TOWARDS p-ADIC GROSS-ZAGIER FORMULA FOR TRIPLE PRODUCT L-SERIES

SHUNSUKE YAMANA

Advanced Mathematical Institute
Osaka City University

ABSTRACT. I will report my joint work with Ming-Lun Hsieh on a (conjectural) description of cyclotomic derivatives of p-adic triple product L-functions in terms of Nekovar's p-adic height of diagonal cycles.

1. THE TRIPLE PRODUCT L-SERIES OF THREE ELLIPTIC CURVES

Let E_1, E_2, E_3 be rational elliptic curves of conductor N_i. Fix an odd prime number p prime to $N_1N_2N_3$. The triple tensor product

$$\rho_p^{\boldsymbol{E}} := T_p(E_1) \otimes T_p(E_2) \otimes T_p(E_3)(-3)$$

is a geometric p-adic Galois representation realized in the middle cohomology of the abelian variety $\boldsymbol{E} = E_1 \times E_2 \times E_3$, where $T_p(E_i) = \varprojlim_n E_i[p^n]$ is the Tate module of E_i. Let $G_{\mathbf{Q}} \supset G_{\mathbf{Q}_\ell} \supset I_\ell$ be the absolute Galois group, its decomposition group at ℓ and its inertia subgroup at ℓ. We consider the central critical twist

$$V_p^{\boldsymbol{E}} := \rho_p^{\boldsymbol{E}}(2) : G_{\mathbf{Q}} \to \mathrm{GL}_8(\mathbf{Z}_p).$$

Observe that $(V_p^{\boldsymbol{E}})^*(1) \simeq V_p^{\boldsymbol{E}}$.

Fix an embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Let $\mathbf{Q}_\infty$ be the $\mathbf{Z}_p$-extension of $\mathbf{Q}$. Define a character $\langle\cdot\rangle : G_{\mathbf{Q}} \to G_{\mathbf{Q}_p} \to 1 + p\mathbf{Z}_p$ by $\langle x\rangle = x/\boldsymbol{\omega}(x)$, where we identify $G_{\mathbf{Q}_p}$ with $\mathbf{Z}_p^\times$ and denote the p-adic Teichmüller character by $\boldsymbol{\omega}$. The twisted triple product L-series is defined by the Euler product

$$L(\boldsymbol{E} \otimes \hat{\chi}, s+2) = \prod_\ell L_\ell(V_p^{\boldsymbol{E}} \otimes \chi, s)$$

for p-adic characters χ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ of finite order, where $\hat{\chi}$ is the Dirichlet character associated to $\iota_\infty \circ \chi$. If $\ell \neq p$, then

$$L_\ell(V_p^{\boldsymbol{E}} \otimes \chi, s) = \det(\mathbf{1}_8 - \ell^{-s}\iota_\infty(\chi(\ell)^{-1}\mathrm{Frob}_\ell|(V_p^{\boldsymbol{E}})^{I_\ell}))^{-1}.$$

The complete triple product L-series

$$\Lambda(\boldsymbol{E}, s) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s-1)^3 L(\boldsymbol{E}, s)$$

proved to be an entire function which satisfies a simple functional equation

$$\Lambda(\boldsymbol{E}, s) = \varepsilon(\boldsymbol{E}, s)\Lambda(\boldsymbol{E}, 4-s)$$

by the integral representation discovered by Garrett [Gar87] and studied extensively in the literatures [PSR87, Ike89, Ike92, GK92, Ram00]. The global sign is given by the product of local signs $\varepsilon = \varepsilon(\boldsymbol{E}, 2) = -\prod_\ell \varepsilon_\ell(\boldsymbol{E})$. Let D be the unique quaternion algebra over $\mathbf{Q}$ such that $D_\ell \not\simeq \mathrm{M}_2(\mathbf{Q}_\ell)$ if and only if $\varepsilon_\ell(\boldsymbol{E}) = -1$. Here we put $D_\ell = D \otimes \mathbf{Q}_\ell$ and $\widehat{D} = D \otimes \widehat{\mathbf{Q}}$.

If E_1, E_2, E_3 are semistable, then N_1, N_2, N_3 are square-free,

$$\varepsilon(\boldsymbol{E}, s) = \varepsilon N_-^{2-s} N_+^{8-4s}, \qquad\qquad \varepsilon = \prod_{\ell | N_-} \prod_{i=1}^{3} \varepsilon_\ell(E_i),$$

where N_- and N_+ are the greatest common divisor and the least common multiple of N_1, N_2, N_3. Note that $\varepsilon_\ell(E_i) = -1$ if and only if ℓ divides N_i and E_i has split multiplicative reduction at ℓ.

2. Ichino's formula

The theorem of Wiles gives a primitive form

$$f_i = \sum_{n=1}^{\infty} \mathbf{a}(n, f_i) q^n \in S_2(\Gamma_0(N_i))$$

such that all the Fourier coefficients $\mathbf{a}(n, f_i)$ are rational integers and such that E_i is isogeneous to the elliptic curve obtained from f_i via the Eichler–Shimura construction, i.e., the Dirichlet series $\sum_{n=1}^\infty \mathbf{a}(n, f_i) n^{-s}$ coincides with the Hasse-Weil L-series $L(s, E_i)$. Then $\varepsilon_q(E_i) = -\mathbf{a}(q, f_i)$ for each prime factor q of N_i. Let π_i be the automorphic representation of $\mathrm{PGL}_2(\mathbf{A})$ generated by f_i. The eigenform f_i determines an automorphic representation $\pi_i^D \simeq \otimes'_v \pi_{i,v}^D$ of $(D \otimes \mathbf{A})^\times$ via the global correspondence of Jacquet, Langlands and Shimizu. Though π_i^D is self-dual, we write $\pi_i^{D\vee}$ for its dual with future generalizations in view. Let $X = \{X_U\}_U$ denote the projective system of rational curves associated to D indexed by open compact subgroups U of $\widehat{D}^\times$.

For every place v of $\mathbf{Q}$ we define the local trilinear form

$$I_v : \bigotimes_{i=1}^{3} \bigl(\pi_{i,v}^D \otimes \pi_{i,v}^{D\vee}\bigr) \to \mathbf{C}$$

by

$$\text{(2.1)} \quad I_v(h_v \otimes h'_v) = \frac{\prod_{i=1}^3 L(1, \pi_{i,v}, \mathrm{ad})}{\zeta_v(2)^2 L\bigl(\frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}\bigr)} \int_{\mathbf{Q}_v^\times \backslash D_v^\times} B_v((\sigma_{1,v} \otimes \sigma_{2,v} \otimes \sigma_{3,v})(g) h_v \otimes h'_v)\, \mathrm{d}g.$$

The global trilinear form $I : \bigotimes_{i=1}^3 \bigl(\pi_i^D \otimes \pi_i^{D\vee}\bigr) \to \mathbf{C}$ is defined to be the tensor product of the local trilinear forms I_v. This definition depends on the choice of the local invariant pairings $B_v : \bigotimes_{i=1}^3 \left(\pi_{i,v}^D \otimes \pi_{i,v}^{D\vee}\right) \to \mathbf{C}$. Normalize the local pairings by the compatibility

$$\otimes_{i=1}^3 \langle\ ,\ \rangle_i = \otimes_v B_v.$$

Here the Petersson pairing $\langle\ ,\ \rangle_i : \pi_i^D \otimes \pi_i^{D\vee} \to \mathbf{C}$ is defined by

$$\langle h_i, h_i' \rangle_i = \int_{\mathbf{A}^\times D^\times \backslash (D \otimes \mathbf{A})^\times} h_i(g) h_i'(g)\, \mathrm{d}g.$$

Define the period integral $\mathscr{P}^D : \bigotimes_{i=1}^3 \pi_i^D \to \mathbf{C}$ by

$$\mathscr{P}^D(h_1 \otimes h_2 \otimes h_3) = \int_{\mathbf{A}^\times D^\times \backslash (D \otimes \mathbf{A})^\times} h_1(g) h_2(g) h_3(g)\, \mathrm{d}g.$$

For a local reason $\mathscr{P}^{D'}$ vanishes on $\bigotimes_{i=1}^3 \pi_i^{D'}$ unless $D \simeq D'$. Ichino proved the following formula for the central critical value in [Ich08]:

$$\mathscr{P}^D(h)\mathscr{P}^D(h') = 2^{-3}\zeta_{\mathbf{Q}}(2)^2 \frac{\Lambda(\boldsymbol{E}, 2)}{\prod_{i=1}^3 \Lambda(1, \pi_i, \mathrm{ad})} I(h \otimes h'),$$

where $\Lambda(s, \pi_i, \mathrm{ad})$ is the complete adjoint L-series of π_i.

3. The complex derivative

Let $\varepsilon = -1$. Then Ichino's formula is trivial as $L(\boldsymbol{E}, 2)$ is automatically 0 and $\mathscr{P}^D$ vanishes. The main object of study in this case is the central derivative $L'(\boldsymbol{E}, 2)$ of $L(\boldsymbol{E}, s)$. Now D is indefinite and X_U is the (compactified) Shimura curve. We regard X_U as the codimensioin 2 cycle embedded diagonally in the threefold X_U^3. One can modify it to obtain a homologically trivial cycle, following [GS95]. Gross and Kudla conjectured an analogous expression for $L'(\boldsymbol{E}, 2)$ in terms of a height pairing of the (f_1, f_2, f_3)-isotypic component of the modified diagonal cycle.

Let $\mathbb{D}$ be the definite quaternion algebra over $\mathbf{A}$ whose finite part is isomorphic to $\widehat{D}$. Since $\mathbb{D}$ is not the base change of any quaternion algebra over $\mathbf{Q}$, it is incoherent in the sense of Kudla. The projective limit X of $\{X_U\}$ is endowed with the action of $\widehat{D}^\times$. The curve X_U has a Hodge class L_U, which is the line bundle whose global sections are holomorphic modular forms of weight two. Normalize the Hodge class by $\xi_U := \frac{L_U}{\mathrm{vol}(X_U)} |\widehat{\mathbf{Z}}^\times / \mathrm{N}_{\mathbf{Q}}^D(U)|$, where

$$\mathrm{vol}(X_U) := \int_{X_U(\mathbf{C})} \frac{\mathrm{d}x\mathrm{d}y}{2\pi y^2}.$$

It is known that $\deg L_U = \mathrm{vol}(X_U)$ and that the induced action of $\widehat{D}^\times$ on the set of geometrically connected components of X_U factors through the norm map $\mathrm{N}_{\mathbf{Q}}^D : \widehat{D}^\times \to \widehat{\mathbf{Q}}^\times$. Hence the restriction of ξ_U to each geometrically connected component of X_U has degree 1.

For any abelian variety A over $\mathbf{Q}$ the space $\mathrm{Hom}_{\xi_U}^0(X_U, A)$ consists of morphisms in $\mathrm{Hom}_{\mathbf{Q}}(X_U, A) \otimes \mathbf{Q}$ which map the Hodge class ξ_U to zero in A. Since any morphism from X_U to an abelian variety factors through the

Jacobian variety J_U of X_U, we also have $\mathrm{Hom}^0_{\xi_U}(X_U, A) = \mathrm{Hom}^0_{\mathbf{Q}}(J_U, A)$. We consider the $\mathbf{Q}$-vector spaces

$$\sigma_i := \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, E_i), \qquad \sigma_i^\vee := \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, E_i^\vee).$$

The space σ_i admits a natural action by $\mathbb{D}^\times$. Actually, $\sigma_i \otimes_{\mathbf{Q}} \mathbf{C} \simeq \otimes'_q \pi^D_{i,q}$ from which $\pi^D_{i,q}$ gains the structure of a $\mathbf{Q}$-vector space. Here the archimedean part $\mathbb{D}^\times_\infty$ acts trivially on σ_i.

Let $h_{i,U} : J_U \to E_i$ and $h'_{i,U} : J_U \to E_i^\vee$ be $\mathbf{Q}$-morphisms. The morphism $h'^\vee_{i,U} : E_i \to J_U$ represents the homomorphism $h'^*_{i,U} : E_i \simeq \mathrm{Pic}^0(E_i) \to \mathrm{Pic}^0(J_U)$ composed with the canonical isomorphism $\mathrm{Pic}^0(J_U) \simeq J_U$ given by the Abel-Jacobi theorem. By Lemma 3.11 of [YZZ13]

$$B_i^\natural(h_i \otimes h'_i) = \mathrm{vol}(X_U)^{-1} h_{i,U} \circ h'^\vee_{i,U} \in \mathrm{End}^0_{\mathbf{Q}}(E_i) = \mathbf{Q}$$

is a perfect $\mathbb{D}^\times$-invariant pairing $\sigma_i \otimes \sigma_i^\vee \to \mathbf{Q}$. Let $B^\natural := \otimes_{i=1}^3 B_i^\natural$ and define the trilinear form $I^\natural \in \mathrm{Hom}_{\widehat{D}^\times \times \widehat{D}^\times}\big(\bigotimes_{i=1}^3 (\sigma_i \otimes \sigma_i^\vee), \mathbf{Q}\big)$ as in (2.1).

For each U we let Δ_U be the diagonal cycle of X_U^3 as an element in the Chow group $\mathrm{CH}^2(X_U^3)$ of codimension 2 cycles. We obtain a homologically trivial cycle Δ_{U,ξ_U} on X_U^3 by some modification with respect to ξ_U as constructed in [GS95]. The classes $\Delta^\dagger_{U,\xi_U} = \frac{\Delta_{U,\xi_U}}{\mathrm{vol}(X_U)}$ form a projective system and define a class $\Delta^\dagger_\xi \in \varprojlim \mathrm{CH}^2(X_U^3)^0$.

Given $h_i \in \sigma_i$ for $i = 1, 2, 3$, we get a homologically trivial class

$$h_* \Delta^\dagger_\xi \in \mathrm{CH}^2(\boldsymbol{E})^0, \qquad h = h_1 \times h_2 \times h_3.$$

One can consider the Beilinson-Bloch height pairing $\langle\ ,\ \rangle_{\mathrm{BB}}$ between homologically trivial cycles on $\boldsymbol{E}$ and $\boldsymbol{E}^\vee$.

The following formula was first conjectured by Gross-Kudla [GK92] and later refined by Yuan, S. W. Zhang and W. Zhang [YZZ]:

Conjecture 3.1 (Gross-Kudla, Yuan-Zhang-Zhang)**.**

$$\langle h_* \Delta^\dagger_\xi, h'_* \Delta^\dagger_\xi \rangle_{\mathrm{BB}} = 2^3 \zeta_{\mathbf{Q}}(2)^2 \frac{\Lambda'(\boldsymbol{E}, 2)}{\prod_{i=1}^3 \Lambda(1, \pi_i, \mathrm{ad})} I^\natural(h \otimes h').$$

This formula is a higher dimensional analogue of the Gross–Zagier formula. A significant progress was given in [YZZ] .

Remark 3.2. (1) Let $\mathrm{CH}^2(\boldsymbol{E})_0$ be the subgroup of elements with trivial projection onto $E_i \times E_j$. Lemma 5.1.2 of [Zha10a] gives the decomposition

$$\mathrm{CH}^2(\boldsymbol{E})^0 \simeq \mathrm{CH}^2(\boldsymbol{E})_0 \oplus \bigoplus_{i=1}^3 2\mathrm{CH}^1(E_i)^0$$

which is compatible with the Künneth decomposition

$$H^3_{\text{ét}}(\boldsymbol{E}_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p(2)) \simeq \otimes_{i=1}^3 H^1_{\text{ét}}(E_{i/\overline{\mathbf{Q}}}, \mathbf{Q}_p)(2) \oplus \bigoplus_{i=1}^3 2H^1_{\text{ét}}(E_{i/\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1).$$

Since $\mathrm{CH}^1(E_i)^0$ is nothing but the Mordell–Weil group of E_i, the BSD conjecture gives $\mathrm{rank}\mathrm{CH}^1(E_i)^0 = \mathrm{ord}_{s=1} L(H^1_{\text{ét}}(E_{i/\overline{\mathbf{Q}}}, \mathbf{Q}_p), s)$ and the Beilinson-Bloch conjecture gives

$$\mathrm{rank}\mathrm{CH}^2(\boldsymbol{E})^0 = \mathrm{ord}_{s=2} L(H^3_{\text{ét}}(\boldsymbol{E}_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p), s),$$
$$\mathrm{rank}\mathrm{CH}^2(\boldsymbol{E})_0 = \mathrm{ord}_{s=2} L(\boldsymbol{E}, s).$$

If $L'(\boldsymbol{E}, 2) \neq 0$, then $h_*\Delta_\xi^\dagger$ is not zero in $\mathrm{CH}^2(\boldsymbol{E})^0$ for some $h \in \otimes_{i=1}^3 \sigma_i$ by Conjecture 3.1.

(2) Let $E_1 = E_2 = E_3 = E$. Then $L(\boldsymbol{E}, s) = L(\mathrm{Sym}^3 E, s)L(E, s-1)^2$. If it has odd functional equation, then its order at $s = 2$ is greater than 1, which is compatible with Proposition 4.5 of [GS95].

(3) Let $f_1 = f_2 \neq f_3$. Then $L(\boldsymbol{E}, s) = L(\mathrm{Sym}^2 f_1 \times f_3, s)L(f_3, s-1)$ and hence $L'(\boldsymbol{E}, 2) = L(\mathrm{Sym}^2 f_1 \times f_3, 2)L'(f_3, 1)$ (see §5.3 of [Zha10b]).

4. Cyclotomic p-adic triple product L-series

Fix an odd prime number p which does not divide N^+ and such that none of $\mathbf{a}(p, f_i)$ is divisible by p. Equivalently, E_1, E_2, E_3 have good ordinary reduction at p. The $G_{\mathbf{Q}_p}$-invariant subspace

$$\mathrm{Fil}^0 T_p(E_i) := T_p(E_i)^{I_p} = \mathrm{Ker}(T_p(E_i) \to T_p(E_i/\mathbb{F}_p)$$

fixed by I_p is one-dimensional, where $E_i/\mathbb{F}_p$ denotes the mod p reduction of the Neron model of E_i.

The Galois representation $V_p^{\boldsymbol{E}}$ satisfies the Panchishkin condition in [Gre94, page 217], i.e., we define the rank four $G_{\mathbf{Q}_p}$-invariant subspace of $V_p^{\boldsymbol{E}}$ by

$$\begin{aligned}\mathrm{Fil}^+ V_p^{\boldsymbol{E}} :=& \mathrm{Fil}^0 T_p(E_1) \otimes \mathrm{Fil}^0 T_p(E_2) \otimes T_p(E_3)(-1)\\ &+ T_p(E_1) \otimes \mathrm{Fil}^0 T_p(E_2) \otimes \mathrm{Fil}^0 T_p(E_3)(-1)\\ &+ \mathrm{Fil}^0 T_p(E_1) \otimes T_p(E_2) \otimes \mathrm{Fil}^0 T_p(E_3)(-1).\end{aligned}$$

The Hodge-Tate numbers of $\mathrm{Fil}^+ V_p^{\boldsymbol{E}}$ are all positive, while none of the Hodge-Tate numbers of $V_p^{\boldsymbol{E}}/\mathrm{Fil}^+ V_p^{\boldsymbol{E}}$ is positive.

The author and Ming-Lun Hsieh have constructed a function $L_p(\boldsymbol{E})$ on the space of continuous characters $\chi : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \to \overline{\mathbf{Q}}_p^\times$ having the following interpolation property

$$L_p(\boldsymbol{E}, \hat{\chi}) = \frac{\Lambda(\boldsymbol{E} \otimes \hat{\chi}, 2)}{\prod_{i=1}^3 \Lambda(1, \pi_i, \mathrm{ad})} (\sqrt{-1})^3 \mathcal{E}_p(\mathrm{Fil}^+ V_p^{\boldsymbol{E}} \otimes \chi)$$

for all finite-order characters $\hat{\chi}$ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ in Corollary 7.9 of [HY], where the modified p-Euler factor is defined by

$$\mathcal{E}_p(\mathrm{Fil}^+ V_p^{\boldsymbol{E}} \otimes \chi) = \frac{L(\mathrm{Fil}^+ V_p^{\boldsymbol{E}} \otimes \chi, 0)}{\varepsilon(\mathrm{Fil}^+ V_p^{\boldsymbol{E}} \otimes \chi) \cdot L((\mathrm{Fil}^+ V_p^{\boldsymbol{E}} \otimes \chi)^\vee, 1)} \cdot \frac{1}{L_p(V_p^{\boldsymbol{E}} \otimes \chi, 0)}.$$

It satisfies the functional equation

$$L_p(\boldsymbol{E}, T) = \varepsilon \langle N_- \rangle_T^{-1} \langle N_+ \rangle_T^{-4} L_p(\boldsymbol{E}, (1+T)^{-1} - 1).$$

5. The p-adic derivative

Letting $\varepsilon = -1$ and $T = 0$, we get

$$L_p(\boldsymbol{E}, \mathbb{1}) = 0.$$

We consider the cyclotomic derivative

$$L_p'(\boldsymbol{E}, \mathbb{1}) := \lim_{s \to 0} \frac{L_p(\boldsymbol{E}, \langle \cdot \rangle^s)}{s}.$$

The conjectural formula for this cyclotomic derivative has the same shape but the real valued height is replaced by a p-adic valued height.

The theory of the p-adic height pairing was developed by Néron, Zarhin, Schneider, Mazur-Tate, Perrin-Riou, Nekovář. The p-adic height pairing depends on a choice of the p-adic logarithm on the idéle class group $\mathbf{A}^\times/\mathbf{Q}^\times$ and a choice of a splitting as $\mathbf{Q}_p$-vector spaces of the Hodge filtration of the de Rham cohomology of $\boldsymbol{E}$ over $\mathbf{Q}_p$. We take the Iwasawa logarithm $l_{\mathbf{Q}} : \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{Q}_p$. Since $V_p^{\boldsymbol{E}}$ satisfies the Panchishkin condition, we have a natural choice of the splitting obtained from $\mathrm{Fil}^+ V_p^{\boldsymbol{E}}$. We may therefore say that there is a canonical p-adic height pairing $\langle\ ,\ \rangle_{\mathrm{Nek}}$ on homologically trivial cycles on $\boldsymbol{E}$.

Conjecture 5.1.

$$\langle h_* \Delta_\xi^\dagger, h'_* \Delta_\xi^\dagger \rangle_{\mathrm{Nek}} \cdot 2^8 \tilde{\zeta}_{\mathbf{Q}}(2)^2 (\sqrt{-1})^3 \mathcal{E}_p(\mathrm{Fil}^+ V_p^{\boldsymbol{E}}) = L_p'(\boldsymbol{E}, \mathbb{1}) I^\natural(h \otimes h')$$

for all $h \in \bigotimes_{i=1}^3 (\sigma_i \otimes \sigma_i^\vee)$, where $\tilde{\zeta}_{\mathbf{Q}}(s) = 2(2\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty n^{-s}$.

Remark 5.2. The p-adic height factors through the Abel-Jacobi map

$$\mathrm{CH}^2(\boldsymbol{E})^0 \otimes \mathbf{Q}_p \to H_f^1(\mathbf{Q}, H^3_{\text{ét}}(\boldsymbol{E}_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p(2))).$$

When $L_p'(\boldsymbol{E}, \mathbb{1}) \neq 0$, Conjecture 5.1 gives a nonzero element of the Bloch-Kato Selmer group of the Galois representation $V_p^{\boldsymbol{E}}$.

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Email address: `yamana@sci.osaka-cu.ac.jp`

Department of Mathematics, Graduate School of Science, Osaka City University