EQUIDISTRIBUTION THEOREM FOR HOLOMORPHIC SIEGEL MODULAR FORMS

HENRY H. KIM, SATOSHI WAKATSUKI AND TAKUYA YAMAUCHI

ABSTRACT. This is a summary of results on our papers [2, 3], where we proved the equidistribution theorem for a family of holomorphic Siegel cusp forms for GSp_4. We deal with only the vertical Sato-Tate theorem and low-lying zeros for degree 4 spinor and degree 5 standard L-functions of holomorphic Siegel cusp forms.

1. Vertical Sato-Tate theorem

Let S_k be the space of elliptic cusp forms of weight k with the trivial central character. For a Hecke eigenform $f \in S_k$, let π_f be the associated cuspidal representation of GL_2. Denote by $\{\alpha_{f,p}, \alpha_{f,p}^{-1}\}$ be the Satake parameter at p. Let $a_{f,p} = \alpha_{f,p} + \alpha_{f,p}^{-1} = 2\cos\theta_{f,p}$. Then $\theta_{f,p} \in [0, \pi]$. Sato-Tate distribution is that for a continuous function h on $[-2, 2]$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} h(a_{f,p}) = \frac{1}{2\pi}\int_{-2}^{2} h(t)\sqrt{4-t^2}\, dt.$$

i.e., $\{\theta_{f,p}\}$ is uniformly distributed with respect to the measure $\frac{2}{\pi}\sin^2\theta$ on $[0, \pi]$. Or more familiar formulation is, for $0 \le a < b \le \pi$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x,\, a \le \theta_{f,p} \le b\} = \frac{2}{\pi}\int_a^b \sin^2\theta\, d\theta.$$

For vertical Sato-Tate distribution, let $\mathcal{F}_k$ be the set of orthogonal basis in S_k. Then for a large prime p,

$$\sum_{f\in\mathcal{F}_k} a_{f,p}^n = \frac{k}{6\pi}(1+\frac{1}{p})\int_0^{\pi} (2\cos\theta)^n \frac{\sin^2\theta}{(1-\frac{1}{p})^2 + \frac{4}{p}\sin^2\theta}\, d\theta + O(p^{\frac{n}{2}+\epsilon}).$$

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Since $\#\mathcal{F}_k \sim \frac{k}{12}$, $\{\theta_{f,p}, f\in\mathcal{F}_k\}$ is uniformly distributed with respect to the measure

$$\frac{2}{\pi}(1+\frac{1}{p})\frac{\sin^2\theta}{(1-\frac{1}{p})^2+\frac{4}{p}\sin^2\theta}\,d\theta = \frac{p+1}{\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)^2-x^2}\cdot\frac{2}{\pi}\sqrt{4-x^2}.$$

Let $S_{\underline{k}}(\Gamma(N))$ be the space of classical holomorphic Siegel cusp forms of level $\Gamma(N)$ with the trivial central character (for simplicity) and weight $\underline{k}=(k_1,k_2), k_1\geq k_2\geq 3$.

Let $HE_{\underline{k}}(\Gamma(N))$ be a basis of $S_{\underline{k}}(\Gamma(N))$ consisting of Hecke eigen forms outside N.

For a Hecke eigen form $F\in S_{\underline{k}}(\Gamma(N))$, let $\pi_F=\otimes\pi_{F,p}$ be the associated cuspidal representation of GSp_4. (We assume Ramanujan conjecture, i.e., each local component $\pi_{F,p}$ is tempered.)

Denote by $\{\alpha^{\pm}_{F,p},\beta^{\pm}_{F,p}\}$ the Satake parameter of $\pi_{F,p}$ at $p\nmid N$. Then it follows from the temperedness that if we set

$$a_{F,p}:=\alpha_{F,p}+\alpha^{-1}_{F,p},\;\; b_{F,p}:=\beta_{F,p}+\beta^{-1}_{F,p},$$

then

$$a_{F,p}, b_{F,p}\in[-2,2].$$

We introduce a suitable measure

$$\mu_p=\frac{(p+1)^4}{p^4}f_p(x,y)g_p^+(x,y)g_p^-(x,y)\cdot\mu_\infty^{\mathrm{ST}}$$

on $\Omega:=[-2,2]\times[-2,2]$, where

$$f_p(x,y)=\frac{1}{\left(\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)^2-x^2\right)\left(\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)^2-y^2\right)},\quad \mu_\infty^{\mathrm{ST}}=\frac{4(x-y)^2}{\pi^2}\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{4}},$$

$$g_p^{\pm}(x,y)=\frac{1}{\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)^2-2\left(1+\frac{xy}{4}\pm\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{4}}\right)}.$$

By setting $x=2\cos\theta_1$, $y=2\cos\theta_2$, we can see that

$$\mu_p=\frac{(p+1)^4}{p^4\pi^2}\left|\frac{(1-e^{2i\theta_1})(1-e^{2i\theta_2})(1-e^{i(\theta_1+\theta_2)})(1-e^{i(\theta_1-\theta_2)})}{(1-p^{-1}e^{2i\theta_1})(1-p^{-1}e^{2i\theta_2})(1-p^{-1}e^{i(\theta_1+\theta_2)})(1-p^{-1}e^{i(\theta_1-\theta_2)})}\right|^2 d\theta_1 d\theta_2.$$

Then we expect

Conjecture 1.1. *For $h\in C^0(\Omega,\mathbb{R})$, the space of $\mathbb{R}$-valued continuous functions on Ω,*

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\sum_{p\leq x}h(a_{F,p},b_{F,p})=\int_\Omega h(x,y)\,d\mu_\infty^{ST}.$$

This is out of reach since we need analytic continuation and non-vanishing for $Re(s) \geq 1$ for all L-functions $L(s, \pi_F, \rho)$ for any irreducible representation $\rho : GSp_4(\mathbb{C}) \longrightarrow GL_N(\mathbb{C})$.

Instead we look at the vertical Sato-Tate distribution.

Put $d_{\underline{k},N} = \dim S_{\underline{k}}(\Gamma(N))$. Then

$$\dim S_{\underline{k}}(\Gamma(N)) \sim C \cdot N^{10}(k_1 - 1)(k_2 - 2)(k_1 - k_2 + 1)(k_1 + k_2 - 3).$$

Theorem 1.1. *(Vertical Sato-Tate) Let $p \nmid N$, $k_1 \geq k_2 \geq 3$, and $N + k_1 + k_2 \longrightarrow \infty$. Then the set*

$$\{(a_{F,p}, b_{F,p}) \in \Omega \mid F \in HE_{\underline{k}}(\Gamma(N))\}$$

is μ_p-equidistributed in Ω, namely, for any $f \in C^0(\Omega, \mathbb{R})$

$$\lim_{N+k_1+k_2 \to \infty} \frac{1}{d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(\Gamma(N))} f(a_{F,p}, b_{F,p}) = \int_\Omega f(x, y)\, d\mu_p.$$

2. Low-lying zeros and n-level density

Next, we consider the distribution of the low-lying zeros of the spinor L-functions for our family.

For $F \in S_{\underline{k}}(\Gamma(N))$, let $L(s, \pi_F, Spin)$ be the degree 4 spinor L-function; it satisfies the functional equation; Let $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

$$\Lambda(s, \pi_F, Spin) = q(F)^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s+\frac{k_1 + k_2 - 3}{2})\Gamma_{\mathbb{C}}(s+\frac{k_1 - k_2 + 1}{2})L(s, \pi_F, Spin) = \epsilon(\pi_F)\Lambda(1-s, \pi_F, Spin),$$

where $\epsilon(\pi_F) \in \{\pm 1\}$, and $N \leq q(F) \leq N^4$. The analytic conductor is $c(F) = (k_1 + k_2)^2(k_1 - k_2 + 1)^2 q(F)$.

Consider the distribution of the zeros of $L(s, \pi_F, Spin)$ around $s = \frac{1}{2}$. Let $\sigma_F = \frac{1}{2} + \sqrt{-1}\gamma_F$ be a non-trivial zero of $L(s, \pi_F, Spin)$. We do not assume GRH, and hence γ_F can be a complex number. Define $D(\pi_F, Spin, \phi)$ for an even Schwartz class function ϕ,

$$D(\pi_F, Spin, \phi) = \sum_{\gamma_F} \phi\left(\frac{\gamma_F}{2\pi} \log c_{\underline{k},N}\right),$$

where $\log c_{\underline{k},N} = \dfrac{1}{d_{\underline{k},N}} \displaystyle\sum_{F \in HE_{\underline{k}}(\Gamma(N))} \log c(F)$.

Since ϕ decays rapidly as $|x| \to \infty$, $D(\pi_F, Spin, \phi)$ measures the density of zeros of $L(s, \pi_F, Spin)$ which are in the radius $O(\frac{1}{\log c_{\underline{k},N}})$ at $s = \frac{1}{2}$. We call them the low-lying zeros.

Katz-Sarnak philosophy is that even though each individual L-functions behave randomly, as a family, low-lying zeros behave according to one of 5 symmetry types: U, SO(even), SO(odd), O, Sp.

In our case, we prove

Theorem 2.1. *Let the notations be as above. Let ϕ be an even Schwartz function for which its Fourier transform has a support sufficiently smaller than $(-1,1)$. Then*

$$\lim_{N+k_1+k_2\to\infty} \frac{1}{d_{\underline{k},N}} \sum_{F\in HE_{\underline{k}}(\Gamma(N))} D(\pi_F, \mathrm{Spin}, \phi) = \hat{\phi}(0) + \frac{1}{2}\phi(0) = \int_{\mathbb{R}} \phi(x) W(G)(x)\, dx,$$

where $G = SO(\text{even})$, $SO(\text{odd})$, or O type, and the corresponding density functions $W(G)$ are

$$W(SO(\text{even}))(x) = 1 + \frac{\sin 2\pi x}{2\pi x},\ W(SO(\text{odd}))(x) = 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x),$$

$W(O)(x) = 1 + \frac{1}{2}\delta_0(x)$, *and* $W(Sp)(x) = 1 - \frac{\sin 2\pi x}{2\pi x}$.

Remark 2.2. *Kowalski-Saha-Tsimerman (in level one case) and M. Dickson considered weighted equidistribution and one-level density of spinor L-functions of scalar-valued Siegel cusp forms, namely, let $\mathcal{F}_k(N)$ be a basis of the space of Siegel eigen cusp forms of weight k with respect to $\Gamma_0(N)$. Then*

$$\lim_{N+k\to\infty} \frac{1}{\sum_{F\in\mathcal{F}_k(N)} \omega_{F,N,k}} f(a_{F,p}, b_{F,p}) = \int_\Omega f(x,y)\, d\mu_p';$$

$$\lim_{N+k\to\infty} \frac{1}{\sum_{F\in\mathcal{F}_k(N)} \omega_{F,N,k}} \sum_{F\in\mathcal{F}_k(N)} \omega_{F,N,k} D(\pi_F, \phi) = \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int_{\mathbb{R}} \phi(x) W(Sp)(x)\, dx.$$

where $d\mu_p'$ is some Plancherel measure, and

$$\omega_{F,N,k} = \frac{\sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-\frac{3}{2})\Gamma(k-2)|A(F;I_2)|^2}{vol(\Gamma_0(N)\backslash\mathbb{H}_2)4\langle F,F\rangle},$$

and $F(Z) = \sum_{T>0} A(F;T) e^{2\pi i Tr(TZ)}$. So the symmetry type is Sp. Notice that the symmetry type is changed due to the weighted sum.

Let $-\frac{L'}{L}(s, \pi_F, Spin) = \sum_{n=1}^\infty \frac{a_F(n)\Lambda(n)}{n^s}$, where $\Lambda(n)$ is the von Mangoldt function: $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and } k>0 \\ 0, & \text{otherwise} \end{cases}.$

Then the standard calculation shows that

$$\frac{1}{d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} D(\pi_F, Spin, \phi) = \widehat{\phi}(0)$$
$$-\frac{2}{(\log c_{\underline{k},N}) d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} \sum_p \frac{a_F(p)\log p}{\sqrt{p}} \widehat{\phi}\left(\frac{\log p}{\log c_{k,N}}\right)$$
$$-\frac{2}{(\log c_{\underline{k},N}) d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} \sum_p \frac{a_F(p^2)\log p}{p} \widehat{\phi}\left(\frac{2\log p}{\log c_{\underline{k},N}}\right) + O\left(\frac{1}{\log c_{\underline{k},N}}\right).$$

By exchanging sums, we need to study the sums:

$$\sum_{F \in HE_{\underline{k}}(\Gamma(N))} a_F(p), \qquad \sum_{F \in HE_{\underline{k}}(\Gamma(N))} a_F(p^2).$$

Since $\sum_{F \in HE_{\underline{k}}(\Gamma(N))} a_F(p) = Tr\, T(p)|_{S_{\underline{k}}(\Gamma(N))}$, where $T(p)$ is the Hecke operator, Arthur's invariant trace formula is tailor-made to provide such sums (invariant under the conjugate of f). Similarly, $a_F(p^2)$ can be written as a sum of the trace of Hecke operators.

For simplicity, let $k_1 \geq k_2 \geq 4$. Fix $p \nmid N$. Let $K(N)$ for $N \in \mathbb{Z}_{>0}$ be the kernel of the natural quotient map from $G(\widehat{\mathbb{Z}})$ to $G(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}})$. Then $\Gamma(N) = Sp_4(\mathbb{Q}) \cap K(N)$.

Consider the algebraic representation $\xi = \xi_{\underline{k}}$ for $\underline{k} = (k_1, k_2)$, and let $D^{\text{hol}}_{l_1,l_2}$ be the holomorphic discrete series of $G(\mathbb{R})$ with the Harish-Chandra parameter $(l_1, l_2) = (k_1 - 1, k_2 - 2)$. We choose the test function $f = f_\xi f_N$ such that f_ξ is a pseudo-coefficient of $D^{\text{hol}}_{l_1,l_2}$. Then

$$\text{tr}(\pi_\infty(f_\xi)) = \begin{cases} -1, & \text{if } \pi_\infty = D^{\text{hol}}_{l_1,l_2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have Arthur's invariant trace formula,

$$I_{\text{spec}}(f_\xi f_N) = I_{\text{geom}}(f_\xi f_N).$$

Then the left hand side is a sum over $HE_{\underline{k}}(\Gamma(N))$, namely, we pick up only holomorphic Siegel cusp forms. So we transfer our calculation to the geometric side. The main term is $\widehat{f}(1) = \widehat{\mu}^{\text{pl}}_p(\hat{f})$ by Plancherel formula. Since we take f_ξ to be a pseudo-coefficient of holomorphic discrete series, there are contributions from unipotent orbits.

There are seven geometric terms. We can compute them explicitly.

Proposition 2.3. *Assume* $(N, 11!) = 1$. *Put* $\underline{k} = (k_1, k_2)$, $k_1 \geq k_2 \geq 3$ *and* $d_{\underline{k},N} := \dim S_{\underline{k}}(N)$. *There exist constants* $a_1'', a_2'', b_1'', b_2'', c_1'', c_2'', v_1, v_1', w_1, w_1'$ *depending only on* G *such that*

(1) (a) *(level-aspect) Fix* k_1, k_2. *Then as* $N \to \infty$,

$$\frac{1}{d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} a_F(p) = O(p^{-\frac{1}{2}} N^{-2}) + O(p^{v_1} N^{-3});$$

(b) *(weight-aspect) Fix* N. *Then as* $k_1 + k_2 \to \infty$,

$$\frac{1}{d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} a_F(p) = B_1 + B_2 + O(\frac{p^{v_1'}}{(k_1 - k_2 + 1)(k_1 - 1)(k_2 - 2)}),$$

$$B_1 = O(\frac{p^{-\frac{1}{2}}}{(k_1 - 1)(k_2 - 2)}),\ B_2 = O(\frac{p^{-\frac{1}{2}}}{(k_1 - k_2 + 1)(k_1 + k_2 - 3)})$$

(2) (a) *(level-aspect) Fix* k_1, k_2. *Then as* $N \to \infty$,

$$\frac{1}{d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} a_F(p^2) = -\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{1}{p^2}\Big) + O\left((a_1'' p^{-\frac{1}{2}} + a_2'' p^{\frac{1}{2}}) N^{-2}\right) + O\left(p^{w_1} N^{-3}\right).$$

(b) *(weight-aspect) Fix* N. *Then as* $k_1 + k_2 \to \infty$,

$$\frac{1}{d_{\underline{k},N}} \sum_{F \in HE_{\underline{k}}(N)} a_F(p^2) = -\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{1}{p^2}\Big) + B_1 + B_2 + O(\frac{p^{w_1'}}{(k_1 - k_2 + 1)(k_1 - 1)(k_2 - 2)}),$$

$$B_1 = O(\frac{p^{-\frac{1}{2}} b_1'' + p^{\frac{1}{2}} b_2''}{(k_1 - 1)(k_2 - 2)}),\ B_2 = O(\frac{p^{-\frac{1}{2}} c_1'' + p^{\frac{1}{2}} c_2''}{(k_1 - k_2 + 1)(k_1 + k_2 - 3)}).$$

Remark 2.4. *Here the second main terms* A, B_1 *come from non-semisimple contributions* $M = G$, $\gamma = u_{min} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$, *while* B_2 *comes from semisimple contributions* $M = G$, $\delta_1 = diag(1, -1, 1, -1)$, *since we use a pseudo-coefficient of holomorphic discrete series. We expect that the second terms* A, B_1 *correspond to non-holomorphic endoscopic representations whose global L-packet does not contain holomorphic Siegel cusp forms. However* B_2 *is mysterious and it seems interesting to figure out what kind of representations contribute to* B_2.

2.1. **Comparison with Shin-Templier** [5]**.** Let $D^{\rm large}_{l_1,l_2}$ be the large discrete series of $GSp_4(\mathbb{R})$ so that $\{D^{\rm hol}_{l_1,l_2}, D^{\rm large}_{l_1,l_2}\}$ makes up an L-packet of $\prod(G(\mathbb{R}))$. For $* \in \{\text{hol}, \text{large}\}$ and each $D^*_{l_1,l_2}$, we choose a pseudo coefficient $f^*_{\xi_{\underline{k}}} \in C^\infty_c(G(\mathbb{R}))$. Put $f^{\rm tot}_{\xi_{\underline{k}}} := f^{\rm hol}_{\xi_{\underline{k}}} + f^{\rm large}_{\xi_{\underline{k}}}$, where we may call it "stable" pseudo-coefficient. Then $f^{\rm tot}_{\xi_{\underline{k}}}$ is called Euler-Poincaré function, which was considered by Shin and Templier [5]. The trace $\mathrm{tr}(f^{\rm tot}_{\xi_{\underline{k}}})$ collects various automorphic forms including those generating holomorphic and large discrete series representations. On the other hand, all unipotent orbit contributions are zero. Only elliptic orbital integrals contribute. We have uniform estimates.

Let $\widetilde{S}_{\underline{k}}(N)$ be the set of cuspidal representations with the given infinity type in the local packet $\{D^{\rm hol}_{l_1,l_2}, D^{\rm large}_{l_1,l_2}\}$. Let $\widetilde{HE}_{\underline{k}}(\Gamma(N))$ be a basis. Then Shin and Templier showed

$$\lim_{N+k_1+k_2\to\infty} \frac{1}{d_{\underline{k},N}} \sum_{F\in \widetilde{HE}_{\underline{k}}(\Gamma(N))} D(\pi_F, \mathrm{Spin}, \phi) = \hat{\phi}(0) + \frac{1}{2}\phi(0).$$

Their result is exactly the same as ours. It is because (1) L-functions of cuspidal representations in the same L-packet are the same; (2) Contribution from endoscopic non-holomorphic forms is negligible.

2.2. **Classification of cuspidal representations of GSp_4.** By Laumon and Weissauer, non-CAP, non-endoscopic holomorphic Siegel cusp forms always appear in pairs with non-holomorphic Siegel cusp forms (evil twin). More precisely, if $\pi = \pi_\infty \otimes \pi_f$, $\pi_\infty = D^{\rm hol}_{l_1,l_2}$ (π is non-CAP, non-endoscopic), there exists a cuspidal representation $\pi' = \pi'_\infty \otimes \pi'_f$ such that $\pi'_\infty \simeq D^{\rm large}_{l_1,l_2}$ and $\pi'_f \simeq \pi_f$. The converse is true.

Now suppose π is endoscopic and $\pi_\infty \simeq D^{\rm hol}_{l_1,l_2}$. Then by Roberts, there exists a cuspidal representation $\pi' = \pi'_\infty \otimes \pi'_f$ such that $\pi'_\infty \simeq D^{\rm large}_{l_1,l_2}$ and $\pi'_f \sim \pi_f$. (Here $\sim$ means weak equivalence, and in fact equivalent outside the ramification of π.)

However, if π is endoscopic and $\pi_\infty \simeq D^{\rm large}_{l_1,l_2}$, there does not exist a cuspidal representation π' such that $\pi'_\infty \simeq D^{\rm large}_{l_1,l_2}$ and $\pi'_f \sim \pi_f$. (For example, we cannot construct a holomorphic Siegel cusp form from a pair of two elliptic cusp forms of level 1, but we can construct a cuspidal representation with the infinity type $D^{\rm large}_{l_1,l_2}$.)

Let $S_{\underline{k}}(N)^{\rm en,large}$ be the space generated by Hecke eigen forms F such that Π_F is endoscopic and $(\Pi_F)_\infty$ is isomorphic to the large discrete series $D^{\rm large}_{l_1,l_2}$. Then we have

$$\dim S_{\underline{k}}(N)^{\rm en,large} = O((k_1-k_2+1)(k_1+k_2-3)N^{8+\epsilon}), \quad \text{as } k_1+k_2+N\to\infty.$$

Therefore

$$\frac{\dim S_{\underline{k}}(N)^{\text{en,large}}}{\dim S_{\underline{k}}(N)} = O(((k_1-1)(k_1-2))^{-1}N^{-2+\epsilon}), \text{ as } k_1+k_2+N\to\infty.$$

Also if $k_1 \geq k_2 \geq 3$, the only CAP forms are CAP forms from Siegel parabolic subgroup and $k_1 = k_2$. If $S_k(\Gamma(N))^{\text{CAP}}$ is the subset of CAP representations of weight (k,k), $k \geq 3$, then

$$\dim S_k(\Gamma(N))^{\text{CAP}} = O(N^{7+\epsilon}k).$$

2.3. n**-level density.** In Theorem 2.1, since the support is smaller than $(-1,1)$, we cannot distinguish the symmetry type among SO(even), SO(odd) and O by one-level density. We need to compute n-level density.

Let $\phi(x_1,...,x_n) = \prod_{i=1}^n \phi_i(x_i)$ be an even Schwartz class function in each variables whose Fourier transform $\hat{\phi}(u_1,...,u_n)$ is compactly supported. We define

$$D^{(n)}(\pi,\phi) = \sum\nolimits^*_{j_1,\cdots,j_n} \phi\left(\gamma_{j_1}\frac{\log c_\pi}{2\pi}, \gamma_{j_2}\frac{\log c_\pi}{2\pi}, \ldots, \gamma_{j_n}\frac{\log c_\pi}{2\pi}\right)$$

where $\sum^*_{j_1,\ldots,j_n}$ is over $j_i \in \mathbb{Z}$ (if the root number is -1) or $\mathbb{Z}\backslash\{0\}$ with $j_a \neq \pm j_b$ for $a \neq b$, and c_π is the analytic conductor of $L(s,\pi)$.

Let $\mathfrak{F}(X)$ be the set of L-functions in $\mathfrak{F}$ such that $X < c_\pi < 2X$. The n-level density conjecture says that

$$\lim_{X\to\infty} \frac{1}{|\mathfrak{F}(X)|} \sum_{\pi\in\mathfrak{F}(X)} D^{(n)}(\pi,\phi) = \int_{\mathbb{R}^n} \phi(x) W(G(\mathfrak{F}))\, dx,$$

where $W(G(\mathfrak{F}))$ is the n-level density function.

There are five possible symmetry types of families of L-functions: U, SO(even), SO(odd), O, and Sp. The corresponding density functions $W(G)$ are determined in [?] (cf. [?]). They are

$$W(\text{U})(x) = \det(K_0(x_j,x_k))_{\substack{1\leq j\leq n\\1\leq k\leq n}},$$

$$W(\text{SO(even)})(x) = \det(K_1(x_j,x_k))_{\substack{1\leq j\leq n\\1\leq k\leq n}},$$

$$W(\text{SO(odd)})(x) = \det(K_{-1}(x_j,x_k))_{\substack{1\leq j\leq n\\1\leq k\leq n}} + \sum_{\nu=1}^n \delta(x_\nu)\det(K_{-1}(x_j,x_k))_{\substack{1\leq j\neq\nu\leq n\\1\leq k\neq\nu\leq n}},$$

$$W(\text{Sp})(x) = \det(K_{-1}(x_j,x_k))_{\substack{1\leq j\leq n\\1\leq k\leq n}}, \quad W(\text{O})(x) = \frac{1}{2}(W(\text{SO(even)})(x) + W(\text{SO(odd)})(x)),$$

where $\delta(x)$ is the Dirac delta function, and $K_\epsilon(x,y) = \dfrac{\sin\pi(x-y)}{\pi(x-y)} + \epsilon\dfrac{\sin\pi(x+y)}{\pi(x+y)}$, $\epsilon \in \{\pm1,0\}$.

For n-level density, the root numbers play a role. If $\epsilon(F) = -1$, $L(s,\pi_F,Spin)$ has a family zero at $s = \frac{1}{2}$. Let $S^\pm_{\underline{k}}(\Gamma(N))$ be the subspace of $S_{\underline{k}}(\Gamma(N))$ with $\epsilon(F) = \pm1$, resp.

Let $S_{\underline{k}}^{\pm}(N)$ be the subspace of $S_{\underline{k}}(N)$ with the root number $\epsilon(\pi_F)=\pm 1$. Let $HE_{\underline{k}}^{\pm}(N)=S_{\underline{k}}^{\pm}(N)\cap HE_{\underline{k}}(N)$, and denote $|HE_{\underline{k}}^{\pm}(N)|=d_{\underline{k},N}^{\pm}$.

When $N=1$ (i.e., level one case), we have $\epsilon(\pi_F)=(-1)^{k_2}$. (In this case, k_1-k_2 should be even.) Hence $HE_{\underline{k}}^{+}(1)=HE_{\underline{k}}(1)$ when k_2 is even, and $HE_{\underline{k}}^{-}(1)=HE_{\underline{k}}(1)$ when k_2 is odd.

However, when $N+k_1+k_2\to\infty$, we expect

Conjecture 2.1. *Put $\underline{k}=(k_1,k_2)$, $k_1\geq k_2\geq 3$.*

(1) (level-aspect) Fix k_1,k_2. Then there exists a constant $\delta>0$ such that $d_{\underline{k},N}^{\pm}=\frac{1}{2}d_{\underline{k},N}+O_{\underline{k}}(N^{9-\delta})$, and for $m=\prod_p p^{v_p(m)}$,

$$\frac{1}{d_{\underline{k},N}^{+}}\sum_{F\in HE_{\underline{k}}^{+}(N)}\tilde{\lambda}_F(m)=\delta_{\square}m^{-\frac{1}{2}}\prod_{p|m}(1+p^{-2}+\cdots+p^{-v_p(m)})+O_{\underline{k}}(N^{-2}m^{c}),$$

where $\delta_{\square}=\begin{cases}1, & \text{if } m \text{ is a square}\\ 0, & \text{otherwise}\end{cases}.$

(2) (weight-aspect) Fix N. Then $d_{\underline{k},N}^{\pm}=\frac{1}{2}d_{\underline{k},N}+O_N((k_1-1)(k_2-2))+O((k_1-k_2+1)(k_1+k_2-3))$, and for $m=\prod_p p^{v_p(m)}$,

$$\frac{1}{d_{\underline{k},N}^{+}}\sum_{F\in HE_{\underline{k}}^{+}(N)}\tilde{\lambda}_F(m)=\delta_{\square}m^{-\frac{1}{2}}\prod_{p|m}(1+p^{-2}+\cdots+p^{-v_p(m)})$$
$$+O_N\left(\frac{m^c}{(k_1-1)(k_2-2)}\right)+O_N\left(\frac{m^d}{(k_1-k_2+1)(k_1+k_2-3)}\right).$$

Under the conjecture, we can show that $\{L(s,\pi_F,Spin):F\in HE_{\underline{k}}^{+}(N)\}$ is SO(even) type, and $\{L(s,\pi_F,Spin):F\in HE_{\underline{k}}^{-}(N)\}$ is SO(odd) type, i.e.,

$$\frac{1}{d_{\underline{k},N}^{\pm}}\sum_{F\in HE_{\underline{k},N}^{\pm}}D^{(n)}(\pi_F,\phi,\mathrm{Spin})=\int_{\mathbb{R}^n}\phi(x)W(G^{\pm})(x)\,dx+O\left(\frac{\omega(N)}{\log c_{\underline{k},N}}\right),$$

where G^{+}=SO(even), and G^{-}=SO(odd).

We proved Conjecture 2.1 for paramodular forms.

3. Standard L-functions of general degree; work in progress

Let F be a Siegel cusp form of weight $\underline{k}=(k_1,...,k_r)$ with respect to $\Gamma=\Gamma(N)$, $k_1\geq k_2\geq\cdots\geq k_r>r$, and let π_F be a cuspidal representation of $\mathrm{Sp}(2r)/\mathbb{Q}$ associated to F. Due to the work of Arthur [1], there exists an automorphic representation Π of $GL_{2r+1}/\mathbb{Q}$ which is the transfer of π_F. We define $L(s,\pi_F,\mathrm{St})$, the degree $2r+1$ standard L-function of π_F, to be $L(s,\Pi)$.

Let $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$. For $p \nmid N$, π_p is the spherical representation of $Sp(2n, \mathbb{Q}_p)$ with the Satake parameter $\{\alpha_{1p}, ..., \alpha_{rp}, 1, \alpha_{rp}^{-1}, ..., \alpha_{1p}^{-1}\}$. Then

$$L(s, \pi_F, \mathrm{St}) = \prod_p L(s, \pi_p, \mathrm{St}),$$

where if $p \nmid N$,

$$L(s, \pi_p, \mathrm{St})^{-1} = (1 - p^{-s}) \prod_{i=1}^{r} (1 - \alpha_{ip} p^{-s})(1 - \alpha_{ip}^{-1} p^{-s}).$$

Theorem 3.1. *$L(s, \pi_F, \mathrm{St})$ has a meromorphic continuation to all of $\mathbb{C}$. Let*

$$\Lambda(s, \pi_F, \mathrm{St}) = q(F)^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s + \epsilon) \Gamma_{\mathbb{C}}(s + k_1 - 1) \cdots \Gamma_{\mathbb{C}}(s + k_r - r) L(s, \pi_F, \mathrm{St}),$$

where $\epsilon = \begin{cases} 0, & \text{if } r \text{ is even} \\ 1, & \text{if } r \text{ is odd} \end{cases}$*, and $q(F)$ is the conductor of π_F and it satisfies $N \leq q(F) \leq N^{2r+1}$. Then*

$$\Lambda(s, \pi_F, \mathrm{St}) = \epsilon(F) \Lambda(1 - s, \pi_F, \mathrm{St}),$$

where $\epsilon(F) \in \{\pm 1\}$.

Lapid showed that $\epsilon(F) = 1$ always. Let f_ξ be the pseudo-coefficient of $D^{\mathrm{hol}}_{l_1,...,l_r}$, where $l_1 = k_1 - 1, ..., l_r = k_r - r$. Put a test function $f = f_\xi f_N$ in Arthur's invariant trace formula. Then the unipotent orbital integrals in the geometric side are estimated by Shintani double zeta functions [4]. We can show that n-level density of the low-lying zeros of $L(s, \pi_F, St)$ is Sp type.

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Henry H. Kim, Department of mathematics, University of Toronto, Toronto, Ontario M5S 2E4, CANADA, and Korea Institute for Advanced Study, Seoul, KOREA

E-mail address: henrykim@math.toronto.edu

Satoshi Wakatsuki, Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kakumamachi, Kanazawa, Ishikawa, 920-1192, JAPAN

E-mail address: wakatsuk@staff.kanazawa-u.ac.jp

Takuya Yamauchi, Mathematical Inst. Tohoku University, 6-3,Aoba, Aramaki, Aoba-Ku, Sendai 980-8578, JAPAN

E-mail address: yamauchi@math.tohoku.ac.jp