Alternating groups and Borsuk-Ulam groups

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1 Introduction

For a compact Lie group G, a G-map $f\colon X \to Y$ is said to be a G-isovariant map if f preserves the isotropy subgroups: $G_x = G_{f(x)}$ for any $x \in X$, where G_x is the isotropy subgroup, that is, $G_x = \{g \in G \mid g \cdot x = x\}$. We call a group G is a BUG (Borsuk-Ulam group) [8] if

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

for any isovariant G-map $f\colon V \to W$ between G-representation spaces V and W. For example, any finite solvable group is a BUG. So, we expect that any group is a BUG. In this paper, we always assume that a group is a finite group. Since a group extension of BUGs is also a BUG, if every simple group is a BUG, then any group is a BUG. Nagasaki and Ushitaki [4] showed that projective special linear group $\mathrm{PSL}(2,q)$ of 2×2 matrices over a finite field $\mathbb{F}_q$ consisting of q elements is a BUG. Let $f\colon V \to W$ be a G-map between G-representation spaces. For a subgroup H of G, let

$$g_f(H) = (\dim W - \dim W^H) - (\dim V - \dim V^H).$$

The map g_f is a class function $\mathcal{S}(G) \to \mathbb{Z}$, where $\mathcal{S}(G)$ is the set of subgroups of G. If f is isovariant and G is cyclic, then $g_f(G) \geq 0$ by (mod p) Borsuk-Ulam theorem [5, 3]. Nagasaki and Ushitaki used the Möbius function and showed $g_f(\mathrm{PSL}(2,q))$ can be written as a conical combination of $g_f(C)$'s for cyclic subgroups C of $\mathrm{PSL}(2,q)$, that is, a linear combination of $g_f(C)$'s with nonnegative coefficients.

Last year in [7] we gave a sufficient condition CCG for a group G to be a BUG and showed that $\mathrm{PSL}(3,q)$ for $q \leq 33$ and A_n for $n \leq 21$ are BUGs. In particular, we showed that the alternating group A_n for $n \leq 21$ is a CCG but A_{22} is not. This paper consists of 2 parts. The first part is for $\mathrm{PSL}(3,q)$ and $\mathrm{PSU}(3,q)$ and we show they are BUGs. The second part is for A_n and we propose a new condition and show that A_n for $22 \leq n \leq 27$ is a BUG.

2 Some families of finite groups

Let $\mu\colon \mathbb{N} \to \{0, \pm 1\}$ be the Möbius function defined as

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^r & n = p_1 p_2 \cdots p_r \text{ for distinct primes } p_1, p_2, \ldots, p_r. \end{cases} .$$

Let $\mathrm{RCycl}(G)$ be the set of representatives of conjugacy classes of all cyclic subgroups of G and let $\mathrm{RCycl}_1(G)$ be the set of representatives of conjugacy classes of all nontrivial cyclic subgroups of G. Recall that $g_f(\{e\}) = 0$. We define $\tilde{\mu}$ as

$$\tilde{\mu}(C, D) = \begin{cases} \mu(\frac{|D|}{|C|}), & (C) \leq (D) \\ 0, & \text{otherwise,} \end{cases}$$

where (C) denotes the conjugacy class of C. Let

$$\beta_G(C, D) = \frac{|C|\tilde{\mu}(C, D)}{|N_G(D)|}$$

and

$$\beta_G(C) = \sum_{D \in \mathrm{RCycl}(G)} \beta_G(C, D).$$

Proposition 2.1 (cf. [7, Proposition 6])

$$g_f(G) = \sum_{C \in \mathrm{RCycl}(G)} \beta_G(C) g_f(C). \tag{1}$$

We recall that G is a Borsuk-Ulam group (BUG) if $g_f(G) \geq 0$ for any isovariant G-map f between G-representation spaces.

From now on, let $f\colon V \to W$ be an isovariant G-map between G-representation spaces. We abbreviate to write $g_f(G)$ as $g(G)$ if f is obvious.

Theorem 2.2 (Fundamental properties [8], [7, Proposition 3.1]) (1) *A finite cyclic group is a* BUG.

(2) *For a subgroup H_1, H_2 of G with $H_1 \triangleleft H_2$, $g_f(H_2) - g_f(H_1) = g_{f^{H_1}}(H_2/H_1)$ and if H_2/H_1 is a* BUG *then $g_f(H_2) \geq g_f(H_1)$. In particular, a finite group which is a group extension of a* BUG *by a* BUG *is also a* BUG.

(3) *If G is a* BUG, *then any factor group of G is a* BUG.

In [7] we proposed that G is a CCG (cyclic condition group), if for an arbitrary map $\gamma_G\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ such that $\gamma_G(C) \leq \gamma_G(D)$ if $(C) \leq (D)$, $\sum_{C \in \mathrm{RCycl}_1(G)} \beta_G(C)\gamma_G(C) \geq 0$.

Proposition 2.3 *A* CCG *is a* BUG.

Proof Let G be a CCG and f a G-map between representation G-spaces. The map $g_f|_{\mathrm{RCycl}(G)}\colon \mathrm{RCycl}(G) \to \mathbb{Z}$ satisfies that $g_f(C) \leq g_f(D)$ if $(C) \leq (D)$ by Theorem 2.2, since a cyclic group is a BUG. Thus we have

$$g_f(G) = \sum_{C \in \mathrm{RCycl}_1(G)} \beta_G(C) g_f(C) \geq 0,$$

which implies G is a BUG. ∎

Let $\mathrm{RCycl}_1^+(G)$ and $\mathrm{RCycl}_1^-(G)$ be the subsets of $\mathrm{RCycl}_1(G)$ consisting of C with $\beta_G(C) > 0$ and $\beta_G(C) < 0$, respectively.

We consider the following linear programming:

$$\underset{\psi\colon \mathrm{RCycl}_1^-(G) \times \mathrm{RCycl}_1^+(G) \to \mathbb{Q}_{\leq 0}}{\text{Maximize}} \quad \min_{D \in \mathrm{RCycl}_1^+(G)} \left(\beta_G(D) + \sum_{C \in \mathrm{RCycl}_1^-(G)} \psi(C, D) \right)$$

$$\text{subject to} \begin{cases} \psi(C, D) \leq 0 \\ \psi(C, D) = 0 \text{ if } (C) \not\leq (D) \\ \sum_{D \in \mathrm{RCycl}_1^+(G)} \psi(C, D) \leq \beta_G(C) \text{ for } C \in \mathrm{RCycl}_1^-(G) \\ \sum_{C \in \mathrm{RCycl}_1^-(G)} \psi(C, D) \geq -\beta_G(D) \text{ for } D \in \mathrm{RCycl}_1^+(G) \end{cases}$$

and had the following theorem by using the software GAP [2].

Theorem 2.4 ([7]) (1) *Alternating groups A_n and symmetric groups S_n for $n \leq 21$ are* CCG*s.*

(2) *A_{22} is not a* CCG *although S_{22} is a* CCG.

(3) *All sporadic groups and automorphism groups of all sporadic groups are* CCG*s.*

(4) *$(C_{30})^5$ is not a* CCG.

Let $\gamma_G\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ be a map such that $\gamma_G(C) \leq \gamma_G(D)$ if C is subconjugate to D in G and let $\mathrm{Cycl}_1(G)$ be the set of all nontrivial cyclic subgroups of G. We define $\bar{\gamma}_G\colon \mathrm{Cycl}_1(G) \to \mathbb{Q}_{\geq 0}$ as a class function which sends a cyclic subgroup C of G to $\gamma_G(C')$ such that $C' \in \mathrm{RCycl}_1(G)$ is conjugate to C in G. Let

$$\mathcal{S} = \{(C, D) \mid C, D \in \mathrm{RCycl}_1(G),\ C \text{ is subconjugate to } D, \text{ and } |D|/|C| \text{ is a prime}\}.$$

Let $C, D \in \mathrm{Cycl}_1(G)$ with $D > C$. We take $D_0, D_1, \ldots, D_k \in \mathrm{RCycl}_1(G)$ of G such that D_0 and D_k are conjugate to D and C in G respectively and $(D_i, D_{i-1}) \in \mathcal{S}$ for $i = 1, \ldots, k$. Then

$$\bar{\gamma}_G(D) - \bar{\gamma}_G(C) = \sum_{i=1}^{k} (\gamma_G(D_{i-1}) - \gamma_G(D_i)).$$

Therefore, we obtain the following proposition.

Proposition 2.5 *A finite group G is a* CCG *if and only if it can be detected by* $\{(C, D) \mid C, D \in \mathrm{RCycl}(G),\ (D) \subset (C),\ |D|/|C| \text{ is a prime}\}$.

By Theorem 2.4 (2), CCG is not closed under extensions although BUG is closed.

We say that a finite group G has subgroup-condition property (SCP) if $g_f(G)$ is equal to a conical combination of $\{g_f(K_2) - g_f(K_1) \mid K_2/K_1 \text{ is a BUG with } K_1 \lhd K_2 < G\}$ for any isovariant G-map f between representation spaces.

Proposition 2.6 *A group having* SCP *is a* BUG.

Proof Let $f\colon V \to W$ be an isovariant G-map between representation spaces. Let $K_1 \lhd K_2 < G$. Note that

$$g_f(K_2) - g_f(K_1) = g_{f^{K_1}}(K_2/K_1).$$

Thus if K_2/K_1 is a BUG, then $g_{f^{K_1}}(K_2/K_1) \geq 0$. Therefore $g_f(G)$ is a sum of nonnegative integers. ∎

Proposition 2.7 *The family of groups having* SCP *is closed under the group extension.*

Proof Let $1 \to H \to G \to K \to 1$ be a short exact sequence and f an isovariant G-map. Suppose H and K have SCP. There are (H_{i1}, H_{i2}), $a_i > 0$ for $i \in I$ and (K_{j1}, K_{j2}), $b_j > 0$ for $j \in J$ such that $H_{i1} \lhd H_{i2} < H$ for $i \in I$, $K_{j1} \lhd K_{j2} < K$ for $j \in J$, and H_{i2}/H_{i1}, K_{j2}/K_{j1} are BUGs, $g_f(G) = \sum_{i \in I} a_i(g(H_{i2}) - g(H_{i1}))$, and $g_{f^H}(K) = \sum_{j \in J} b_j(g(K_{i2}) - g(K_{i1}))$. Let $\pi\colon G \to K$ be a canonical projection. Since

$$\begin{aligned} g_f(G) &= g_f(H) + g_{f^H}(K) \\ &= \sum_{i \in I} a_i \left(g_f(H_{i2}) - g_f(H_{i1})\right) + \sum_{j \in J} b_j \left(g_{f^H}(K_{j2}) - g_{f^H}(K_{j1})\right) \\ &= \sum_{i \in I} a_i \left(g_f(H_{i2}) - g_f(H_{i1})\right) + \sum_{j \in J} b_j \left(g_f(\pi^{-1}(K_{j2})) - g_f(\pi^{-1}(K_{j1}))\right) \end{aligned}$$

and $\pi^{-1}(K_{j2})/\pi^{-1}(K_{j1}) \cong K_{j2}/K_{j1}$, the group G has SCP. ∎

3 Projective special linear groups

The projective special linear group $\mathrm{PSL}(2,q)$ over the 2-dimensional vector space over a finite field F_q is a BUG [4] and a CCG [7]. In this section, we show that the projective special linear group $\mathrm{PSL}(3,q)$ over the 3-dimensional vector space over a finite field F_q is a SCG.

The group $\mathrm{SL}(3,q)$ is of order $q^3(q^2-1)(q^3-1)$. Let $\phi\colon \mathrm{SL}(3,q) \to \mathrm{PSL}(3,q)$ be a natural surjective homomorphism. Put $q = p^u$ for a prime p, $G = \mathrm{PSL}(3,q)$, $r = q-1$, $d = \gcd(3,r)$, $\rho^r = 1$, $r' = r/d$, $s = q+1$, $t = q^2+q+1$, $t' = t/d$, $\sigma^s = \rho = \tau^t$. A maximal cyclic subgroup of $\mathrm{PSL}(3,q)$ is conjugate to one of the followings, whose generator is represented by a corresponding to Jordan canonical form over a suitable extension field:

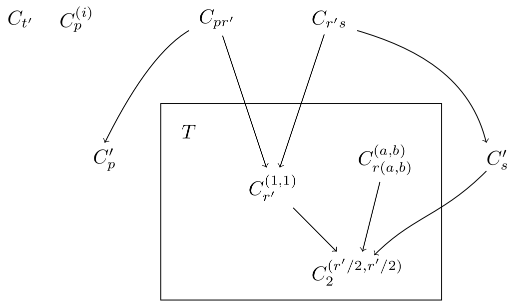
$$C_{pr'} = \langle \phi \begin{pmatrix} \rho & 1 & \\ & \rho & \\ & & \rho^{-2} \end{pmatrix} \rangle, \quad C_{r's} = \langle \phi \begin{pmatrix} \sigma & & \\ & \sigma^q & \\ & & \rho^{-1} \end{pmatrix} \rangle, \quad C_{t'} = \langle \phi \begin{pmatrix} \tau^r & & \\ & \tau^{qr} & \\ & & \tau^{q^2 r} \end{pmatrix} \rangle,$$

$$C_\ell^{(i)} = \langle \phi \begin{pmatrix} 1 & \theta^i & \\ & 1 & \theta^i \\ & & 1 \end{pmatrix} \rangle \quad (0 \le i < d), \quad \ell = \begin{cases} p, & p > 2 \\ 4, & p = 2 \end{cases},$$

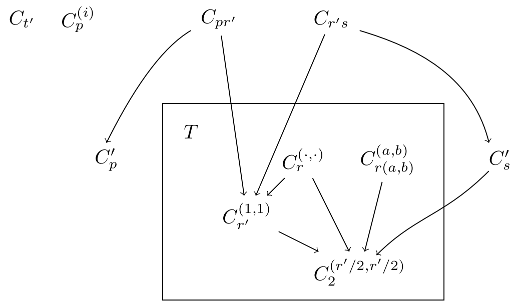
$$C_{r(a,b)}^{(a,b)} = \langle \phi \begin{pmatrix} \rho^a & & \\ & \rho^b & \\ & & \rho^{-a-b} \end{pmatrix} \rangle \quad (0 \le a < r',\ a \le b < r,\ (r,a,b) = 1),$$

where $r(a,b) = r'$ if $d = 3$ and $r'a \equiv rb/d \equiv -r'(a+b) \mod r$, and $r(a,b) = r$ otherwise [6, Table 1a]. Note that there may contain a duplicated group within the above groups: For example, $C_{10}^{(2,3)}$ and $C_{10}^{(1,5)}$ are conjugate in $\mathrm{PSL}(3,11) \cong \mathrm{SL}(3,11)$. We may assume that $\mathrm{RCycl}(G)$ is a subset of the set of the above cyclic subgroups.

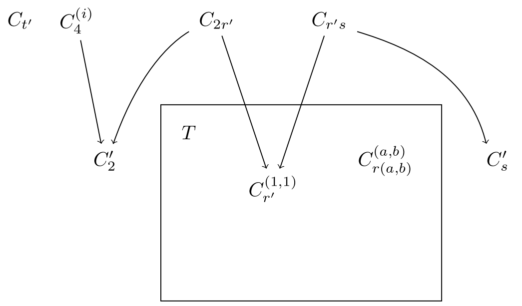
Let T be an abelian subgroup of G of order rr' generated by the image of diagonal matrices of $\mathrm{SL}(3,q)$ by ϕ. Note that any nontrivial subgroup of $C_p, C_{t'}$ is not a subset of (T) and $C_{r(a,b)}^{(a,b)} < T$. We may assume that $(C_{pr'}) \cap T = C_{r'}^{(1,1)} = (C_{r's}) \cap T$. Note that $d = 3$ if and only if $r(1,1) = r/3$. If $d = 3$ then $\langle \mathrm{diag}(\rho^{r'}, \rho^{r'}, \rho^{r'}) \rangle$ is the center of $\mathrm{SL}(3,q)$. In addition if r' is not divisible by 3, then $C_{r'}^{(1,1)}$ is a subgroup of $C_r^{(\frac{1+br'}{d}, \frac{1+br'}{d})}$ with index d, where $1 + br' \equiv 0 \mod d$. $C_{pr'} \cap (C_{r's})$ is a subgroup of $C_{pr'}$ of order r'.



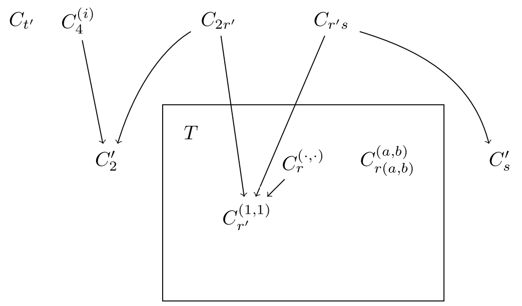
Case where $2|r$, and $3\nmid r$ or $9|r$



Case where $6|r$ and $9\nmid r$



Case where $p=2$, and $3\nmid r$ or $9|r$



Case where $p=2$, $3|r$ and $9\nmid r$

An arrow $A \to B$ means that B is a subgroup of A and C'_n for $n = 2, p, s$ denotes a cyclic group of order n.

Let $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ be a map satisfying that $\gamma(H_1) \leq \gamma(H_2)$ for subgroups $H_1 \trianglelefteq H_2 \leq G$ with H_2/H_1 a BUG.

We see

$$\gamma(G) = n_1 + n_2 + n_3 + n_4 + n_5 \tag{2}$$

where

$$\begin{aligned}
n_1 &= \sum_{D \leq C_{t'}} \sum_{C \leq D} \beta_G(C,D)\gamma(C),\\
n_2 &= \sum_{\substack{D \in \mathrm{RCycl}(G)\\ p \mid |D|}} \sum_{C \leq D} \beta_G(C,D)\gamma(C),\\
n_3 &= \sum_{\substack{D \leq C_{r's}\\ C_{r'}^{(1,1)} \not\leq D}} \sum_{C \leq D} \beta_G(C,D)\gamma(C),\\
n_4 &= \sum_{D \leq C_{r'}^{(1,1)}} \sum_{C \leq D} \beta_G(C,D)\gamma(C), \text{ and}\\
n_5 &= \sum_{\substack{D \in \mathrm{RCycl}(G)\\ D \leq T\\ D \not\leq C_{r'}^{(1,1)}}} \sum_{C \leq D} \beta_G(C,D)\gamma(C).
\end{aligned}$$

We show each of n_1, n_2, n_3, n_4, n_5 is nonnegative.

Lemma 3.1 ([7, Lemma 12]) *Let C be a cyclic subgroup of a finite group K. Suppose that there is a unique maximal cyclic subgroup D of K with $C < D$. Then $N_K(C) = N_K(D)$, $\beta_K(C) = 0$, and $\beta_K(D) = \frac{|D|}{|N_K(D)|} > 0$.*

By Lemma 3.1, we have

$$n_1 = \frac{t'}{|N_G(C_{t'})|}\gamma(C_{t'}) = \frac{\gamma(C_{t'})}{3} \geq 0 \tag{3}$$

and

$$\begin{aligned}
\sum_{\substack{D\leq C_{pr'}\\ D\not\leq C_{r'}^{(1,1)}}}\sum_{C\leq D}\beta_G(C,D)\gamma(C) &= \sum_{\substack{D\leq C_{pr'}\\ D\not\leq C_{r'}^{(1,1)}}}\sum_{C\leq D}\frac{|C|\gamma(C)}{|N_G(C_{pr'})|}\mu(C,D) \\
&= \sum_{C\leq C_{pr'}}\frac{|C|\gamma(C)}{|N_G(C_{pr'})|}\left(\sum_{D\leq C_{pr'}} - \sum_{D\leq C_{r'}^{(1,1)}}\right)\mu(C,D) \\
&= \frac{pr'}{|N_G(C_{pr'})|}\gamma(C_{pr'}) - \frac{r'}{|N_G(C_{pr'})|}\gamma(C_{r'}^{(1,1)}) \geq 0.
\end{aligned} \tag{4}$$

Therefore

$$n_2 = \sum_{\substack{D\leq C_{pr'}\\ D\not\leq C_{r'}^{(1,1)}}}\sum_{C\leq D}\beta_G(C,D)\gamma(C) + \alpha \geq \alpha \tag{5}$$

where

$$\alpha = \sum_{\substack{p\,|\,|D|\\ D\not\leq C_{pr'}}}\sum_{C\leq D}\beta_G(C,D)\gamma(C).$$

If p is odd then

$$\alpha = \sum_{i=0}^{d-1}\beta_G(C_p^{(i)})\gamma(C_p^{(i)}) \geq 0 \tag{6}$$

and otherwise

$$\begin{aligned}
\alpha &= \sum_{i=0}^{d-1}(\beta_G(C_4^{(i)}, C_4^{(i)})\gamma(C_4^{(i)}) + \beta_G(C_2', C_4^{(i)})\gamma(C_2')) \\
&= \sum_{i=0}^{d-1}\frac{4\gamma(C_4^{(i)}) - 2\gamma(C_2')}{|N_G(C_4^{(i)})|} \geq 0.
\end{aligned} \tag{7}$$

Lemma 3.2 *Let C_1 and C_2 be cyclic subgroups of a finite group K with $C_1 < C_2$. Suppose that $N_K(D) = N_K(C_2)$ for any D with $D \leq C_2$ and $D \not\leq C_1$. Then*

$$\sum_{C\leq C_2}\sum_{\substack{D\leq C_2\\ D\not\leq C_1}}\beta_K(C,D)\gamma(C) = \frac{|C_2|}{|N_K(C_2)|}\left(\gamma(C_2) - \frac{\gamma(C_1)}{|C_2/C_1|}\right).$$

Proof We straightforwardly see

$$\begin{aligned}
&\sum_{C\le C_2}\sum_{\substack{D\le C_2\\ D\not\le C_1}} \beta_K(C,D)\gamma(C)\\
&\quad= \sum_{C\le C_2}\sum_{\substack{D\le C_2\\ D\not\le C_1}} \frac{|C|\mu(C,D)\gamma(C)}{|N_K(D)|}\\
&\quad= \sum_{C\le C_2}\frac{|C|\gamma(C)}{|N_K(C_2)|}(\sum_{D\le C_2}-\sum_{D\le C_1})\mu(C,D)\\
&\quad= \sum_{C\le C_2}\frac{|C|\gamma(C)}{|N_K(C_2)|}\sum_{D\le C_2}\mu(C,D)-\sum_{C\le C_1}\frac{|C|\gamma(C)}{|N_K(C_2)|}\sum_{D\le C_1}\mu(C,D)\\
&\quad= \frac{|C_2|\gamma(C_2)}{|N_K(C_2)|}-\frac{|C_1|\gamma(C_1)}{|N_K(C_2)|}.
\end{aligned}$$

∎

Under the assumption of Lemma 3.2, since $\gamma(C_2)\ge\gamma(C_1)\ge 0$, we have

$$\sum_{\substack{D\le C_2\\ C_1\not\le D}}\sum_{C\le D}\beta_K(C,D)\gamma(C)\ge 0.$$

By seeing the eigenvalues of the preimage by ϕ of the generator of $C_{r's}$, for any cyclic subgroup D of $C_{r's}$ with $D\not\le C_{r'}^{(1,1)}$ the equality $N_K(D)=N_K(C_{r's})$ holds. By Lemma 3.2,

$$n_3=\frac{r's}{|N_G(C_{r's})|}\gamma(C_{r's})-\frac{r'}{|N_G(C_{r's})|}\gamma(C_{r'}^{(1,1)})\ge 0. \tag{8}$$

We see $|N_G(C)|=|\mathrm{GL}(2,q)|/d$ for $\{1\}<C\le C_{r'}^{(1,1)}$ and thus

$$n_4=\frac{d}{|\mathrm{GL}(2,q)|}\sum_{C\le C_{r'}^{(1,1)}}|C|\gamma(C)\sum_{D\le C_{r'}^{(1,1)}}\mu(C,D)=\frac{r}{|\mathrm{GL}(2,q)|}\gamma(C_{r'}^{(1,1)})\ge 0. \tag{9}$$

We put

$$\hat{T}=\langle t,\begin{pmatrix} & 1 & \\ 1 & & \\ & & -1\end{pmatrix},\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}\mid t\in T\rangle.$$

Note that T is a normal subgroup of $\hat{T}$ with index 6. For a nontrivial cyclic subgroup $D\le T$, we see $N_G(D)=N_{\hat{T}}(D)$ and the conjugacy class of D in $\hat{T}$ is the union of $6|T|/|N_{\hat{T}}(D)|$ conjugacy classes of D in T. The conjugation action preserves the set of eigenvalues. For a cyclic subgroup D of T with $(D)\not\le(C_{r'}^{(1,1)})$, any matrix of the preimage of the generator of D has distinct diagonal elements and thus $N_G(D)=N_{\hat{T}}(D)$.

Therefore we see

$$
\begin{aligned}
n_5 &= \sum_{\substack{D\le \mathrm{RCycl}(\hat{T})\\ D\le T\\ D\not\le C_{r'}^{(1,1)}}} \sum_{C\le D} \frac{|C|\mu(C,D)\gamma(C)}{|N_{\hat{T}}(D)|}\\
&= \sum_{\substack{D\le \mathrm{RCycl}(T)\\ D\le T\\ D\not\le C_{r'}^{(1,1)}}} \left(\frac{6rr'}{|N_{\hat{T}}(D)|}\right)^{-1} \sum_{C\le D} \frac{|C|\mu(C,D)\gamma(C)}{|N_{\hat{T}}(D)|}\\
&= \tfrac{1}{6rr'}\Big(\sum_{\substack{D\le \mathrm{RCycl}(T)\\ D\le T}} - \sum_{\substack{D\le \mathrm{RCycl}(T)\\ D\le C_{r'}^{(1,1)}}}\Big) \sum_{C\le D} |C|\mu(C,D)\gamma(C)\\
&= \tfrac{1}{6}\left(\sum_{C\in \mathrm{RCycl}(T)} \beta_T(C)\gamma(C) - \tfrac{1}{r}\sum_{C\in \mathrm{RCycl}(C_{r'}^{(1,1)})} \beta_{C_{r'}^{(1,1)}}(C)\gamma(C)\right)\\
&= \tfrac{1}{6}\left(\gamma(T) - \tfrac{1}{r}\gamma(C_{r'}^{(1,1)})\right)
\end{aligned}
$$

Since T is an extension of a cyclic group by a cyclic group and then solvable. Therefore, we conclude

$$
n_5 \ge \frac{r-1}{6r}\gamma(C_{r'}^{(1,1)}) \ge 0. \tag{10}
$$

The equality (2) and inequalities (3)–(10) for $\gamma = g_f$ complete the proof of the following.

Theorem 3.3 $\mathrm{PSL}(3,q)$ *has* SCP.

Therefore, $\mathrm{PSL}(3,q)$ is a BUG by Proposition 2.6.

Lemma 3.4 *Let L be a cyclic subgroup of a finite group K and let C_1 and C_2 be distinct proper subgroups of L. Suppose that $N_K(D) = N_K(L)$ for any D with $D \le L$, $D \not\le C_1$ and $D \not\le C_2$. Then*

$$
\sum_{C\le L} \sum_{\substack{D\le (L)\\ D\not\le C_1\\ D\not\le C_2}} \beta_K(C,D)\gamma(C) = \frac{|L|}{|N_K(L)|}\left(\gamma(L) - \frac{\gamma(C_1)}{|L/C_1|} - \frac{\gamma(C_2)}{|L/C_2|} + \frac{\gamma(C_1\cap C_2)}{|L/(C_1\cap C_2)|}\right).
$$

Proof Let $C_3 = C_1 \cap C_2$. We see

$$
\begin{aligned}
&\sum_{C\le L} \sum_{\substack{D\le L\\ D\not\le C_1\\ D\not\le C_2}} \beta_K(C,D)\gamma(C)\\
&\qquad = \sum_{C\le L} \sum_{\substack{D\le L\\ D\not\le C_1\\ D\not\le C_2}} \frac{|C|\mu(C,D)\gamma(C)}{|N_K(D)|}\\
&\qquad = \sum_{C\le L} \frac{|C|\gamma(C)}{|N_K(L)|}\Big(\sum_{D\le L} - \sum_{D\le C_1} - \sum_{D\le C_2} + \sum_{D\le C_3}\Big)\mu(C,D)\\
&\qquad = \sum_{C\le L} \frac{|C|\gamma(C)}{|N_K(L)|}\sum_{D\le L}\mu(C,D) - \sum_{C\le C_1} \frac{|C|\gamma(C)}{|N_K(L)|}\sum_{D\le C_1}\mu(C,D)\\
&\qquad\quad - \sum_{C\le C_2} \frac{|C|\gamma(C)}{|N_K(L)|}\sum_{D\le C_2}\mu(C,D) + \sum_{C\le C_3} \frac{|C|\gamma(C)}{|N_K(L)|}\sum_{D\le C_3}\mu(C,D)\\
&\qquad = \frac{|L|\gamma(L)}{|N_K(L)|} - \frac{|C_1|\gamma(C_1)}{|N_K(L)|} - \frac{|C_2|\gamma(C_2)}{|N_K(L)|} + \frac{|C_3|\gamma(C_3)}{|N_K(L)|}. \quad \blacksquare
\end{aligned}
$$

Under the assumption of Lemma 3.4, we have

$$\sum_{C\le L}\sum_{\substack{D\le L\\ D\not\le C_1\\ D\not\le C_2}}\beta_K(C,D)\gamma(C)\ge\frac{|L|\gamma(L)}{|N_K(L)|}\left(1-\frac{|C_1|}{|L|}-\frac{|C_2|}{|L|}\right)+\frac{|C_3|\gamma(C_3)}{|N_K(L)|}\ge 0.$$

4 Projective special unitary groups

Let σ be an automorphism of a finite field F_{q^2} defined by $\sigma(x)=x^q$. For a matrix $A=(a_{ij})$ over F_{q^2}, let $A^*=(a_{ji}^{\sigma})$ and $U(n,q)=\{A\in \mathrm{GL}(n,q^2)\mid AA^*=I_n\}$. The unitary group $U(n,q)$ has order $q^{n(n-1)/2}\prod_{i=1}^{n}(q^i-(-1)^i)$. The special unitary group $\mathrm{SU}(n,q)$ is defined by $U(n,q)\cap \mathrm{SL}(n,q)$ whose order is $q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)$. The projective special unitary group $\mathrm{PSU}(n,q)$ has order $|\mathrm{SU}(n,q)|/\gcd(n,q+1)$. In particular, $\mathrm{SU}(3,q)$ is a subgroup of $\mathrm{SL}(3,q^2)$ of order $q^3(q^2-1)(q^3+1)$ and $\mathrm{PSU}(3,q)$ has order $q^3(q^2-1)(q^3+1)/\gcd(3,q+1)$.

Note that $\mathrm{PSU}(2,q)$ is isomorphic to $\mathrm{PSL}(2,q)$. In this section, we show that $\mathrm{PSU}(3,q)$ is a SCG. The argument is quite similar as those of the projective special linear groups $\mathrm{PSU}(3,q)$.

Let $\phi\colon \mathrm{SU}(3,q)\to \mathrm{PSU}(3,q)$ be a natural surjective homomorphism. Put $q=p^u$ for a prime p, $G=\mathrm{PSU}(3,q)$, $r=q+1$, $d=\gcd(3,r)$, $\rho^r=1$, $r'=r/d$, $s=q-1$, $t=q^2-q+1$, $t'=t/d$, $\sigma^s=\rho=\tau^t$. A maximal cyclic subgroup of $\mathrm{PSU}(3,q)$ is conjugate to one of the followings, whose generator is represented by a corresponding to Jordan canonical form in $\mathrm{GL}(3,\mathbb{F})$ over a suitable extension field $\mathbb{F}$:

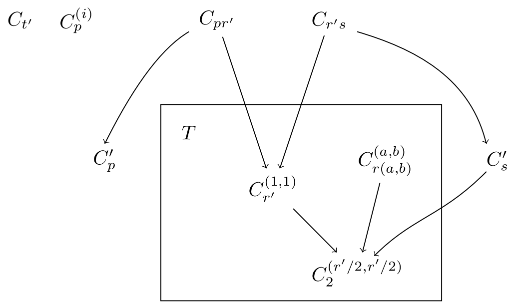
$$C_{pr'}=\langle\phi\begin{pmatrix}\rho&1&\\&\rho&\\&&\rho^{-2}\end{pmatrix}\rangle,\quad C_{r's}=\langle\phi\begin{pmatrix}\sigma^{-1}&&\\&\sigma^{q}&\\&&\rho^{-1}\end{pmatrix}\rangle,\quad C_{t'}=\langle\phi\begin{pmatrix}\tau^{r}&&\\&\tau^{-qr}&\\&&\tau^{q^2r}\end{pmatrix}\rangle,$$

$$C_{\ell}^{(i)}=\langle\phi\begin{pmatrix}1&\theta^i&\\&1&\theta^i\\&&1\end{pmatrix}\rangle\quad(0\le i<d),\quad \ell=\begin{cases}p,&p>2\\4,&p=2\end{cases},$$

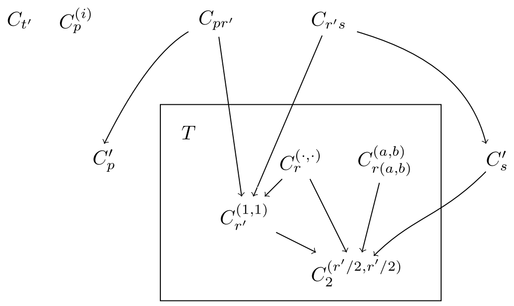
$$C_{r(a,b)}^{(a,b)}=\langle\phi\begin{pmatrix}\rho^{a}&&\\&\rho^{b}&\\&&\rho^{-a-b}\end{pmatrix}\rangle\quad(0\le a<r',\ a\le b<r,\ (r,a,b)=1),$$

where $r(a,b)=r'$ if $d=3$ and $r'a\equiv rb/d\equiv -r'(a+b)\mod r$, and $r(a,b)=r$ otherwise [6, Table 1a]. Note that there may contain a duplicated group within the above groups. We may assume that $\mathrm{RCycl}(G)$ is a subset of the set of the above cyclic subgroups.

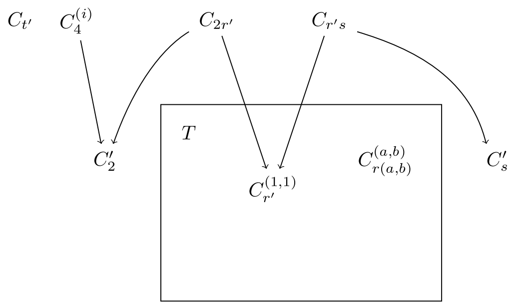
Let T be an abelian subgroup of G of order rr' generated by the image of diagonal matrices of $\mathrm{SU}(3,q)$ by ϕ. Note that any nontrivial subgroup of $C_p, C_{t'}$ is not a subset of (T) and $C_r^{(0,1)}, C_{r(a,b)}^{(a,b)} < T$. We may assume that $(C_{pr'}) \cap T = C_{r'}^{(1,1)} = (C_{r's}) \cap T$. Note that $d = 3$ if and only if $r(1,1) = r/3$. If $d = 3$ then $\langle \mathrm{diag}(\rho^{r'}, \rho^{r'}, \rho^{r'}) \rangle$ is the center of $\mathrm{SU}(3,q)$. In addition if r' is not divisible by 3, then $C_{r'}^{(1,1)}$ is a subgroup of $C_r^{(\frac{1+br'}{d}, \frac{1+br'}{d})}$ with index d, where $1 + br' \equiv 0 \mod d$. $C_{pr'} \cap (C_{r's})$ is a subgroup of $C_{pr'}$ of order r'.



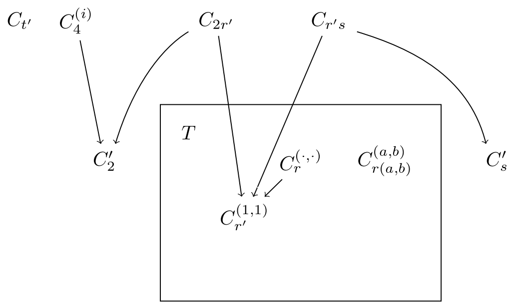
Case where $2|r$, and $3\not| r$ or $9|r$



Case where $6|r$ and $9\not| r$



Case where $p = 2$, and $3\not| r$ or $9|r$



Case where $p = 2$, $3|r$ and $9\not| r$

Let $\gamma\colon \mathrm{RCycl}_1(G) \to \mathbb{Q}_{\geq 0}$ be a map satisfying that $\gamma(H_1) \leq \gamma(H_2)$ for subgroups $H_1 \trianglelefteq H_2 \leq G$ with H_2/H_1 a BUG. We see

$$\gamma(G) = n_1 + n_2 + n_3 + n_4 + n_5 \tag{11}$$

where

$$n_1 = \sum_{D \leq C_{t'}} \sum_{C \leq D} \beta_G(C,D)\gamma(C), \quad n_2 = \sum_{\substack{D \in \mathrm{RCycl}(G) \\ p||D|}} \sum_{C \leq D} \beta_G(C,D)\gamma(C),$$

$$n_3 = \sum_{\substack{D \leq C_{r's} \\ C_{r'}^{(1,1)} \not\leq D}} \sum_{C \leq D} \beta_G(C,D)\gamma(C), \quad n_4 = \sum_{D \leq C_{r'}^{(1,1)}} \sum_{C \leq D} \beta_G(C,D)\gamma(C), \text{ and}$$

$$n_5 = \sum_{\substack{D \in \mathrm{RCycl}(G) \\ D \leq T \\ D \not\leq C_{r'}^{(1,1)}}} \sum_{C \leq D} \beta_G(C,D)\gamma(C).$$

We show each of n_1, n_2, n_3, n_4, n_5 is nonnegative. By Lemma 3.1, we have

$$n_1 = \frac{t'}{|N_G(C_{t'})|}\gamma(C_{t'}) \geq 0 \tag{12}$$

and

$$\begin{aligned} n_2 &= \sum_{\substack{D \leq C_{pr'} \\ D \not\leq C_{r'}}} \sum_{C \leq D} \beta_G(C, D)\gamma(C) + \sum_{i=0}^{d-1} \beta_G(C_\ell^{(i)})\gamma(C_\ell^{(i)}) \\ &\geq \sum_{C \leq C_{pr'}} \frac{|C|\gamma(C)}{|N_G(C_{pr'})|} \left(\sum_{D \leq C_{pr'}} - \sum_{D \leq C_{r'}} \right) \mu(C, D) \\ &= \frac{pr'}{|N_G(C_{pr'})|}\gamma(C_{pr'}) - \frac{r'}{|N_G(C_{pr'})|}\gamma(C_{r'}) \geq 0. \end{aligned} \tag{13}$$

By seeing the eigenvalues of the preimage by ϕ of the generator of $C_{r's}$, for any cyclic subgroup D of $C_{r's}$ with $D \not\leq C_{r'}^{(1,1)}$ the equality $N_K(D) = N_K(C_{r's})$ holds. By Lemma 3.2,

$$n_3 = \frac{r's}{|N_G(C_{r's})|}\gamma(C_{r's}) - \frac{r'}{|N_G(C_{r's})|}\gamma(C_{r'}^{(1,1)}) \geq 0. \tag{14}$$

We see $|N_G(C)| = |U(2, q)|/d$ for $\{1\} < C \leq C_{r'}^{(1,1)}$ and thus

$$n_4 = \frac{d}{|U(2, q)|} \sum_{C \leq C_{r'}^{(1,1)}} |C|\gamma(C) \sum_{D \leq C_{r'}^{(1,1)}} \mu(C, D) = \frac{r}{|U(2, q)|}\gamma(C_{r'}^{(1,1)}) \geq 0. \tag{15}$$

We put

$$\hat{T} = \langle t, \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \mid t \in T \rangle.$$

For a nontrivial cyclic subgroup $D \leq T$, we see $N_G(D) = N_{\hat{T}}(D)$ and the conjugacy class of D in $\hat{T}$ is the union of $|\hat{T}/T|/|N_{\hat{T}}(D)/N_T(D)|$ conjugacy classes of D in T. The conjugation action preserves the set of eigenvalues. For a cyclic subgroup D of T with $(D) \not\leq (C_{r'}^{(1,1)})$, any matrix of the preimage of the generator of D has distinct diagonal elements and thus $N_G(D) = N_{\hat{T}}(D)$. Therefore we see

$$\begin{aligned}
n_5 &= \sum_{\substack{D \le \mathrm{RCycl}(\hat{T}) \\ D \le T \\ D \not\le C_{r'}^{(1,1)}}} \sum_{C \le D} \frac{|C|\mu(C,D)\gamma(C)}{|N_{\hat{T}}(D)|} \\
&= \sum_{\substack{D \le \mathrm{RCycl}(T) \\ D \le T \\ D \not\le C_{r'}^{(1,1)}}} \left(\frac{2}{|N_{\hat{T}}(D)/N_T(D)|}\right)^{-1} \sum_{C \le D} \frac{|C|\mu(C,D)\gamma(C)}{|N_{\hat{T}}(D)|} \\
&= \frac{1}{2}\Big(\sum_{\substack{D \le \mathrm{RCycl}(T) \\ D \le T}} - \sum_{\substack{D \le \mathrm{RCycl}(T) \\ D \le C_{r'}^{(1,1)}}} \Big) \sum_{C \le D} \frac{|C|\mu(C,D)\gamma(C)}{|N_T(D)|} \\
&= \frac{1}{2}\left(\sum_{C \in \mathrm{RCycl}(T)} \beta_T(C)\gamma(C) - \frac{r'}{|T|} \sum_{C \in \mathrm{RCycl}(C_{r'}^{(1,1)})} \beta_{C_{r'}^{(1,1)}}(C)\gamma(C) \right) \\
&= \frac{1}{2}\gamma(T) - \frac{1}{2r}\gamma(C_{r'}^{(1,1)})
\end{aligned}$$

Since T is an extension of a cyclic group by a cyclic group and then solvable. Therefore, we conclude

$$n_5 \ge \frac{r-1}{2r}\gamma(C_{r'}^{(1,1)}) \ge 0. \tag{16}$$

The equality (11) and inequalities (12)–(16) for $\gamma = g_f$ complete the proof of the following.

Theorem 4.1 $\mathrm{PSU}(3, q)$ *has* SCP.

Therefore, $\mathrm{PSU}(3, q)$ is a BUG by Proposition 2.6.

5 Alternating groups

Let A_n be an alternating group on letters $1, 2, \ldots, n$. In this section we show that A_n, $22 \le n \le 27$ have SCP and in particular are BUGs.

Let $\mathcal{S}_0(n) = \{(C, D) \mid C, D \in \mathrm{RCycl}_1(A_n),\ (D) > (C),\ |D/C| \text{ is a prime}\}$. By using its character table and computer, we get the following result.

Example 5.1 *Let* $\mathcal{S}_1(n, k) = \{(A_j, \langle A_j, (1, 2)^{n-j+1}(j+1, \ldots, n)\rangle) \mid k \le j \le n-2\}$. $g(A_n)$ *is written as a conical combination of* $\{g(H_2) - g(H_1) \mid (H_1, H_2) \in \mathcal{S}_0(n) \cup \mathcal{S}_1(n, k_1(n))\}$ *and is not a conical combination of* $\{g(H_2) - g(H_1) \mid (H_1, H_2) \in \mathcal{S}_0(n) \cup \mathcal{S}_1(n, k_1(n) + 1)\}$ *for* $n = 22, 23, 24, 25, 26, 27$, *where*

n	22	23	24	25	26	27
$k_1(n)$	20	18	18	22	21	23

Theorem 5.2 *The alternating groups* A_{22}, A_{23}, A_{24}, A_{25}, A_{26}, *and* A_{27} *have* SCP.

Proof Recall that A_k is a BUG since it is a CCG for $k \leq 21$ by Theorem 2.4. Let $n = 22$. For $(H_1, H_2) \in \mathcal{S}_1(n, k)$, groups H_1 and H_2 are BUGs, H_1 is a normal subgroup of H_2, H_2/H_1 is cyclic, and $g(H_1) - g(H_2) \geq 0$. Therefore, A_n has SCP by Example 5.1.

Now, let $22 < n \leq 27$. As the induction hypothesis, we suppose A_k is a BUG for $k < n$. By the similar argument as above, we see that A_n has SCP. ∎

Example 5.3 *Let* $\mathcal{S}_2(n, k) = \{(A_j, A_j \times A_{n-j}) \mid k \leq j \leq n-3\}$. $g(A_n)$ *is written as a conical combination of* $\{g(H_2) - g(H_1) \mid (H_1, H_2) \in \mathcal{S}_0(n) \cup \mathcal{S}_2(n, k_2(n))\}$ *and is not a conical combination of* $\{g(H_2) - g(H_1) \mid (H_1, H_2) \in \mathcal{S}_0(n) \cup \mathcal{S}_2(n, k_2(n)+1)\}$ *for* $n = 22, 23, 24, 25, 26, 27$, *where*

n	22	23	24	25	26	27
$k_2(n)$	19	18	17	20	21	22

The vector $(\cdots, \dim V_j, \cdots)$ is not a conical combination of

$$\{(\cdots, \dim V_j^{H_1} - \dim V_j^{H_2}, \cdots) \mid (H_1, H_2) \in \mathcal{S}_0(28) \cup \mathcal{S}_2(28, 14)\},$$

where V_j runs over nontrivial irreducible representation spaces.

Question 5.4 *Does* A_{28} *have* SCP*?*

To attack this problem we may assume that any proper subgroup of A_{28} is a BUG. However there are quite many subgroups (even up to conjugate). By the following theorem supports that the number of necessary subgroups has upper limit.

Theorem 5.5 (Carathéodory's theorem [1]) *If a point* x *of* $\mathbb{R}^d$ *lies in the convex hull of a set* P, x *lies in an* r*-simplex with vertices in* P, *where* $r \leq d$.

By Carathéodory's theorem, if a point x of $\mathbb{R}^d$ lies in the conical hull of P, then x can be written as the conical combination of at most $d+1$ points in P. Therefore, we can choose some pairs (H_1, H_2) of subgroups with $H_1 \triangleleft H_2$ whose number is less than or equal to the cardinality of $\mathrm{RCycl}(G)$, that is, the number of conjugacy classes of cyclic subgroups.

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