})$$

of an oriented representation  $V$  with  $\dim V = 2n$ . A unitary representation  $V$  naturally become an oriented representation and hence the Euler class of  $V$  is defined. In general, by [8], if there exists a  $G$ -map  $h : S(V) \rightarrow S(W)$  for fixed-point-free representations with the same dimension, then  $e(W) = (\deg h)e(V)$  holds.

By an argument of the Serre spectral sequence, we see that

$$\text{Res} : H^*(B\widetilde{M}_p; \mathbb{Q}) \rightarrow H^*(BS^1 Q) \cong \mathbb{Q}[t]$$

is a ring isomorphism. Using this isomorphism, we see  $e(V) \neq 0$  and  $e(W) = 0$  for the representations defined as above. Hence we obtain  $\deg h = 0$ .  $\square$

Let  $V$  and  $W$  be as before. Set

$$\widetilde{V} = 2V \oplus V_{1,p-1}, \quad W_1 = V \oplus W \quad \text{and} \quad \widetilde{W} = 2W.$$

Note  $\dim S(\widetilde{V}) = 8p + 1$  and  $\dim S(W_1) = \dim S(\widetilde{W}) = 8p - 1$ . By an obstruction theoretic argument, we see that there exists an  $\widetilde{M}_p$ -map  $f_{(8p)} : S(\widetilde{V})_{(8p)} \rightarrow S(W_1)$ , where  $S(\widetilde{V})_{(8p)}$  is the  $8p$ -skeleton of  $S(\widetilde{V})$ . We would like to extend  $f_{(8p)}$  on  $S(\widetilde{V})$ , however, there is an obstruction to do it in general. In order to avoid this difficulty, we consider a degree 0  $\widetilde{M}_p$ -map  $\widetilde{h} := h * id : S(W_1) \rightarrow S(\widetilde{W})$ . Since the composite map  $\widetilde{h} \circ f_{(8p)}$  is null-homotopic, we can extend  $\widetilde{h} \circ f_{(8p)}$  on  $S(\widetilde{V})_{(8p+1)} = S(\widetilde{V})$ . Thus we obtain an  $\widetilde{M}_p$ -map  $f : S(\widetilde{V}) \rightarrow S(\widetilde{W})$ , which provides a counterexample to BUP (II).

#### 4. FROM THE POINT OF VIEW OF THE BORSUK-ULAM CONSTANT

Bartsch [2] and Meyer [9] introduced the following constant  $a_G$  which we call the (*equivariant*) *Borsuk-Ulam constant* of  $G$ .

*Definition.*  $a_G = \sup\{a \in \mathbb{R} \mid \text{a constant } a \text{ satisfies condition (WBU)}\}$ .

(WBU):  $a \dim V \leq \dim W$  holds whenever there exists a  $G$ -map  $f : S(V) \rightarrow S(W)$ .

**Proposition 4.1.** *The following hold.*

- (1)  $a_G = \inf\{\dim W / \dim V \mid \text{there exists a } G\text{-map } f : S(V) \rightarrow S(W)\}$ .
- (2)  $0 \leq a_G \leq 1$ .
- (3)  $a_G = 1$  if and only if  $G$  has BUP (II).

Theorem 1.3 shows that  $a_G = 1$  if and only if  $G$  is an elementary abelian group. By a result of [1], if  $G$  is not a  $p$ -group, then  $a_G = 0$ . When  $G$  is a  $p$ -group, strict values of  $a_G$  are not known other than cyclic  $p$ -groups. Meyer [9] (and also Bartsch [2]) showed  $a_G = 1/p^{k-1}$  for  $G = C_{p^k}$ .

Bartsch conjectures in [2] that  $a_G = 1/p^{k-1}$  for a  $p$ -group  $G$  of exponent  $p^k$ . However our result shows that this conjecture is false. Indeed, the exponent of  $G = M_p$  is  $p$  and the conjecture asserts that  $a_G = 1$ , however, we see  $a_G \leq \frac{4p}{4p+1}$  by an existence of a  $G$ -map  $\text{Res } f : S(\text{Res } \tilde{V}) \rightarrow S(\text{Res } \tilde{W})$  in section 3.

We finally pose some unsolved problems.

- Problem.**
- (1) Give a lower estimate of  $a_{M_p}$ .
  - (2) Is it true that  $a_G > 0$  for a  $p$ -group  $G$ ?

*Remark.* If  $a_G > 0$ , the weak Borsuk-Ulam theorem in the sense of Bartsch [1]. In finite group case, Bartsch shows that the weak Borsuk-Ulam theorem holds if and only if  $G$  is a  $p$ -group. Therefore, if problem (2) is true, then the weak Borsuk-Ulam theorem holds if and only if  $a_G > 0$ .

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