THE EQUIVARIANT SPECTRAL FUNCTION AND NON–SPHERICAL SUBCONVEX BOUNDS FOR HECKE–MAASS FORMS

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Let M be a closed Riemannian manifold M of dimension d and $P_0 : \mathrm{C}^\infty(M) \to \mathrm{L}^2(M)$ an elliptic classical pseudodifferential operator on M of degree m, where $\mathrm{C}^\infty(M)$ denotes the space of smooth functions on M and $\mathrm{L}^2(M)$ the space of square-integrable functions on M. Assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P with the m-th Sobolev space as domain, and let $\{\phi_j\}_{j\geq 0}$ be an orthonormal basis of $\mathrm{L}^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j\geq 0}$ repeated according to their multiplicity. By a classical result of Avacumovic, Levitan, and Hörmander [1, 9, 7] one has for any $j \in \mathbb{N}$ the *convex bound*

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{d-1}{2m}}. \tag{0.1}$$

If the ϕ_j are eigenfunctions of a larger family of commuting differential operators on M containing P_0, this bound can be improved. Thus, assume that M carries an isometric action of a compact Lie group K such that all orbits have the same dimension $\kappa \leq d-1$. Denote by $\widehat{K}$ the set of equivalence classes of irreducible unitary representations of K, which can be identified with the set of irreducible characters of K. Suppose further that P commutes with the family of differential operators generated by the action of K, so that the eigenfunctions ϕ_j can be chosen to be compatible with the Peter–Weyl decomposition of $\mathrm{L}^2(M)$ into σ-isotypic components $\mathrm{L}^2_\sigma(M)$, where $\sigma \in \widehat{K}$. It was then shown in [14, 15] that the *equivariant convex bound*

$$\|\phi_j\|_\infty \ll \Big(d_\sigma \sup_{u\leq \lfloor \kappa/2+1\rfloor} \|D^u\sigma\|_\infty\Big)^{1/2} \lambda_j^{\frac{d-\kappa-1}{2m}}, \qquad \phi_j \in \mathrm{L}^2_\sigma(M), \tag{0.2}$$

holds, where d_σ denotes the dimension of a representation of class σ, and D^u is a differential operator of order u on K. If $K = T$ is a torus, one actually has the almost sharp estimate

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{d-\kappa-1}{2m}}, \qquad \phi_j \in \mathrm{L}^2_\sigma(M), \qquad \sigma \in \mathcal{W}_{\lambda_j}, \tag{0.3}$$

where $\mathcal{W}_\lambda$ denotes the subset of K-types occuring in the Peter-Weyl decomposition of $\mathrm{L}^2(M)$ that grow at most with rate $\lambda^{1/m}/\log\lambda$.

The bounds (0.1) and (0.2) are known to be sharp in the eigenvalue aspect on the standard d-sphere, but if the considered eigenfunctions are joint eigenfunctions of an even larger family of commuting operators, they can be improved. Thus, let G be a semisimple real Lie group, K a maximal compact subgroup of G, $\Gamma \subset G$ a lattice, and $Y := \Gamma\backslash G/K$ the corresponding locally symmetric space of dimension d and rank r. If $\{\psi_j\}_{j\geq 0}$ constitutes an orthonormal basis in $\mathrm{L}^2(Y)$ of simultaneous eigenfunctions of the full ring of invariant differential operators on Y, which is isomorphic to a finitely generated polynomial ring in r variables and contains the Beltrami–Laplace operator Δ, Sarnak [18] was able to show the *spherical convex bound*

$$\|\psi_j|_\Omega\|_\infty \ll_\Omega \lambda_j^{\frac{d-r}{4}} \tag{0.4}$$

for arbitrary compacta $\Omega \subset Y$, λ_j being the Beltrami–Laplace eigenvalue of ψ_j. From an arithmetic point of view, there is still an additional family of commuting operators on Y given by the Hecke operators, and in the case $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2)$, Iwaniec and Sarnak [8] were able to strengthen the bound (0.4) for certain compact locally symmetric spaces $Y = \Gamma\backslash\mathbb{H}$ of rank $r = 1$, given as quotients

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of the complex upper half plane $\mathbb{H} \simeq G/K$ by suitable congruence arithmetic lattices Γ, and proved for any $\varepsilon > 0$ and $j \in \mathbb{N}$ the substantially stronger *spherical subconvex bound*

$$\|\psi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24}+\varepsilon}, \tag{0.5}$$

provided that the ψ_j are also eigenfunctions of the ring of Hecke operators on $\mathrm{L}^2(\Gamma\backslash\mathbb{H})$. More generally, if H is a semisimple algebraic group over $\mathbb{Q}$ satisfying certain conditions, $\Gamma \subset H(\mathbb{Q})$ an arithmetic congruence lattice, and $G = H(\mathbb{R})$, Marshall [12] was able to strengthen the bound (0.4) and prove *spherical subconvex bounds* of the form

$$\|\psi_j|_\Omega\|_\infty \ll_\Omega \lambda_j^{\frac{d-r}{4}-\delta} \tag{0.6}$$

for some $\delta > 0$ and arbitrary compacta $\Omega \subset Y$, if the ψ_j are also eigenfunctions of the ring of Hecke operators on $\mathrm{L}^2(Y)$, generalizing previous work of Blomer-Maga [3, 4] and Blomer-Pohl [5], among others. In fact, for negatively curved manifolds, much better bounds are expected to hold generically, the bound (0.5) being the strongest known bound up to now. The estimates (0.4)–(0.6) represent bounds for automorphic forms on G which are right K-invariant, and for this reason are called *spherical*.

Recently, together with Satoshi Wakatsuki, the author succeeded in extending the spherical subconvex bounds (0.5) and (0.6) to non-spherical situations, that is, to non-trivial K-types in the Peter-Weyl decomposition of $\mathrm{L}^2(\Gamma\backslash G)$ for a large class of compact arithmetic quotients $\Gamma\backslash G$, sharpening the bounds (0.1) and (0.2) in case that the eigenfunctions ϕ_j are Hecke–Maass forms. Here and in the following, left Γ-invariant functions on G which are simultaneous eigenfunctions of an invariant elliptic differential operator and some module of Hecke operators are called *Hecke–Maass forms of rank* 1. This class encompasses the usual concept of an automorphic form on G, and coincides with it in the rank 1 case.

More precisely, let $\mathcal{R}$ be an Eichler order in an indefinite division quaternion algebra A over $\mathbb{Q}$. Denote by $N(x)$ the reduced norm of an element $x \in A$, and write $\mathcal{R}(m) := \{\alpha \in \mathcal{R} \mid N(\alpha) = m\}$ for any $m \in \mathbb{N}_*$. Choose an embedding $\theta : \sqcup_{m=1}^\infty \mathcal{R}(m) \to G$, and set $\Gamma := \theta(\mathcal{R}(1))$. Then Γ constitutes a congruence arithmetic subgroup, and $\Gamma\backslash\mathbb{H} \simeq \Gamma\backslash G/K$ becomes a compact hyperbolic surface. Now, let χ be a Nebentypus character on Γ, and denote by $\mathrm{L}^2_\chi(\Gamma\backslash G)$ the Hilbert space of measurable functions on G such that

$$f(\gamma x) = \chi(\gamma)\, f(x), \qquad \gamma \in \Gamma,\, x \in G, \qquad \|f\| := \left(\int_{\Gamma\backslash G} |f(x)|^2 dx\right)^{1/2} < \infty.$$

The space $\mathrm{L}^2_\chi(\Gamma\backslash G)$ can be regarded as a closed subspace in $\mathrm{L}^2(\Gamma_\chi\backslash G)$, where $\Gamma_\chi := \ker \chi$. Identifying $\mathcal{R}(n)$ with its image $\theta(\mathcal{R}(n))$ for each n prime to a fixed natural number which depends only on $\mathcal{R}$, the finite cosets $\Gamma\backslash\mathcal{R}(n)$ give rise to Hecke operators on $\mathrm{L}^2_\chi(\Gamma\backslash G)$. Now, with the identification $K \simeq S^1 \simeq [0, 2\pi)$, any K-type $\sigma_l \in \widehat{K}$ can be realized as the character $\sigma_l(\theta) = e^{il\theta}$, $\theta \in [0, 2\pi)$, $l \in \mathbb{Z}$, and we denote by $\mathrm{L}^2_{\sigma_l,\chi}(\Gamma\backslash G)$ the σ_l-isotypic component of $\mathrm{L}^2_\chi(\Gamma\backslash G)$. It is then shown in [16, Theorem 5.5] that for any orthonormal basis $\{\phi_j\}_{j\geq 0}$ of $\mathrm{L}^2(\Gamma_\chi\backslash G)$ consisting of Hecke–Maass forms (of rank 1) with Beltrami–Laplace eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and compatible with the Peter-Weyl decomposition one has the *hybrid subconvex bound*

$$\|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24}+\varepsilon}, \qquad \phi_j \in \mathrm{L}^2_\chi(\Gamma\backslash G), \tag{0.7}$$

for arbitrary small $\varepsilon > 0$ in the eigenvalue and isotypic aspect. This bound is the first sharpening the bound (0.3) for arbitrary K-types. If σ_l and χ are trivial, one recovers the spherical subconvex bound (0.5). Note that (0.7) is a subconvex bounds on a manifold which does have both positive and negative sectional curvature. It is stated from the perspective of elliptic operator theory, which is the natural one in our approach, while in the theory of automorphic forms it is more common to work within a representation-theoretic framework, and use the Casimir operator $\mathcal{C}$ of G instead of the Beltrami–Laplace operator Δ, the former being no longer elliptic. But since on $\mathrm{L}^2_{\sigma_l,\chi}(\Gamma\backslash G)$ the operators in question are related according to $\Delta = -\mathcal{C} + \frac{l^2}{4}\mathrm{id}$, the bound (0.7) can be rephrased accordingly. Thus, for any Hecke eigenform $\phi \in \mathrm{L}^2_{\sigma_l,\chi}(\Gamma\backslash G)$ satisfying $\|\phi\|_{\mathrm{L}^2} = 1$ and $\mathcal{C}\phi = \frac{s^2-1}{8}\phi$ one has the *hybrid subconvex bound*

$$\|\phi\|_\infty \ll_\varepsilon (1 - s^2 + 2l^2)^{\frac{5}{24}+\varepsilon},$$

see [16, Theorem 5.8]. In this way, we obtain subconvex bounds for new classes of automorphic representations, in particular for the discrete series D_s and their limits $D_{\pm,0}$, as well as the principal series $H(1,s)$. Let us note that for fixed s we obtain the bound $\|\phi_j\|_\infty \ll_\varepsilon (1+|l|)^{\frac{5}{12}+\varepsilon}$ for any $\phi_j \in \mathrm{L}^2_{\sigma_l,\chi}(\Gamma\backslash G)$. This agrees with results of Venkatesh [22, p. 993], though by work of Reznikov [17, Theorem 1.5] one has in this case the much better bound $\|\phi_j\|_\infty \ll_\varepsilon (1+|l|)^{\frac{1}{3}+\varepsilon}$. Nevertheless, our results do imply new results for a classical automorphic form $f : \mathbb{H} \to \mathbb{C}$ of weight $l \in \mathbb{N}$ and arbitrary Nebentypus character, for which we show in [16, Eq. (5.11)] the subconvex bound

$$\|f\|_\infty \ll_\varepsilon l^{\frac{5}{12}+\varepsilon}$$

in the weight aspect. The best previously known subconvex bound, proved by Das and Sengupta [6], had the exponent $\frac{1}{2} - \frac{1}{33} = \frac{31}{66}$.

Our second main result concerns bounds of the form (0.6). As before, let H be a semisimple algebraic group over $\mathbb{Q}$ which is assumed to be connected in the sense of Zariski. Write $\mathbb{A}_{\mathrm{fin}}$ for the finite adele ring of $\mathbb{Q}$ and $\mathbb{A} := \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$ for the adele ring. Choosing an open compact subgroup K_0 in $H(\mathbb{A}_{\mathrm{fin}})$, we obtain an arithmetic subgroup $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$ in the semisimple Lie group $G = H(\mathbb{R})$. Assume that $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$ and that $H(\mathbb{Q})\backslash H(\mathbb{A})$ is compact, so that $\Gamma\backslash G$ is also compact. From the point of view of automorphic representations, one has a suitable family of Hecke operators on $\mathrm{L}^2(\Gamma\backslash G)$, which is given by unramified Hecke algebras over $\mathbb{Q}_p$ for infinitely many primes p [12]. Now, let K be a maximal compact subgroup of G and $\{\phi_j\}_{j\geq 0}$ an orthonormal basis of $\mathrm{L}^2(\Gamma\backslash G)$ consisting of Hecke–Maass forms of rank 1 with respect to an elliptic left-invariant differential operator P_0 on $\Gamma\backslash G$ of order m which commutes with the right regular representation of K. Assume that P_0 is positive and symmetric, and that the cosphere bundle defined by its principal symbol is strictly convex. Then, assuming the condition (WS) made in [12], we show in [16, Theorem 7.4] that there exists a constant $\delta > 0$ independent of σ such that one has the *equivariant subconvex bound*

$$\|\phi_j\|_\infty \ll \sqrt{d_\sigma \sup_{u \leq \lfloor \frac{\dim K}{2}+1 \rfloor} \|D^u\sigma\|_\infty}\, \lambda_j^{\frac{\dim G/K-1}{2m}-\delta}, \qquad \phi_j \in \mathrm{L}^2_\sigma(\Gamma\backslash G), \tag{0.8}$$

where λ_j denotes the spectral eigenvalue of ϕ_j with respect to P_0; if $K = T$ is a torus, one has the stronger estimate

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{\dim G/K-1}{2m}-\delta}, \qquad \phi_j \in \mathrm{L}^2(\Gamma\backslash G).$$

The bound (0.8) bound sharpens the bound (0.2) for a large class of examples. If σ is trivial, it is implied by (0.6). An example would be given by $H = \mathrm{SL}(1,D)$, where D is any central division algebra of index n over $\mathbb{Q}$, and $G = \mathrm{SL}(n,\mathbb{R})$. Furthermore, we show in [16, Theorem 7.9] for some $\delta > 0$ the weaker *non-equivariant subconvex bound*

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{\dim G-1}{2m}-\delta}, \qquad \phi_j \in \mathrm{L}^2(\Gamma\backslash G), \tag{0.9}$$

for an orthonormal basis of $\mathrm{L}^2(\Gamma\backslash G)$ consisting of suitable Hecke–Maass forms, sharpening the bound (0.1), but without assuming the condition (WS) of [12]. An example is again $H = \mathrm{SL}(1,D)$, where now D is any central division algebra over $\mathbb{Q}$, except when $G = \mathrm{SL}(1,\mathbb{H})$. As before, (0.8) and (0.9) constitute first arithmetic subconvex bounds on a large class of manifolds which are both positively and negatively curved, and if P_0 is the Beltrami–Laplace operator, the bounds can be rephrased in terms of the eigenvalues of the Casimir operator of G. Indeed, by [16, Theorem 7.12] we have for each $\phi_j \in \mathrm{L}^2_\sigma(\Gamma\backslash G)$ with Casimir eigenvalue μ_j the bound

$$\|\phi_j\|_\infty \ll \sqrt{d_\sigma \sup_{u \leq \lfloor \frac{\dim K}{2}+1 \rfloor} \|D^u\sigma\|_\infty}\, (-\mu_j + 2\mu_\sigma)^{\frac{\dim G/K-1}{4}-\delta}, \qquad \phi_j \in \mathrm{L}^2_\sigma(\Gamma\backslash G),$$

provided that $H = \mathrm{Res}_{F/\mathbb{Q}}\underline{G}$ and (WS) is fulfilled, while in general

$$\|\phi_j\|_\infty \ll (-\mu_j + 2\mu_\sigma)^{\frac{\dim G-1}{4}-\delta},$$

μ_σ being the eigenvalue of the Casimir operator of K on σ. If $K = T$ is a torus,

$$\|\phi_j\|_\infty \ll (-\mu_j + 2\mu_\sigma)^{\frac{\dim G/K-1}{4}-\delta}.$$

Let us briefly say a few words about the methods employed. While in the theory of automorphic forms representation-theoretic tools prevail, our analysis is mainly based on the spectral theory of elliptic operators, and uses Fourier integral operator methods. Thus, let P be an elliptic pseudodifferential operator on a closed Riemannian manifold M as above. Our main tool is the *spectral function* $e(x, y, \mu)$ of the m-th root $Q := \sqrt[m]{P}$ of P given by

$$e(x, y, \mu) := \sum_{\mu_j \leq \mu} \phi_j(x)\overline{\phi_j(y)} \in \mathrm{C}^\infty(M \times M), \qquad \mu \in \mathbb{R}, \quad \mu_j := \sqrt[m]{\lambda_j}.$$

From the asymptotic behaviour of

$$s_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu),$$

which represents the Schwartz kernel of the spectral projection s_μ onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu + 1]$, statements about the growth of eigenfunctions can be deduced, yielding in particular the bound (0.1). In the presence of symmetries, these can be refined. More precisely, if M carries an effective and isometric action of a compact Lie group K and $\sigma \in \widehat{K}$, denote by Π_σ the projector onto the σ-isotypic component in the Peter-Weyl decomposition of $\mathrm{L}^2(M)$. In order to show the L^∞-bounds (0.2), and analogous equivariant convex L^p-bounds, an asymptotic formula for the Schwartz kernel of $s_\mu \circ \Pi_\sigma$, or rather of $\widetilde{s}_\mu \circ \Pi_\sigma$, where $\widetilde{s}_\mu$ represents certain smooth approximation to s_μ, was derived in [14, Corollary 2.2. and Theorem 3.3] in a neighbourhood of the diagonal relying on the theory of Fourier integral operators. Now, let G be a semisimple Lie group with finite center, Γ a discrete cocompact subgroup, and K a maximal compact subgroup of G. Let $\widehat{\Gamma}$ denote the set consisting of characters of Γ of finite order. For $\chi \in \widehat{\Gamma}$, introduce on $\mathrm{L}^2(\Gamma_\chi\backslash G)$ the Hecke operators $\mathcal{T}^\chi_{\Gamma\beta\Gamma}$

$$(\mathcal{T}^\chi_{\Gamma\beta\Gamma} f)(x) := [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi\backslash\Gamma\beta\Gamma} \overline{\chi(\alpha)}\, f(\alpha \cdot x),$$

where β belongs to a certain set containing the commensurator $C(\Gamma)$ of Γ. Based on the asymptotics for the kernel of $\widetilde{s}_\mu \circ \Pi_\sigma$ mentioned above, where in this case $M = \Gamma_\chi\backslash G$, we deduce for any small $\delta > 0$ and some constant $C > 0$ the equivariant bound

$$K_{\mathcal{T}^\chi_{\Gamma\beta\Gamma}\circ\tilde{s}_\mu\circ\Pi_\sigma}(x, x) \ll \frac{d_\sigma}{[\Gamma : \Gamma_\chi]}\, \mu^{\dim G/K-1} \sup_{u \leq \lfloor \frac{\dim K}{2}+1 \rfloor} \|D^u\sigma\|_\infty\, M(x, \beta, \delta)$$
$$+ \frac{d_\sigma}{[\Gamma : \Gamma_\chi]}\, \mu^{\frac{\dim G/K-1}{2}} \sup_{u \leq \left\lfloor \frac{\dim K}{2}+1 \right\rfloor} \|D^u\sigma\|_\infty \int_\delta^C s^{-\frac{1}{2}}\, dM(s)$$

uniformly in $x \in \Gamma_\chi\backslash G$ for the Schwartz kernel of $\mathcal{T}^\chi_{\Gamma\beta\Gamma} \circ \tilde{s}_\mu \circ \Pi_\sigma$, where we introduced the lattice point counting function

$$M(\delta) := M(x, \beta, \delta) := \#\left\{\alpha \in \Gamma_\chi\backslash\Gamma\beta\Gamma \mid \mathrm{dist}\,(xK, \alpha \cdot xK)^{\dim G/K-1} < \delta\right\}$$

given in terms of the distance function on the Riemannian symmetric space G/K. In case that $K = T$ is a torus, a corresponding better estimate holds. From this, we obtain the subconvex bounds (0.7) and (0.8) by using known uniform upper bounds [8, 12] for $M(x, \beta, \delta)$ combined with arithmetic amplification. The bound (0.9) is inferred by analogous methods. In both cases, it is important to control the caustic behaviour of the kernels of $\tilde{s}_\mu \circ \Pi_\sigma$ and $\tilde{s}_\mu$ near the diagonal as $\mu \to +\infty$, respectively. Note that in the spherical situations [8, 3, 4, 5, 12] examined before, a crucial role is played by asymptotics for spherical kernel functions, see [8, Eq. (1.3)] and [12, Eq. (8)]. The latter constitute particular smooth approximations to the kernel of the spectral projection s_μ for the Laplace-Beltrami operator on $\Gamma_\chi \backslash G/K$. The kernels of the spectral projections $s_\mu \circ \Pi_\sigma$ and their smooth approximations considered by us are non-spherical generalizations of the former.

Let us close with some comments. There exist several variants of the bounds (0.5), beginning with [8, Appendix], where the non compact hyperbolic surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is considered. On the other hand, bounds in the level aspect are shown in [19] for compact locally symmetric spaces of arithmetic type, while bounds in the eigenvalue and level aspect are derived for the modular surfaces $\Gamma_0(N)\backslash\mathbb{H}$ in [2, 20] and other papers. It is likely that our work can be extended to these settings. Also, we intend to widen our results to Hecke–Maass forms of rank r, that is, simultaneous eigenfunctions of the Hecke operators and the full ring of invariant differential operators associated to the center of the universal envelopping algebra of the complexification of the Lie algebra of G. For such forms, the exponent $-1/2m$ in (0.8) and (0.9) should be improvable by a factor r.

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