On equivariant cohomology rings of flag Bott towers

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1. Introduction

This article is mainly an overview of the paper [**KKLS**] which we generalize the flag Bott tower by two ways and compute their equivariant cohomology. We first briefly recall a history of the generalizations of the Bott tower.

A *Bott tower* is the following tower of bundles:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots B_2 \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} \{pt\}$$

where each π_i is the projectivization of sum of two line bundles. By definition, this is the iterated $\mathbb{C}P^1$-bundles. Since each stage is obtained by the sum of line bundles, the total space has the maximal T^m-action and this becomes a toric manifold (cf. $\mathbb{C}P$-tower defined in [**KS1, KS2**]). This object is introduced by Grossberg-Karshon in [**GK**] with the aim of the comparison between the Bott-Samelson variety and its maximal torus action which does not preserve the complex structure defined by Raoul Bott, i.e., a Bott tower. Grossberg-Karshon construct a one-parameter family between the Bott-Samelson variety and the Bott tower in [**GK**].

In [**MS**], Masuda-Suh define a *generalized Bott tower* which is the following tower of bundles:

$$GB_m \xrightarrow{\pi_m} GB_{m-1} \xrightarrow{\pi_{m-1}} \cdots GB_2 \xrightarrow{\pi_2} GB_1 \xrightarrow{\pi_1} \{pt\}$$

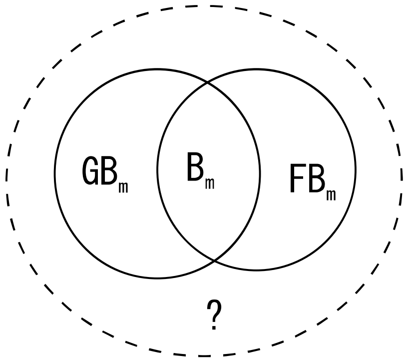
where each π_i is the projectivization of sum of several (may not be two) line bundles. By definition, the generalized Bott tower is an iterated complex projective space bundles and has the structure of a toric manifold. Masuda-Suh in [**MS**] ask the cohomological rigidity problem to this class of toric manifolds, i.e., if the cohomology rings of two generalized Bott towers are isomorphic, then are they diffeomorphic? The cohomological rigidity problem is still open and now it is regarded as one of the central problems in toric topology.

On the other hand, there seems to be no natural Bott-Samelson counterpart of a generalized Bott tower. Under this motivation, in [**KLSS**], Kuroki-Lee-Suh-Song define a *flag Bott tower* by the following tower of bundles:

$$FB_m \xrightarrow{\pi_m} FB_{m-1} \xrightarrow{\pi_{m-1}} \cdots FB_2 \xrightarrow{\pi_2} FB_1 \xrightarrow{\pi_1} \{pt\}$$

where each π_i is the flagification of sum of several line bundles. Since $Fl(\mathbb{C}^2) \cong \mathbb{C}P^1$, the flag Bott tower is a generalization of the Bott tower. Moreover, this generalization is more direct generalization of the Bott tower in [**GK**] than the generalized Bott tower. In fact, there is a Bott-Samelson counterparts of a flag Bott tower, called a *flag Bott-Samelson variety* (sse [**FLS**] or [**Ku**] also). We also note that there is a nice torus action on the flag Bott tower, and the flag Bott tower has the structure of a GKM manifold.

Now we have two generalizations of Bott towers. The following figure shows the relations among them:



The purpose of the paper [**KKLS**] is to define a (nice) class which contains both of the generalized Bott towers and the flag Bott towers. In [**KKLS**], we introduce the flag *Bott tower of general Lie types* by the following two constructions:

(1) Iterated pull-back bundles from the universal bundles;
(2) Quotient of the product of compact connected Lie groups.

We introduce them in Section 2 with the other generalizations. One of the merits of the first construction is to compute the (equivariant) cohomology by using the classical method introduced by Borel and Leray-Hirsh. On the other hand, one of the merits of the second construction is to define the torus action explicitly. The second main theorem of the paper [**KKLS**] is to compute the equivariant cohomology of the flag Bott towers of general Lie types for the explicit definition of torus actions (see Section 3). We show an example of the computation of an equivariant cohomology ring in Section 4. In the final section, Section 5, we also introduce the other construction of the symplectic flag bundles. We can also define the flag Bott tower of type C by this way.

2. Flag Bott tower of general Lie types

In this section, we introduce the two definitions of the flag Bott tower of general Lie types. We also introduce the other generalizations of Bott towers.

2.1. Two definitions. Let K_i be a compact, connected Lie group, T_i be its maximal torus, and $Z_i \subset K_i$ be the centralizer of a circle subgroup of T_i, $1 \leq i \leq m$. Recall that K_i/Z_i is often called a (generalized) flag manifold.

The first definition is the pull-back definition (see [**KKLS**, Definition 3.1]):

Definition 2.1. An *m-stage flag Bott tower $F_\bullet = \{F_j \mid 0 \le j \le m\}$ of general Lie type (or an m-stage flag Bott tower) associated to* $(K_\bullet, Z_\bullet) = \{(K_j, Z_j) \mid 1 \le j \le m\}$ is the following tower of bundles:

$$F_m \xrightarrow{\pi_m} F_{m-1} \xrightarrow{\pi_{m-1}} \cdots F_2 \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} \{pt\}$$

where each $\pi_i : F_i \to F_{i-1}$ $(i = 1, \ldots, m)$ is a K_i/Z_i-bundle associated to a characteristic map

$$f_i : F_{i-1} \to BT_i \to BK_i.$$

The assumption of f_i in this definition means that each K_i/Z_i-bundle admits the reduction of the structure group K_i to its maximal torus T_i. For the (flag and generalized) Bott tower this assumption corresponds to the assumption that each vector bundles split into line bundles.

The second definition is the quotient definition (see [**KKLS**, Definition 3.5]):

Definition 2.2. Let $(K_\bullet, Z_\bullet) = \{(K_j, Z_j) \mid 1 \le j \le m\}$. Given a family of homomorphisms $\{\varphi_j^{(\ell)} \colon Z_j \to T_\ell \mid 1 \le j < \ell \le m\}$, the space F_m^φ is defined by the orbit space

$$F_m^\varphi := (K_1 \times \cdots \times K_m)/(Z_1 \times \cdots \times Z_m),$$

where $(z_1, \ldots, z_m) \in Z_1 \times \cdots \times Z_m$ acts on $(g_1, \ldots, g_m) \in K_1 \times \cdots \times K_m$ from the right by

$$\begin{aligned}&(g_1, \ldots, g_m) \cdot (z_1, \ldots, z_m)\\ &:= \Big(g_1 z_1, \quad \varphi_1^{(2)}(z_1)^{-1} g_2 z_2, \quad \prod_{j=1}^{2} \varphi_j^{(3)}(z_j)^{-1} g_3 z_3, \quad \ldots, \quad \prod_{j=1}^{m-1} \varphi_j^{(m)}(z_j)^{-1} g_m z_m\Big).\end{aligned}$$

It is easy to check that F_m^φ admits the tower of generalized flag bundles. Moreover, since the image of $\varphi_j^{(\ell)}$ commutes with T_ℓ, we have that F_m^φ admits the natural $T_1 \times \cdots \times T_m$-action by the left multiplication.

We have the following proposition (see [**KKLS**, Proposition 3.6]):

Proposition 2.3. Let $F_\bullet = \{F_j \mid 0 \le j \le m\}$ be an m-stage flag Bott tower of general Lie type associated to $(K_\bullet, Z_\bullet)$, where K_j are simply-connected for all $1 \le j \le m$. Then there exists a family of homomorphisms $\{\varphi_j^{(\ell)} \colon Z_j \to T_\ell \mid 1 \le j < \ell \le m\}$ such that $F_\bullet$ and $F_\bullet^\varphi$ are isomorphic as flag Bott towers.

Remark 2.4. In the proof of [**KKLS**, Proposition 3.6], we need to assume all K_j's are simply-connected (e.g. $U(n)$ is not simply-connected). However, we can weaken this condition under some technical assumptions (see [**KKLS**, Remark 3.8]).

By taking an appropriate circle subgroups, we can take K_j/Z_j as both of flag manifolds and complex projective spaces (see Examples in [**KKLS**, Section 3]). Therefore, these constructions may be regarded as the generalization of both of GB_m and FB_m.

Remark 2.5. Some of the flag Bott towers of general Lie types also can be constructed from the vector bundles like (flag or generalized) Bott tower; e.g. the tower of partial flag bundles or the tower of flag manifolds of type C. In Section 5, we will give a construction of the symplectic flag bundle, i.e., the flag bundle of type C, from the even-dimensional complex vector bundle.

Remark 2.6. Note that the centralizer of $T^1 \subset Sp(1)$ is T^1 itself. Therefore, for example, the quaternionic projective space $Sp(n)/Sp(n-1) \times Sp(1)$ can not be obtained by the generalized flag manifold K_j/Z_j. Hence, the tower of quaternionic projective spaces is not the flag Bott tower of general Lie types (see the next subsection and Section 5).

2.2. Other generalizations. One can consider the several generalizations of Bott towers. For example, the $\mathbb{C}P$-towers defined in [**KS1, KS2**] may also be regarded as the generalization of Bott towers, though they do not have a nice torus action. In this subsection, we introduce two straightforward generalizations which may fit in toric topology.

2.2.1. *The homogeneous space Bott tower.* Let H_j be a maximal rank subgroup of K_j, i.e., a maximal torus $T_j \subset K_j$ is also a maximal torus of H_j. If we change Z_j into H_j in the definitions above, then it is easy to obtain the generalized object whose fibres are the homogeneous spaces K_j/H_j. We call this object a *homogeneous space Bott tower (with maximal ranks).* The homogeneous space Bott tower also contains the tower of quaternionic projective spaces.

With the method similar to that demonstrated in the quotient construction F_m^{φ}, we can define a natural torus action on the homogeneous space Bott tower defined by the quotient construction. Since the T_j-action on K_j/H_j admits the structure of a GKM manifold (see [**GHZ**]), the $T_1 \times \cdots \times T_m$-action on the homogeneous space Bott tower also admits the structure of a GKM manifold.

2.2.2. *The GKM Bott tower.* Let M_j be a manifold with a torus T_j-action, $1 \leq j \leq m$. Consider the following tower of bundles:

$$X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \cdots X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \{pt\}$$

where each $\pi_i : X_i \to X_{i-1}$ $(i = 1, \ldots, m)$ is a M_i-bundle associated to a characteristic map

$$f_i : X_{i-1} \to BT_i,$$

i.e., the pull back of the following bundle:

$$\begin{array}{ccc} X_i & \longrightarrow & ET_i \times_{T_i} M_i \\ \pi_i \downarrow & & \downarrow \\ X_{i-1} & \xrightarrow{f_i} & BT_i \end{array}$$

If we assume all M_i's are simply connected, by the similar methods in [**HY**], every torus actions on X_{i-1} lift to actions on X_i. Therefore, if all M_i's are simply connected, then X_m has the $T_1 \times \cdots \times T_m$-action which induced from the actions on the lower stages of the tower. Moreover, if all M_i's are simply connected GKM manifolds, then X_m admits the structure of a GKM manifold like a homogeneous space Bott tower. We call such a X_m a *GKM Bott tower.* The GKM fibre bundle is studied in [**GSZ**].

Remark 2.7. In general, there may be no quotient construction of the GKM Bott tower. However, if each M_j is a (quasi)toric manifold, then we can obtain this tower by the quotient construction because all (quasi)toric manifolds can be obtained by the quotients of the free torus actions on the moment-angle manifolds (see [**BP**]).

3. Equivariant cohomology ring

In this section, we introduce the equivariant cohomology ring of the flag Bott tower of general Lie types. We first prepare notations. Let F_m^{φ} be the flag Bott tower of general Lie types in Definition 2.2 and $\mathbb{T} = T_1 \times \cdots \times T_m$, where $T_j \simeq T^{n_j}$. Let R be a PID in which torsion primes of all K_j are invertible. In this paper, we consider the cohomology over R-coefficient. The symbol $W(Z)$ represents the product of Weyl groups $\prod_{j=1}^{m} W(Z_j)$. Let $\mathbf{u}_j$ and $\mathbf{y}_j$ stand for $(u_{j,1}, \ldots, u_{j,n_j})$ and $(y_{j,1}, \ldots, y_{j,n_j})$, respectively for $j = 1, \ldots, m$. The symbol $\mathbf{u}_j$ shall be used for the generators of the cohomology $H^*(BT_j) \simeq R[\mathbf{u}_j]$, where $T_j \simeq T^{n_j}$ is one of the factors of the torus $\mathbb{T}$ acting on F_m^{φ}. On the other hand, the symbol $\mathbf{y}_j$ shall be used for the generators of the cohomology $H^*(BT_j) \simeq R[\mathbf{y}_j]$, where T_j is a maximal torus of $Z_j \subset K_j$. Recall the cohomology ring formula:

$$H^*(BK_j) \simeq R[\mathbf{y}_j]^{W(K_j)}$$

and

$$H^*(BZ_j) \simeq R[\mathbf{y}_j]^{W(Z_j)}.$$

Then, we have the following theorem [**KKLS**, Corollary 4.3]:

Theorem 3.1. *There is the following isomorphism:*

$$H^*_{\mathbb{T}}(F_m^{\varphi}; R) \simeq R[\mathbf{u}_1, \ldots, \mathbf{u}_m] \otimes_R (R[\mathbf{y}_1, \ldots, \mathbf{y}_m])^{W(Z)} / \langle I_1, \ldots, I_m \rangle$$

where I_ℓ, for $1 \leq \ell \leq m$, is the ideal generated by the polynomials

$$h(\mathbf{y}_\ell) - h\left(\mathbf{u}_\ell + \sum_{j=1}^{\ell-1} \Phi_j^{(\ell)}(\mathbf{y}_\ell)\right)$$

for $h \in R[\mathbf{y}_\ell]^{W(K_\ell)}$ and

$$\Phi_j^{(\ell)}(\mathbf{y}_\ell) := \left((\varphi_j^{(\ell)})^*(y_{\ell,1}), \ldots, (\varphi_j^{(\ell)})^*(y_{\ell,n_\ell})\right).$$

Here, in the statement of this theorem, the invariant polynomial $h \in R[\mathbf{y}_\ell]^{W(K_\ell)}$ means

$$h(\mathbf{y}_\ell) = h(y_{\ell,1}, \ldots, y_{\ell,n_\ell})$$

and

$$\begin{aligned} &h\left(\mathbf{u}_\ell + \sum_{j=1}^{\ell-1} \Phi_j^{(\ell)}(\mathbf{y}_\ell)\right) \\ =&h(u_{\ell,1} + (\varphi_1^{(\ell)})^*(y_{\ell,1}) + \cdots + (\varphi_{\ell-1}^{(\ell)})^*(y_{\ell,1}), \ldots \\ &\qquad \ldots, u_{\ell,n_\ell} + (\varphi_1^{(\ell)})^*(y_{\ell,n_\ell}) + \cdots + (\varphi_{\ell-1}^{(\ell)})^*(y_{\ell,n_\ell})), \end{aligned}$$

where $\mathbb{T}' = T_1 \times \cdots \times T_{\ell-1}$ and

$(\varphi_j^{(\ell)})^* : H^*(BT_\ell) \to H^*_{\mathbb{T}'}(F^\varphi_{\ell-1}) \simeq R[\mathbf{u}_1, \ldots, \mathbf{u}_{\ell-1}] \otimes_R (R[\mathbf{y}_1, \ldots, \mathbf{y}_{\ell-1}])^{\prod_{j=1}^{\ell-1} W(Z_j)} / \langle I_1, \ldots, I_{\ell-1} \rangle$

is the induced homomorphism of $B\varphi_j^{(m)} \circ \overline{u}$ which appears in the following composition maps:

$$\widetilde{f}_\ell : E\mathbb{T}' \times_{\mathbb{T}'} F^\varphi_{\ell-1} \xrightarrow{\overline{u}} B\left(\prod_{j=1}^{\ell-1} Z_j\right) \xrightarrow{B(\mathrm{mul})\circ B\varphi} BT_\ell \xrightarrow{B\iota} BK_\ell,$$

where $\overline{u}$ is the characteristic map which determines the following fibre bundle:

$$\prod_{j=1}^{\ell-1} Z_j \to E\mathbb{T}' \times_{\mathbb{T}'} \prod_{j=1}^{\ell-1} K_j \to E\mathbb{T}' \times_{\mathbb{T}'} F^\varphi_{\ell-1}$$

and mul is the multiplication of T_ℓ and $\varphi := \prod_{j=1}^{\ell-1} \varphi_j^{(\ell)} : \prod_{j=1}^{\ell-1} Z_j \to T_\ell$. Note that $\widetilde{f}_\ell$ is the characteristic map which determines the following fibre bundle:

$$K_\ell/Z_\ell \to E\mathbb{T}' \times_{\mathbb{T}'} F^\varphi_\ell \to E\mathbb{T}' \times_{\mathbb{T}'} F^\varphi_{\ell-1}.$$

4. An explicit example

The invariant polynomial rings of the Weyl group $W(SU(n+1))$ is well-known as follows:

$$R[t_1, \ldots t_n]^{W(SU(n+1))} \simeq R[\widetilde{\sigma}_2, \ldots, \widetilde{\sigma}_n]$$

where $\widetilde{\sigma}_i(t_1, \ldots, t_n) := \sigma_i(t_1, \ldots, t_n, t_1 + \cdots + t_n)$ for the degree i elementary symmetric polynomial σ_i with $n+1$ variables. For example,

$$R[t]^{W(SU(2))} \simeq R[t^2];$$
$$R[t_1, t_2]^{W(SU(3))} \simeq R[t_1^2 + 3t_1t_2 + t_2^2, t_1t_2(t_1 + t_2)].$$

Moreover, the invariant polynomial rings of the Weyl group $W(G_2) \simeq D_6$ is as follows (see [**Ke**]):

$$R[t_1, t_2]^{W(G_2)} = R[(t_1 - t_2)^2, t_1^2t_2^2(t_1 + t_2)^2].$$

Define the 2-stage flag Bott tower of general Lie types

$$F^\varphi_2 := G_2 \times_{T^2} (SU(2)/T^1)$$

by the homomorphism $\varphi : T^2 \to T^1$ such that $(t_1, t_2) \mapsto t_1t_2$. Namely, the following fibre bundle:

$$SU(2)/T^1 \to F^\varphi_2 \to G_2/T^2,$$

Then, $T^2 \times T^1$ acts on F^φ_2 naturally. Since $H_*(G_2; \mathbb{Z})$ has 2-torsion, we may apply Theorem 3.1 for the computation of the equivariant cohomology over $\mathbb{F}_3$ (i.e., the finite field of order 3) coefficient. By using the formula of the invariant polynomials as above and Theorem 3.1, we have the following ring isomorphism:

$$H^*_{T^2\times T^1}(F_2^{\varphi};\mathbb{F}_3)\cong \mathbb{F}_3[u_{1,1},u_{1,2},u_{2,1},y_{1,1},y_{1,2},y_{2,1}]/\langle I_1, I_2\rangle$$

where

$$I_1=(1+(y_{1,1}-y_{1,2})^2)(1+y_{1,1}^2y_{1,2}^2(y_{1,1}+y_{1,2})^2)-(1+(u_{1,1}-u_{1,2})^2)(1+u_{1,1}^2u_{1,2}^2(u_{1,1}+u_{1,2})^2)$$

and

$$\begin{aligned} I_2 &= (1+y_{2,1}^2)-(1+(u_{2,1}+\Phi_1^{(2)}(y_{2,1}))^2)\\ &= y_{2,1}^2-u_{2,1}^2-2u_{2,1}(y_{1,1}+y_{1,2})-(y_{1,1}+y_{1,2})^2 \end{aligned}$$

where $\Phi_1^{(2)}$ may be regarded as the homomorphism $H^*(BT^1;\mathbb{F}_3)\to H^*(BT^2;\mathbb{F}_3)$ such that $y_{2,1}\mapsto y_{1,1}+y_{1,2}$ which is the induced homomorphism from $\varphi: T^2\to T^1$.

5. More direct computation of the equivariant cohomology of $Sp(n)/T^n$

In the previous paper [**KS2**], we interpret the flag manifold of type C, i.e., $Sp(n)/T^n$ as the tower of complex projective spaces (which is not the generalized Bott tower). So it is natural to ask how we can compute the equivariant cohomology of $Sp(n)/T^n$ by using the Borel-Hirzebruch formula. In this final section, we briefly introduce the other kind of the construction of $Sp(n)/T^n$ by using the quaternionic flag manifold $Sp(n)/Sp(1)^n$ and compute its equivariant cohomology by using this construction and the Borel-Hirzebruch formula.

5.1. The relation with the quaternionic flag manifold. Let F_n be $Sp(n)/T^n$, and B_n be the quaternionic flag manifold $Sp(n)/Sp(1)^n$. It is easy to check that there is a $Sp(1)^n/T^n\simeq (S^2)^n$ fibration on the natural projection map $q_n: F_n\to B_n$. We first interpret this map as the morphism of two towers, i.e., we will construct the fibration $q_n: F_n\to B_n$ from the following set of maps $q_\bullet=\{q_j: F_j\to B_j\}$, say $q_\bullet: F_\bullet\to B_\bullet$:

$$\begin{tikzcd}
& q_n^*E_{n+1} \arrow[dl] \arrow[dd] & & q_{n-1}^*E_n \arrow[dl] \arrow[dd] & & q_{n-2}^*E_{n-1} \arrow[dl] \arrow[dd] & & & q_1^*E_2 \arrow[dl] \arrow[dd] & & q_0^*E_1 \arrow[dl] \arrow[dd] \\
F_n \arrow[rr, "p_n"] \arrow[dd, "q_n"] & & F_{n-1} \arrow[rr, "p_{n-1}"] \arrow[dd, "q_{n-1}"] & & F_{n-2} \arrow[rr, "p_{n-2}"] \arrow[dd, "q_{n-2}"] & & \cdots \arrow[r, "p_2"] & F_1 \arrow[rr, "p_1"] \arrow[dd, "q_1"] & & F_0 \arrow[dd, "q_0=\mathrm{id}"] & \\
& E_{n+1} \arrow[dl] & & E_n \arrow[dl] & & E_{n-1} \arrow[dl] & & & E_2 \arrow[dl] & & E_1 \arrow[dl] \\
B_n \arrow[rr, "\pi_n"] & & B_{n-1} \arrow[rr, "\pi_{n-1}"] & & B_{n-2} \arrow[rr, "\pi_{n-2}"] & & \cdots \arrow[r, "\pi_2"] & B_1 \arrow[rr, "\pi_1"] & & B_0 &
\end{tikzcd}$$

We first define the rightest block as follows:

$$
\begin{tikzcd}
 & q_1^*E_2 \arrow[dl] \arrow[dd] & & q_0^*\mathbb{H}^n \equiv (\mathbb{C}\oplus\overline{\mathbb{C}})^n \arrow[dl] \arrow[dd] \\
\mathbb{C}P^{2n-1} \arrow[rr, "p_1"] \arrow[dd, "q_1"] & & \{*\} \arrow[dd, "q_0(=)"] & \\
 & E_2 \arrow[dl] & & \mathbb{H}^n \arrow[dl] \\
\mathbb{H}P^{n-1} \arrow[rr, "\pi_1"] & & \{*\} &
\end{tikzcd}
$$

where $\mathbb{H}$ is the quaternionice space and the left vertical map is induced from the following $\mathbb{C}P^1$-bundle:

$$q_1 : \mathbb{C}P^{2n-1} = \mathbb{P}(\mathbb{C}^{2n}) \longrightarrow \mathbb{H}P^{n-1} = \mathbb{H}P(\mathbb{H}^n)$$

which may be regarded (in the homogeneous space languages) as

$$Sp(n)/U(1) \times Sp(n-1) \longrightarrow Sp(n)/Sp(1) \times Sp(n-1).$$

Note that

$$E_2 = \mathbb{H}P^{n-1} \times \mathbb{H}^n$$

and

$$q_1^*E_2 = \mathbb{C}P^{2n-1} \times (\mathbb{C}\oplus\overline{\mathbb{C}})^n.$$

In order to get the next tower (see the following diagram), we first take the tautological quaternionic line bundle of E_2, say γ_2, and the normal bundle $\gamma_2^\perp$ in E_2 with respect to the usual Hermitian metric of the quaternionic space. Then, we can define the following pull-back bundles:

$$
\begin{tikzcd}
 & q_2^*E_3 \arrow[dl] \arrow[dd] & & q_1^*\gamma_2^\perp \subset q_1^*E_2 \arrow[dl] \arrow[dd] \\
\mathbb{P}(q_1^*\gamma_2^\perp) \arrow[rr, "p_2"] \arrow[dd, "q_2"] & & \mathbb{C}P^{2n-1} \arrow[dd, "q_1"] & \\
 & E_3 = \pi_2^*\gamma_2^\perp \arrow[dl] & & \gamma_2^\perp \subset E_2 \arrow[dl] \\
\mathbb{H}P(\gamma_2^\perp) \arrow[rr, "\pi_2"] & & \mathbb{H}P^{n-1} &
\end{tikzcd}
$$

Here, each map is defined as follows:

$$\pi_2 : \mathbb{H}P(\gamma_2^{\perp}) = \bigcup_{y \in \mathbb{H}P^{n-1}} \mathbb{H}P((\gamma_2^{\perp})_y) \longrightarrow \mathbb{H}P^{n-1}$$

is the quaternionic projectivization whose fibre is $\mathbb{H}P^{n-2} \simeq \mathbb{H}P((\gamma_2^{\perp})_y)$, where $(\gamma_2^{\perp})_y$ is the fibre of $\gamma_2^{\perp}$ over $y \in \mathbb{H}P^{n-1}$;

$$p_2 : \mathbb{P}(q_1^*\gamma_2^{\perp}) = \bigcup_{x \in \mathbb{C}P^{2n-1}} \mathbb{P}((q_1^*\gamma_2^{\perp})_x) \longrightarrow \mathbb{C}P^{2n-1}$$

is the complex projectivization whose fibre is $\mathbb{C}P^{2n-3} \simeq \mathbb{P}((q_1^*\gamma_2^{\perp})_x)$, where $(q_1^*\gamma_2^{\perp})_x$ is the fibre of $q_1^*\gamma_2^{\perp}$ over $x \in \mathbb{C}P^{2n-1}$;

$$q_2 : \mathbb{P}(q_1^*\gamma_2^{\perp}) = \bigcup_{x \in \mathbb{C}P^{2n-1}} \mathbb{P}((q_1^*\gamma_2^{\perp})_x) \longrightarrow \mathbb{H}P(\gamma_2^{\perp}) = \bigcup_{y \in \mathbb{H}P^{n-1}} \mathbb{H}P((\gamma_2^{\perp})_y)$$

is defined by

$$q_2(x, \ell) = (q_1(x), [\ell])$$

where $[\ell] \in \mathbb{H}P((\gamma_2^{\perp})_{q_1(x)}) \simeq \mathbb{H}P^{n-2}$ is the image of $\ell \in \mathbb{P}((q_1^*\gamma_2^{\perp})_x) \simeq \mathbb{C}P^{2n-3}$ which induced from the map

$$\mathbb{C}P^{2n-3} = Sp(n-1)/U(1) \times Sp(n-2) \to \mathbb{H}P^{n-2} = Sp(n-1)/Sp(1) \times Sp(n-2).$$

In other words, we construct the map from the fibres of

$$\mathbb{C}P^{2n-3} \longrightarrow F_2 = \mathbb{P}(q_1^*\gamma_2^{\perp}) = Sp(n)/U(1) \times U(1) \times Sp(n-2) \xrightarrow{p_2} \mathbb{C}P^{2n-1} = F_1$$

to the fibres of

$$\mathbb{H}P^{n-2} \longrightarrow B_2 = \mathbb{H}P(\gamma_2^{\perp}) = Sp(n)/Sp(1) \times Sp(1) \times Sp(n-2) \xrightarrow{\pi_2} \mathbb{H}P^{n-1} = B_1$$

Similarly we can define $E_{j+1} = \pi_j^*\gamma_j^{\perp}$ and define the diagram $q_\bullet : F_\bullet \to B_\bullet$ by iteration. Then, from this tower, we get

$$F_n \simeq SpFl(\mathbb{C}^{2n}) \simeq Sp(n)/T^n, \quad B_n \simeq Fl_{\mathbb{H}}(\mathbb{H}^n) \simeq Sp(n)/Sp(1)^n,$$

and the fibre bundle $q_n : F_n \to B_n$ whose fibre is the trivial Bott-tower, where $SpFl(\mathbb{C}^{2n})$ is the set of the symplectic flags in $\mathbb{C}^{2n}$ for some symplcecti structure ω and $Fl_{\mathbb{H}}(\mathbb{H}^n)$ is the set of the quaternionic flags in $\mathbb{H}^n$.

5.2. The symplectic flag bundle and the Bott tower of type C. We next generalize this construction for more general manifolds. Let us define the following diagram:

$$\begin{tikzcd}[column sep=small]
& q_n^*E_{n+1} \arrow[dl] \arrow[dd] & & q_{n-1}^*E_n \arrow[dl] \arrow[dd] & & q_{n-2}^*E_{n-1} \arrow[dl] \arrow[dd] & & & q_1^*E_2 \arrow[dl] \arrow[dd] & & q_0^*E_1 \arrow[dl] \arrow[dd] \\
F_n \arrow[rr, "p_n"] \arrow[dd, "q_n"] & & F_{n-1} \arrow[rr, "p_{n-1}"] \arrow[dd, "q_{n-1}"] & & F_{n-2} \arrow[rr, "p_{n-2}"] \arrow[dd, "q_{n-2}"] & & \cdots \arrow[r, "p_2"] & F_1 \arrow[rr, "p_1"] \arrow[dd, "q_1"] & & F_0 \arrow[dd, "q_0=\mathrm{id}"] & \\
& E_{n+1} \arrow[dl] & & E_n \arrow[dl] & & E_{n-1} \arrow[dl] & & & E_2 \arrow[dl] & & E_1 \arrow[dl] \\
B_n \arrow[rr, "\pi_n"] & & B_{n-1} \arrow[rr, "\pi_{n-1}"] & & B_{n-2} \arrow[rr, "\pi_{n-2}"] & & \cdots \arrow[r, "\pi_2"] & B_1 \arrow[rr, "\pi_1"] & & B_0 &
\end{tikzcd}$$

Here, $B_0 = F_0 = M$ is any smooth manifold (cf. this was the point in the previous construction), E_1 is a quaternionic vector bundle over B_0 and $q_0^*E_1$ is a complex vector bundle over F_0 forgetting the quaternionic structure. Here, similarly, the first and the second blocks are defined by the following pull-back bundles:

$$\begin{tikzcd}
 & q_2^*E_3 \arrow[dl] \arrow[dd] & & q_1^*E_2 \equiv q_1^*\gamma_2 \oplus q_1^*\gamma_2^{\perp} \arrow[dl] \arrow[dd] & & q_0^*E_1 \arrow[dl] \arrow[dd] \\
\mathbb{P}(q_1^*\gamma_2^{\perp}) \arrow[rr, "p_2"] \arrow[dd, "q_2"] & & \mathbb{P}(q_0^*E_1) \arrow[rr, "p_1"] \arrow[dd, "q_1"] & & M \arrow[dd, "q_0"] & \\
 & E_3 := \pi_2^*\gamma_2^{\perp} \arrow[dl] & & E_2 := \pi_1^*E_1 \equiv \gamma_2 \oplus \gamma_2^{\perp} \arrow[dl] & & E_1 \arrow[dl] \\
\mathbb{H}P(\gamma_2^{\perp}) \arrow[rr, "\pi_2"] & & \mathbb{H}P(E_1) \arrow[rr, "\pi_1"] & & M &
\end{tikzcd}$$

where $\gamma_2 \subset E_2$ is the tautological quaternionic line bundle. By iterating this process, we obtain the morphism (of towers) between the symplectic flag bundle $F_n = SpFl(E_1)$ and the quaternionic flag bundle $B_n = Fl_{\mathbb{H}}(E_1)$ over M.

Remark 5.1. Iterating this construction from the point, i.e., $M = \{*\}$, we also obtain the map between the flag Bott tower of type C and the tower of quaternionic flag manifolds.

5.3. Cohomology of symplectic flag bundle. From the previous construction, the $Sp(n)/T^n$-bundle $p : F_n \to M$ is obtained by the symplectic flag of the following sum of quaternionic line bundles:

$$p^*(q_0^*E_1) \equiv \oplus_{j=1}^n q_j^*\gamma_{j+1}$$

where the 1-dim quaternionic vector bundles $q_j^*\gamma_{j+1}$ in the right-hand means the pull-back to the top stage (we abuse the notation). By this decomposition, we can compute its Chern class as follows:

$$c(p^*(q_0^*E_1)) = c(\oplus_{j=1}^n q_j^*\gamma_{j+1}) = \prod_{j=1}^{n} c(q_j^*\gamma_{j+1}) \tag{5.1}$$

Note that γ_{j+1} is the complex 2-dimensional bundle forgetting quaternionic structure. Therefore, γ_{j+1} split into two complex line bundles with different orientations, i.e., $q_j^*\gamma_{j+1} \equiv \xi_{j+1} \oplus \overline{\xi}_{j+1}$ for some line bundle ξ_{j+1}. Thus, we have

$$c(q_j^*\gamma_{j+1}) = c(\xi_{j+1})c(\overline{\xi}_{j+1}) = (1 + x_{j+1})(1 - x_{j+1}) = 1 - x_{j+1}^2 = 1 - p_1(q_j^*\gamma_{j+1})$$

where $x_{j+1} = c_1(\xi_{j+1})$ is the first Chern class of ξ_{j+1} and $p_1(q_j^*\gamma_{j+1})$ is the first Pontrjagin class of $q_j^*\gamma_{j+1}$. Note that $x_{n+1} = 0$. Therefore, by using the Borel-Hirzebruch formula, we have that the equation (5.1) is the unique relation in the cohomology of $H^*(F_n)$ as $H^*(M)$-algebra:

Proposition 5.2. There is the following $H^*(M)$-algebra isomorphism:

$$H^*(F_n) \simeq H^*(M)[x_1, \ldots, x_n]/\langle c(E_1) = \prod_{j=1}^{n}(1 - x_j^2)\rangle$$

for some degree 2 elements x_j and $c(E_1) \in H^*(M)$.

Remark 5.3. Because E_1 has the quaternionic structure, the splitting principle shows that it's Chern class behaves like the vector bundle splits into n complex line bundles and its reversed oriented complex line bundles. Therefore, $c(E_1)$ is written by $\prod_{j=1}^{n}(1 - y_j^2)$ for some $y_j \in H^2(M)$. This gives the same formula with [**FP**, (6.10) in p73].

5.4. Severl cohomology rings. Several formula of cohomology rings can be obtained by Proposition 5.2. For example, if $M = \{*\}$, we have the following well-known formula of $Sp(n)/T^n$:

$$H^*(Sp(n)/T^n) \simeq \mathbb{Z}[x_1, \ldots, x_n]/\langle 1 = \prod_{j=1}^{n}(1 - x_j^2)\rangle$$

If T_1 acts on M and E_1 is an equivariant vector bundle, then the formula in Proposition 5.2 just changes into the equivariant counterpart of the ordinary cohomology as follows:

$$H^*_{T_1}(F_n) \simeq H^*_{T_1}(M)[x_1, \ldots, x_n]/\langle c^{T_1}(E_1) = \prod_{j=1}^{n}(1 - x_j^2)\rangle$$

More explicitly, by the similar reason in Remark 5.3, there exists $\sigma_j \in H^2_{T_1}(M)$ which corresponds to the first Chern class appeared in the decomposition of the pull-back of the equivariant bundle E_1 such that

$$H^*_{T_1}(F_n) \simeq H^*_{T_1}(M)[x_1, \ldots, x_n]/\langle \prod_{j=1}^{n}(1 - \sigma_j^2) = \prod_{j=1}^{n}(1 - x_j^2)\rangle$$

If we consider the $T = T_1 \times T_2$-action on F_n such that T_2 acts only on each fibre (i.e., acts on M trivially), then

$$H^*_T(F_n) \simeq H^*(BT_2) \otimes H^*_{T_1}(M)[x_1, \ldots, x_n]/\langle \prod_{j=1}^{n}(1 - (\sigma_j + t_j)^2) = \prod_{j=1}^{n}(1 - x_j^2)\rangle$$

where $t_j \in H^2(BT_2)$ ($j = 1, \ldots, n$) is determined by the character of the representation $t_j : T_2 \to S^1$ on the line bundle corresponding to σ_j. In particular, if $M = \{*\}$ and $T_1 = \{e\}$, this formula gives the well-known formula of equivariant cohomology of flag manifold of type C:

$$
\begin{aligned}
H_T^*(Sp(n)/T^n) &\simeq H^*(BT)[x_1,\dots,x_n]/\langle\prod_{j=1}^n(1-t_j^2)=\prod_{j=1}^n(1-x_j^2)\rangle \\
&\simeq \mathbb{Z}[x_1,\dots,x_n,t_1,\dots,t_n]/\langle\prod_{j=1}^n(1-t_j^2)=\prod_{j=1}^n(1-x_j^2)\rangle
\end{aligned}
$$

where we may take t_j as the standard basis of $H^*(BT)$ $(T=T^n)$. This formula coincides with our formula in Theorem 3.1 for the 1-stage flag Bott tower of type C.

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