Complexifications of pseudo H-type algebras

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Abstract. We study complexifications of 2-step nilpotent real Lie algebras, intimately connected to the Clifford algebras, that are called pseudo H-type Lie algebras. They are indexed by pairs (r, s) reflecting the signature of the scalar product on the centre. We show that the complexifications of real pseudo H-type Lie algebras of equal dimensions with the same value $n = r + s$ are isomorphic.

Key words: admissible modules, Clifford algebras, 2-step nilpotent Lie algebras, indefinite scalar product, isomorphism.
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1 Introduction

Since the construction of H-type Lie algebras in [8], the study of their properties and generalizations has been an important research topic in the interface of analysis, geometry and algebra. The definition of these algebras in [9] via representations of Clifford algebras defined by positive definite quadratic forms was successfully extended to the non-degenerate case in [2]. In the general situation of scalar products much more care needs to be taken, since one needs to require an additional metric condition on the Clifford modules, referred to as *admissibility*. In recent times, these Lie algebras have been actively studied, see for example [1, 3, 4, 5, 6] and the references therein.

The aim of this short note it to give a hands-on proof of the fact that the complexification of pseudo H-type Lie algebras having equal dimensions of center and horizontal parts yields isomorphic complex Lie algebras. It bases many of its argument on the classification of H-type algebras obtained in [5, 6]. This situation can be seen as a nilpotent analogue of the well-known fact that

$$\mathfrak{sl}(n, \mathbb{C}) \cong \mathfrak{su}(n) \otimes \mathbb{C} \cong \mathfrak{su}(p, q) \otimes \mathbb{C}$$

for any $n \geq 2$ and any positive integers p, q satisfying $p + q = n$.

The structure of the paper is the following. In Section 2 we present some basic definitions and state precisely the main result. In Section 3 we prove the theorem for minimal admissible modules for the so-called basic cases. In Section 4 we explain in detail how to proceed in the case the admissible modules under consideration are non minimal for the basic cases. Finally, in Section 5 we apply periodicity arguments to finish the proof of the main result.

2 Pseudo H-type algebras

Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ be a nilpotent graded Lie algebra of step 2, $\langle\, . \,, . \,\rangle_{r,s}$ a non-degenerate symmetric bilinear form on $\mathfrak{z}$ of index (r, s) and $\langle\, . \,, . \,\rangle$ a non-degenerate symmetric bilinear form on $\mathfrak{v}$ of index (l, l) in the case $s > 0$, and index $(n, 0)$ if $s = 0$. Define the map $J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle\, z, [x, y]\,\rangle_{r,s} = \langle\, J_z x, y\,\rangle, \quad z \in \mathfrak{z}, \quad x, y \in \mathfrak{v}\,. \tag{2.1}$$

Then the map J_z is skew-symmetric with respect to the bilinear form $\langle\, . \,, . \,\rangle$ in the following sense

$$\langle\, J_z x, y\,\rangle + \langle\, x, J_z y\,\rangle = 0. \tag{2.2}$$

Definition 2.1. *A pair* $(\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}, \langle\, . \,, . \,\rangle_{r,s} + \langle\, . \,, . \,\rangle)$ *is called a pseudo H-type Lie algebra if*

$$J_z^2 + \langle\, z\,, z\,\rangle\, \mathrm{Id}_{\mathfrak{v}} = 0 \quad \textit{for all} \quad z \in \mathfrak{z},$$

where $\mathrm{Id}_{\mathfrak{v}} \in \mathrm{End}(\mathfrak{v})$ *is the identity on* $\mathfrak{v}$.

The pseudo H-type Lie algebras are in one to one correspondence with the Clifford $\mathrm{Cl}(\mathfrak{z}, \langle\, . \,, . \,\rangle_{r,s})$-module structures on $\mathfrak{v}$ admitting a linear form satisfying (2.2), see [1, 2, 7]. We call these modules *admissible*. We use the isometries of scalar product vector spaces $(\mathfrak{z}, \langle\, . \,, . \,\rangle_{r,s}) \cong \mathbb{R}^{r,s}$,

$$(\mathfrak{v}, \langle\, . \,, . \,\rangle) \cong \mathbb{R}^{l,l}, \quad \text{if } s > 0,$$
$$(\mathfrak{v}, \langle\, . \,, . \,\rangle) \cong \mathbb{R}^{n,0}, \quad \text{if } s = 0,$$

and the isomorphism of Clifford algebras $\mathrm{Cl}(\mathfrak{z}, \langle\, . \,, . \,\rangle_{r,s}) \cong \mathrm{Cl}_{r,s} = \mathrm{Cl}(\mathbb{R}^{r,s})$. We denote by $\mathfrak{n}_{r,s}$ the pseudo H-type Lie algebras related to the admissible $\mathrm{Cl}_{r,s}$-module of minimal dimension, called *minimal admissible*. If the module is not minimal admissible then we will write $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ indicating the necessary properties of $\mathfrak{v}^{r,s}$. For the rest of this paper, we always denote by $z_1, \ldots, z_r, z_{r+1}, \ldots, z_{r+s}$ an orthonormal basis for $\mathbb{R}^{r,s}$.

The main theorem is the following

Theorem 2.1. *Let $\mathcal{N}$ be a collection of pseudo H-type Lie algebras $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ with equal value $n = r + s$ and equal dimension of the admissible modules $\mathfrak{v}^{r,s}$. Then the complexified Lie algebras from $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s}) \in \mathcal{N}$ are isomorphic.*

3 Minimal admissible modules in the basic cases

Let V be a real finite dimensional vector space, its complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is the complex vector space obtained from V by considering the scalars as complex instead of real numbers. By means of this process, linear maps and bilinear forms defined on V can be defined on $V^{\mathbb{C}}$ in a natural way. For the sake of clarity, we will sometimes stress the complexified objects with an upperscript $^{\mathbb{C}}$.

Consider the complexification $(\mathfrak{n}^{\mathbb{C}}_{r,s}(v^{r,s}), \langle\, . \,, . \,\rangle^{\mathbb{C}}_{r,s} + \langle\, . \,, . \,\rangle^{\mathbb{C}})$ of a real pseudo H-type Lie algebra $(\mathfrak{n}_{r,s}(v^{r,s}), \langle\, . \,, . \,\rangle_{r,s} + \langle\, . \,, . \,\rangle)$. The maps $J^{\mathbb{C}}_{\zeta}$, $\zeta \in \mathfrak{z}^{\mathbb{C}}$, are defined in a similar way as in equation (2.1), since for any $v, w \in \mathfrak{v}$, $z \in \mathfrak{z}$, and $\alpha, \beta, \gamma \in \mathbb{C}$ we have

$$\alpha\beta\gamma\,\langle\, J_z v, w\,\rangle = \langle\, J^{\mathbb{C}}_{z\otimes\alpha}(v \otimes \beta), w \otimes \gamma\,\rangle^{\mathbb{C}} = \langle\, z \otimes \alpha, [v \otimes \beta, w \otimes \gamma]^{\mathbb{C}}\,\rangle^{\mathbb{C}} = \alpha\beta\gamma\,\langle\, z, [v, w]\,\rangle\,.$$

The idea of the proof is to show that for some specific chosen bases of the real Lie algebras in $\mathcal{N}$, the complexified Lie algebras will have the same structure constants. Notice the following

$$\big(J_\zeta^{\mathbb{C}}\big)^2=\Big(\big(J_\alpha^2-J_\beta^2+i\big(J_\alpha J_\beta+J_\beta J_\alpha\big)\Big),\quad \zeta=\alpha+i\beta. \tag{3.1}$$

For $\zeta=\alpha_1+i\beta_1$, $\eta=\alpha_2+i\beta_2$

$$J_\eta^{\mathbb{C}}J_\zeta^{\mathbb{C}}=\Big(\big(J_{\alpha_2}J_{\alpha_1}-J_{\beta_2}J_{\beta_1}+i\big(J_{\alpha_2}J_{\beta_1}+J_{\beta_2}J_{\alpha_1}\big)\Big). \tag{3.2}$$

Now, let $z_1,\dots,z_r,z_{r+1},\dots,z_{r+s}$ be an orthonormal basis for $(\mathbb{R}^{r,s},\langle\,.\,,\,.\,\rangle_{r,s})$. We define

$$\zeta_k=\begin{cases} z_k, & \text{if}\quad \langle z_k,z_k\rangle_{r,s}=1,\\ iz_k, & \text{if}\quad \langle z_k,z_k\rangle_{r,s}=-1.\end{cases} \tag{3.3}$$

Then the set $\{\zeta_k\}$, $k=1,\dots,r+s$, is an orthonormal basis for $\mathbb{C}^{r+s}$ satisfying $\langle\zeta_k,\zeta_k\rangle_{r,s}^{\mathbb{C}}=1$. From now on we write $\mathfrak{v}_{min}^{\mathbb{C}}$ for the complexified minimal admissible module, $(.,.)$ for complexified symmetric bilinear forms, and J_{ζ_k} for the complexification of J_{z_k}. Moreover from (3.1) and (3.2) we obtain

$$\big(J_{\zeta_k}\big)^2=-\operatorname{Id}_{\mathfrak{v}^{\mathbb{C}}},\quad J_{\zeta_2}J_{\zeta_1}=-J_{\zeta_1}J_{\zeta_2},\quad (J_{\zeta_k}v_1,J_{\zeta_k}v_2)=(v_1,v_2),$$

for any $v_1,v_2\in\mathfrak{v}_{min}^{\mathbb{C}}$.

Recall that the orthonormal basis for the minimal admissible modules are cyclic and generated starting from an initial vector $v\in\mathfrak{v}_{min}$ with $\langle v,v\rangle=1$ by Clifford action, see [4, 5, 6]:

$$v,\quad J_{z_k}v,\quad J_{z_k}J_{z_l}v,\quad J_{z_k}J_{z_l}J_{z_m}v,\dots.$$

Definition 3.1. *We say that the bases of two minimal admissible* $\mathrm{Cl}_{r,s}$*-modules* $\mathfrak{v}_1$ *and* $\mathfrak{v}_2$ *have the same lexicographical order if they are obtained by the action of the same ordered collection* $\mathcal{J}=\{J_{z_k},\ \ J_{z_k}J_{z_l},\ \ J_{z_k}J_{z_l}J_{z_m},\dots\}$ *on vectors* $v_1\in\mathfrak{v}_1$ *and* $v_2\in\mathfrak{v}_2$ *with* $(v_1,v_1)=(v_2,v_2)=1$.

The construction of a basis for a minimal admissible module is based on the structure of a collection $\mathcal{I}_{r,s}$ of mutually commuting isometric symmetric involutions, acting on the admissible module $\mathfrak{v}_{min}$ for which the initial vector $v\in\mathfrak{v}_{min}$ is the common eigenvector with eigenvalue 1 of the maps from $\mathcal{I}_{r,s}$, see [4]. The involutions are of two types: $F_a=J_{z_k}J_{z_l}J_{z_m}J_{z_l}$ and $T_b=J_{z_k}J_{z_l}J_{z_m}$, $a,b\in\mathbb{N}$. It is easy to see that the maps $F_a^{\mathbb{C}}=J_{\zeta_k}J_{\zeta_l}J_{\zeta_m}J_{\zeta_l}$ and $T_b^{\mathbb{C}}=J_{\zeta_k}J_{\zeta_l}J_{\zeta_m}$ are also mutually commuting isometric involutions, acting on $\mathfrak{v}_{min}^{\mathbb{C}}$. In some special cases it is possible to consider involutions of type $F_b^{\mathbb{C}}$ instead of those of type $T_b^{\mathbb{C}}$ multiplying $T_b^{\mathbb{C}}$ by J_{ζ_k} for some suitable ζ_k. We denote by $\mathcal{I}_{r,s}^{\mathbb{C}}=\{F_a^{\mathbb{C}},T_b^{\mathbb{C}},\ a,b\in\mathbb{N}\}$ the collection of mutually commuting isometric involutions, acting on $\mathfrak{v}_{min}^{\mathbb{C}}$.

Theorem 3.1. *Consider pseudo H-type Lie algebras* $\mathfrak{n}_{r,s}$ *having equal values* $n=r+s$ *and equal dimensions of the minimal admissible modules* $\mathfrak{v}_{min}^{r,s}$. *If the minimal admissible modules* $\mathfrak{v}_{min}^{r,s}$ *have bases with equal lexicographical order, and admit the same collection* $\mathcal{I}_{r,s}^{\mathbb{C}}$ *of mutually commuting isometric symmetric involutions, acting on* $(\mathfrak{v}_{min}^{r,s})^{\mathbb{C}}$*, then the complexified Lie algebras* $\mathfrak{n}_{r,s}^{\mathbb{C}}$ *are isomorphic.*

Proof. The structure constants on the complexified Lie algebras are calculated by making use of equality

$$(J_{\zeta_k} v_l, v_m) = (\zeta_k, [v_l, v_m]).$$

The calculation only depends on the number of permutations leading to relations of the type $(\zeta_k, [v_l, v_m]) = (\pm P_a v, v) = \pm 1$ for $P_a \in \mathcal{I}^{\mathbb{C}}_{r,s}$ and the generating vector $v \in (\mathfrak{v}^{r,s}_{min})^{\mathbb{C}}$. Since the lexicographical order of the bases and the involutions $\mathcal{I}^{\mathbb{C}}_{r,s}$ coincide, we obtain equal structure constants. □

Corollary 3.2. *If the pseudo H-type Lie algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{n,0}$ with $n = r + s$ have equal dimension of the minimal admissible modules, then their complexifications are isomorphic Lie algebras.*

Proof. Any collection $\mathcal{I}_{r,s}$ of mutually commuting isometric involutions for $\mathfrak{n}_{r,s}$ is also the set of such involutions for $\mathfrak{n}_{n,0}$, see [4]. Thus the lexicographical order of the bases and the collection $\mathcal{I}^{\mathbb{C}}_{r,s}$ are the same. The conclusion follows from Theorem 3.1. □

To facilitate the description of the case-by-case analysis that will follow, we present the table of the dimensions of the minimal admissible modules.

Table 1: Dimensions of minimal admissible modules

8	16	32	64	$64_{\times 2}$	128	128	128	$128_{\times 2}$	256
7	16	32	64	64	128	128	128	128	256
6	16	$16_{\times 2}$	32	32	64	$64_{\times 2}$	128	128	256
5	16	16	16	16	32	64	128	128	256
4	8	8	8	$8_{\times 2}$	16	32	64	$64_{\times 2}$	128
3	8	8	8	8	16	32	64	64	128
2	4	$4_{\times 2}$	8	8	16	$16_{\times 2}$	32	32	64
1	2	4	8	8	16	16	16	16	32
0	1	2	4	$4_{\times 2}$	8	8	8	$8_{\times 2}$	16
s/r	0	1	2	3	4	5	6	7	8

We focus in the present section on low dimensional pseudo H-type algebras, so-called *basic cases*, which correspond to $1 \leq r + s \leq 8$ and

$$(2, 7), (3, 6), (3, 7), (6, 3), (7, 2), (7, 3),$$

where periodicity arguments do not hold. We show that low dimensional pseudo H-type algebras satisfy the conditions of Theorem 3.1. We will start by studying the basic cases for which $r + s > 8$, since the arguments use information that does not appear in Table 1. In fact, we have

$$\mathfrak{n}^{\mathbb{C}}_{3,7} \cong \mathfrak{n}^{\mathbb{C}}_{7,3},$$

since the dimension of the minimal admissible $\mathrm{Cl}_{10,0}$-module is 64. Recall that the Atiyah–Bott periodicity of Clifford algebras, implies

$$\dim(\mathfrak{v}^{r,s}) = 16 \dim(\mathfrak{v}^{r-8,s}) = 16 \dim(\mathfrak{v}^{r,s-8}) = 16 \dim(\mathfrak{v}^{r-4,s-4}).$$

Therefore Corollary 3.2 holds. Similarly, we have that

$$\mathfrak{n}_{3,6}^{\mathbb{C}} \cong \mathfrak{n}_{7,2}^{\mathbb{C}},$$

since the dimension of the minimal admissible $\mathrm{Cl}_{9,0}$-module is 32. The fact that

$$\mathfrak{n}_{2,7}^{\mathbb{C}} \cong \mathfrak{n}_{6,3}^{\mathbb{C}},$$

follows by direct inspection of the set of involutions $\mathcal{I}_{r,s}^{\mathbb{C}}$, that coincide, see [4].

In the following, we omit the cases of the pseudo H-type Lie algebras $\mathfrak{n}_{r,s}$ whose dimensions of the centres and of the minimal admissible modules coincide with $\mathfrak{n}_{n,0}$, since the conclusion follows from Corollary 3.2.

CASE $r+s=3$. In the cases of 8 dimensional minimal admissible modules the orthonormal bases are the following:

$$\{v,\ J_{z_k}v,\ J_{z_k}J_{z_l}v,\ k,l=1,2,3,\ J_{z_1}J_{z_2}J_{z_3}v\}.$$

The collection $\mathcal{I}_{r,s}^{\mathbb{C}}$ is empty.

CASE $r+s=5$. The 16 dimensional minimal admissible modules have the following orthonormal basis:

$$\{v,\ J_{z_k}v,\ k=1,\dots,5\ \ J_{z_l}J_{z_5}v,\ l=1,\dots,4,\ \ J_{z_1}J_{z_m}v,\ J_{z_1}J_{z_m}J_{z_5}v\ m=2,3,4\}.$$

The involution is $F=J_{z_1}J_{z_2}J_{z_3}J_{z_4}$.

CASE $r+s=6$. The pseudo H-type Lie algebras $\mathfrak{n}_{5,1}$ and $\mathfrak{n}_{1,5}$ are isomorphic [5]. The pseudo H-type Lie algebras $\mathfrak{n}_{5,1}$, $\mathfrak{n}_{4,2}$, and $\mathfrak{n}_{0,6}$ have equal collection

$$\mathcal{I}_{r,s}^{\mathbb{C}}=\{F_1^{\mathbb{C}}=J_{\zeta_1}J_{\zeta_2}J_{\zeta_3}J_{\zeta_4},\ \ F_2^{\mathbb{C}}=J_{\zeta_1}J_{\zeta_2}J_{\zeta_5}J_{\zeta_6}\}.$$

Note that the involution $F_2^{\mathbb{C}}$ for the Lie algebra $\mathfrak{n}_{5,1}$ was obtained by modifying the involution $T=J_{z_1}J_{z_2}J_{z_5}$. The multiplication of $T^{\mathbb{C}}=J_{\zeta_1}J_{\zeta_2}J_{\zeta_5}$ by J_{ζ_6} does not change the structural constant of the basis. The orthonormal basis is the following

$$\{v,\ J_{z_k}v,\ k=1,\dots,6\ \ J_{z_l}J_{z_6}v,\ l=1,\dots,5,\ \ J_{z_2}J_{z_m}v,\ J_{z_2}J_{z_m}J_{z_6}v\ m=3,4\}.$$

CASE $r+s=7$. The pairs of real pseudo H-type Lie algebras $\mathfrak{n}_{6,1}$, $\mathfrak{n}_{1,6}$, and $\mathfrak{n}_{5,2}$, $\mathfrak{n}_{2,5}$ are isomorphic [5]. The pseudo H-type Lie algebras $\mathfrak{n}_{4,3}$, and $\mathfrak{n}_{0,7}$ have the set $\mathcal{I}_{r,s}$:

$$F_1=J_{z_1}J_{z_2}J_{z_3}J_{z_4},\ F_2=J_{z_1}J_{z_2}J_{z_5}J_{z_6},\ F_3=J_{z_1}J_{z_3}J_{z_5}J_{z_7},$$

and therefore have the basis with the same lexicographical order. Analogously, the pseudo H-type Lie algebras $\mathfrak{n}_{2,5}$, and $\mathfrak{n}_{6,1}$ have common mutually commuting isometric involutions:

$$F_1=J_{z_1}J_{z_2}J_{z_3}J_{z_4},\ F_2=J_{z_1}J_{z_2}J_{z_5}J_{z_6},\ T=J_{z_1}J_{z_3}J_{z_5},$$

that leads to the basis with the equal lexicographical order. Moreover, the collection

$$\mathcal{I}_{r,s}^{\mathbb{C}}=\{F_1=J_{\zeta_1}J_{\zeta_2}J_{\zeta_3}J_{\zeta_4},\ F_2=J_{\zeta_1}J_{\zeta_2}J_{\zeta_5}J_{\zeta_6},\ F_3=J_{\zeta_1}J_{\zeta_3}J_{\zeta_5}J_{\zeta_7},\}$$

is common for all four last mentioned algebras: $\mathfrak{n}_{2,5}$, $\mathfrak{n}_{6,1}$, $\mathfrak{n}_{4,3}$, and $\mathfrak{n}_{0,7}$. The two previous groups $\mathfrak{n}_{6,1},\mathfrak{n}_{1,6},\mathfrak{n}_{5,2},\mathfrak{n}_{2,5}$ and $\mathfrak{n}_{0,7},\mathfrak{n}_{4,3}$ have the isomorphic complexifications since the pseudo H-type Lie algebras $\mathfrak{n}_{6,1}$, and $\mathfrak{n}_{0,7}$ have bases

$$\{v,\ J_{z_k}v,\ k=1,\dots,7\ \ J_{z_m}J_{z_7}v,\ m=1,\dots,6,\ \ J_{z_1}J_{z_2}v,\ J_{z_1}J_{z_2}J_{z_7}v\}$$

with the same lexicographical order.

CASE $r+s=8$. The pseudo H-type Lie algebras $\mathfrak{n}_{6,2}$ and $\mathfrak{n}_{2,6}$ are isomorphic [5]. The pseudo H-type Lie algebras $\mathfrak{n}_{6,2}$, $\mathfrak{n}_{5,3}$ and $\mathfrak{n}_{1,7}$ have the same set of mutually commuting isometric involutions:

$$\mathcal{I}_{r,s}=\{F_1=J_{z_2}J_{z_3}J_{z_4}z_5,\quad F_2=J_{z_2}J_{z_3}J_{z_7}J_{z_8},\quad T=J_{z_1}J_{z_2}J_{z_3}\}.$$

Therefore, we obtain orthonormal bases with equal lexicographical order and the same collection $\mathcal{I}^{\mathbb{C}}_{r,s}$.

4 Non minimal admissible modules in the basic cases

Before we proceed to study the isomorphism between the pseudo H-type Lie algebras $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ and $\mathfrak{n}_{p,q}(\mathfrak{v}^{p,q})$ with different indices $(r,s)\neq(p,q)$, $r+s=p+q$, we need to revise whether the real Lie algebras $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ and $\mathfrak{n}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$ are isomorphic for admissible modules $\mathfrak{v}^{r,s}$ and $\tilde{\mathfrak{v}}^{r,s}$ having different decompositions into minimal admissible modules.

4.1 Automorphisms of $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s})$ and $\mathfrak{n}^{\mathbb{C}}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$

In the rest of the paper we use the upper index $^{\pm}$ to indicate the scalar products on admissible modules that differ by sign: $\mathfrak{v}^{r,s;+}_{min}=(\mathfrak{v}^{r,s}_{min},\langle\cdot\,,\cdot\rangle_{\mathfrak{v}^{r,s}_{min}})$ and $\mathfrak{v}^{r,s;-}_{min}=(\mathfrak{v}^{r,s}_{min},-\langle\cdot\,,\cdot\rangle_{\mathfrak{v}^{r,s}_{min}})$. We also use the lower index $_{\pm}$ to distinguish the minimal admissible modules, corresponding to non equivalent irreducible modules, $\mathfrak{v}^{r,s;+}_{min;\pm}=(\mathfrak{v}^{r,s}_{min;\pm},\langle\cdot\,,\cdot\rangle_{\mathfrak{v}^{r,s}_{min;\pm}})$ and $\mathfrak{v}^{r,s;-}_{min;\pm}=(\mathfrak{v}^{r,s}_{min;\pm},-\langle\cdot\,,\cdot\rangle_{\mathfrak{v}^{r,s}_{min;\pm}})$, see [6, Corollary 2.2.4].

Recall that Clifford modules are completely reducible and any admissible module can be decomposed into the orthogonal sum of minimal admissible modules. We decompose an admissible module $\mathfrak{v}^{r,s}$ of the Clifford algebra $\mathrm{Cl}_{r,s}$ into the direct sum of, possibly different, minimal admissible modules. We distinguish the following possibilities.

If $r-s\not\equiv 3\,(\mathrm{mod}\ 4)$ and s is arbitrary or $r-s\equiv 3\,(\mathrm{mod}\ 4)$ and s is odd then

$$\mathfrak{v}^{r,s}=\Big(\overset{p^+}{\oplus}\,\mathfrak{v}^{r,s;+}_{min}\Big)\bigoplus\Big(\overset{p^-}{\oplus}\,\mathfrak{v}^{r,s;-}_{min}\Big).\tag{4.1}$$

If $r-s\equiv 3\,(\mathrm{mod}\ 4)$ and s is even, then

$$\mathfrak{v}^{r,s}=\Big(\overset{p^+_+}{\oplus}\,\mathfrak{v}^{r,s;+}_{min;+}\Big)\bigoplus\Big(\overset{p^-_+}{\oplus}\,\mathfrak{v}^{r,s;-}_{min;+}\Big)\bigoplus\Big(\overset{p^+_-}{\oplus}\,\mathfrak{v}^{r,s;+}_{min;-}\Big)\bigoplus\Big(\overset{p^-_-}{\oplus}\,\mathfrak{v}^{r,s;-}_{min;-}\Big).\tag{4.2}$$

We formulate the main results of the classification, see [6].

Theorem 4.1. *[6, Theorem 4.1.1] Let $\mathfrak{v}^{r,s}=(\mathfrak{v},\langle\cdot\,,\cdot\rangle_{\mathfrak{v}})$ and $\tilde{\mathfrak{v}}^{r,s}=(\tilde{\mathfrak{v}},\langle\cdot\,,\cdot\rangle_{\tilde{\mathfrak{v}}})$ be admissible modules of a Clifford algebra $\mathrm{Cl}_{r,s}$. If $r\equiv 0,1,2 \bmod 4$, then the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ is determined by the dimension of the admissible module $\mathfrak{v}^{r,s}$ and does not depend on the choice of an admissible scalar product. Thus $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})\cong\mathfrak{n}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$, if and only if $\dim(\mathfrak{v}^{r,s})=\dim(\tilde{\mathfrak{v}}^{r,s})$.*

If $r \equiv 3 \,(\mathrm{mod}\ 4)$, then the pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ is determined by the dimension of $\mathfrak{v}^{r,s}$ and by the value of the index s.

Theorem 4.2. *[6, Theorem 4.1.2] Let $r \equiv 3 \,(\mathrm{mod}\ 4)$ and $s \equiv 0 \,(\mathrm{mod}\ 4)$ and let the admissible modules be decomposed into the direct sums:*

$$\mathfrak{v}^{r,s} = (\overset{p_+^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min;+}) \bigoplus (\overset{p_+^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min;+}) \bigoplus (\overset{p_-^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min;-}) \bigoplus (\overset{p_-^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min;-}),$$
$$\tilde{\mathfrak{v}}^{r,s} = (\overset{\tilde{p}_+^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min;+}) \bigoplus (\overset{\tilde{p}_+^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min;+}) \bigoplus (\overset{\tilde{p}_-^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min;-}) \bigoplus (\overset{\tilde{p}_-^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min;-}).$$

Then the Lie algebras $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s})$ and $\mathfrak{n}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$ are isomorphic, if and only if,

$$p = p_+^+ + p_-^- = \tilde{p}_+^+ + \tilde{p}_-^- = \tilde{p} \quad \textit{and} \quad q = p_+^- + p_-^+ = \tilde{p}_+^- + \tilde{p}_-^+ = \tilde{q}$$

or

$$p = p_+^+ + p_-^- = \tilde{p}_+^- + \tilde{p}_-^+ = \tilde{q} \quad \textit{and} \quad q = p_+^- + p_-^+ = \tilde{p}_+^+ + \tilde{p}_-^- = \tilde{p}.$$

Theorem 4.3. *[6, Theorem 4.1.3] Let $r \equiv 3 \,(\mathrm{mod}\ 4)$ and $s \equiv 1, 2, 3 \,(\mathrm{mod}\ 4)$ and let $\mathfrak{v}^{r,s}$ and $\tilde{\mathfrak{v}}^{r,s}$ be decomposed into the direct sums*

$$\mathfrak{v}^{r,s} = (\overset{p^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min}) \bigoplus (\overset{p^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min}), \qquad \tilde{\mathfrak{v}}^{r,s} = (\overset{\tilde{p}^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min}) \bigoplus (\overset{\tilde{p}^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min}).$$

Then $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s}) \cong \mathfrak{n}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$, if and only if $p = p^+ = \tilde{p}^+ = \tilde{p}$ and $q = p^- = \tilde{p}^- = \tilde{q}$, or $p = p^+ = \tilde{p}^- = \tilde{q}$ and $q = p^- = \tilde{p}^+ = \tilde{p}$.

Let us show the following statement.

Theorem 4.4. *Let $\mathfrak{v}^{r,s}$ and $\tilde{\mathfrak{v}}^{r,s}$ be admissible modules decomposed into different direct sums of type* (4.1) *or* (4.2)*. Then $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s}) \cong \mathfrak{n}^{\mathbb{C}}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$, if and only if $\dim(\mathfrak{v}^{r,s}) = \dim(\tilde{\mathfrak{v}}^{r,s})$.*

Proof. If $r \equiv 0, 1, 2 \bmod 4$, then $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s}) \cong \mathfrak{n}^{\mathbb{C}}_{r,s}(\tilde{\mathfrak{v}}^{r,s})$, since the real pseudo H-type Lie algebras are isomorphic.

Let $r \equiv 3 \,(mod\ 4)$ and $s \equiv 1, 2, 3 \,(mod\ 4)$. Consider first the real pseudo H-type Lie algebras

$$n_{r,s}(\mathfrak{v}^{r,s;+}_{min}) \quad \text{and} \quad n_{r,s}(\mathfrak{v}^{r,s;-}_{min})$$

and construct the isomorphism map for their complexifications that will act as identity on their centres. We choose $v \in \mathfrak{v}^{r,s;+}_{min}$ with $\langle\, v, v \,\rangle_{\mathfrak{v}^{r,s;+}_{min}} = 1$ and $\tilde{v} \in \mathfrak{v}^{r,s;-}_{min}$ with $\langle\, \tilde{v}, \tilde{v} \,\rangle_{\mathfrak{v}^{r,s;-}_{min}} = -1$. Then the structure constants will differ by sign. For the complexified algebras we choose the complex basis on the centres as in (3.3) and construct the basis for the complexified $\mathfrak{v}^{r,s;+}_{min}$ starting from v and the basis for complexified $\mathfrak{v}^{r,s;-}_{min}$ starting from $i\tilde{v}$, by making use the same collection $\mathcal{I}_{r,s}$. Since $(i\tilde{v}, i\tilde{v}) = \langle\, \tilde{v}, \tilde{v} \,\rangle^{\mathbb{C}}_{\mathfrak{v}^{r,s;-}_{min}} = 1$ the structure constants on the complexified algebras will coincide.

This shows that for any decomposition

$$\mathfrak{v}^{r,s} = (\overset{p^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min}) \bigoplus (\overset{p^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min}), \qquad \tilde{\mathfrak{v}}^{r,s} = (\overset{\tilde{p}^+}{\oplus}\ \mathfrak{v}^{r,s;+}_{min}) \bigoplus (\overset{\tilde{p}^-}{\oplus}\ \mathfrak{v}^{r,s;-}_{min}).$$

we can construct the basis generated from

$$v = \oplus_{l=1}^{p^+} v_l \bigoplus \oplus_{j=1}^{p^-} i u_j, \qquad \langle\, v_l, v_l \,\rangle_{\mathfrak{v}^{r,s;+}_{min}} = 1, \quad \langle\, u_j, u_j \,\rangle_{\mathfrak{v}^{r,s;-}_{min}} = -1$$

and

$$\tilde{v} = \oplus_{l=1}^{\tilde{p}^+}\tilde{v}_l \bigoplus \oplus_{j=1}^{\tilde{p}^-} i\tilde{u}_j, \qquad \langle \tilde{v}_l, \tilde{v}_l \rangle_{\tilde{\mathfrak{v}}^{r,s;+}_{min}} = 1, \quad \langle \tilde{u}_j, \tilde{u}_j \rangle_{\tilde{\mathfrak{v}}^{r,s;-}_{min}} = -1$$

Since the structure constants in any block will coincide and the blocks are commute, we finish the proof.

Let now $r \equiv 3\,(mod\,4)$ and $s \equiv 0\,(mod\,4)$. We start from considering the real pseudo H-type Lie algebras

$$n_{r,s}(\mathfrak{v}^{r,s;+}_{min;+}) \quad \text{and} \quad n_{r,s}(\mathfrak{v}^{r,s;+}_{min;-})$$

and construct the isomorphism map for their complexifications. We restrict the consideration to the cases $(r,s) \in \{(3,0),(3,4),(7,0)\}$ because of the periodicity. We choose $v \in \mathfrak{v}^{r,s;+}_{min;+}$ with $\langle v, v \rangle_{\mathfrak{v}^{r,s;+}_{min;+}} = 1$ and $u \in \mathfrak{v}^{r,s;+}_{min;-}$ with $\langle u, u \rangle_{\mathfrak{v}^{r,s;+}_{min;-}} = 1$. Let $(r,s) = (3,0)$. We consider the respective orthonormal bases

$$\begin{array}{llll} x_0 = v, & x_k = J_{z_k} v & \text{for} & n_{r,s}(\mathfrak{v}^{r,s;+}_{min;+}), \\ y_0 = u, & y_k = -J_{z_k} u & \text{for} & n_{r,s}(\mathfrak{v}^{r,s;+}_{min;-}). \end{array} \tag{4.3}$$

For the complexified Lie algebras we choose the bases for the centres as in (3.3) and for the admissible modules generated as in (4.3) with initial vectors v for $\left(\mathfrak{v}^{r,s;+}_{min;+}\right)^{\mathbb{C}}$ and iu for $\left(\mathfrak{v}^{r,s;+}_{min;-}\right)^{\mathbb{C}}$. The direct calculations shows that structure constants coincide for

$$n^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s;+}_{min;+}) \quad \text{and} \quad n^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s;+}_{min;-}).$$

Note that we used

$$\Omega_{r,s} x = \prod_{k=1}^{r+s} J_{z_i} x = x, \quad \text{for any} \quad x \in vv^{r,s;+}_{min;+}$$

and

$$\Omega_{r,s} y = \prod_{k=1}^{r+s} J_{z_i} y = -y, \quad \text{for any} \quad y \in vv^{r,s;-}_{min;+}$$

in calculation of the structure constants. Since $n^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s;+}_{min;\pm}) \cong \mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s;-}_{min;\pm})$ by the arguments at the beginning of the proof, we conclude that for any decomposition of modules as in Theorem 4.2 the respective Lie algebras will be isomorphic.

It finishes the proof of the theorem. □

4.2 Isomorphisms of $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s})$ and $\mathfrak{n}^{\mathbb{C}}_{p,q}(\tilde{\mathfrak{v}}^{p,q})$

Before we turn to the general case we also mention that for some situations the real Lie algebras $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s}_{min})$ and $\mathfrak{n}_{s,r}(\tilde{\mathfrak{v}}^{s,r}_{min})$ are not isomorphic by simple reason that the module has different dimensions. In this case one module always has twice the dimension of the other. Careful study of this cases was done in [5] with the conclusion that $\mathfrak{n}_{r,s}(\mathfrak{v}^{r,s}_{min}) \cong \mathfrak{n}_{s,r}(\tilde{\mathfrak{v}}^{s,r})$ if $\dim(\mathfrak{v}^{r,s}_{min}) = 2\dim(\mathfrak{v}^{s,r}_{min})$.

Summarising the considerations above we conclude that any admissible module $\mathfrak{v}^{r,s}$ can be decomposed in into direct sum of minimal admissible and the complexification $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s})$ does not depend niether of the type of irreducible module nor on the choice of metric on it.

Now if $r+s=p+q$ and $\dim(\mathfrak{v}^{r,s}_{min})=\dim(\mathfrak{v}^{p,q}_{min})$, then $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s}_{min})$ is isomorphic to $\mathfrak{n}^{\mathbb{C}}_{p,q}(\mathfrak{v}^{p,q}_{min})$ for any choice of scalar product and type of irreducible module. It shows that $\mathfrak{n}^{\mathbb{C}}_{r,s}(\mathfrak{v}^{r,s})$ is isomorphic to $\mathfrak{n}^{\mathbb{C}}_{p,q}(\mathfrak{v}^{p,q})$ if $\dim(\mathfrak{v}^{r,s})=\dim(\mathfrak{v}^{p,q})$ for any type of decomposition into minimal admissible modules. The isomorphism in both cases will have equal action on the centres.

This observation finishes the proof of Theorem 2.1.

5 Periodicity argument for the non basic cases

It was shown in [4, 5] that the module $\mathfrak{v}^{r+8,s}$ is the tensor product of the module $\mathfrak{u}^{r,s}$ and the module $\mathfrak{w}^{8,0}_{min}$. The representations are constructed as follows. Let $\{\bar{z}_1,\ldots,\bar{z}_8\}$ be an orthonormal bases for $\mathbb{R}^{8,0}$ and $\bar{J}_{\bar{z}_\alpha}$, $\alpha=1,\ldots,8$ be the respective representations. Let J_{z_j}, $j=1,\ldots,r+s$ be representations of an orthonormal basis for $\mathbb{R}^{r,s}$. Set

$$\begin{aligned}\hat{J}_{z_j} &= J_{z_j}\otimes\prod_{\alpha=1}^{8}\bar{J}_{\bar{z}_\alpha} \quad \text{for} \quad j=1,\ldots,r+s,\\ \hat{J}_{\bar{z}_\alpha} &= \mathrm{Id}_{\mathfrak{u}^{r,s}}\otimes\bar{J}_{\bar{z}_\alpha} \quad \text{for} \quad \alpha=1,\ldots,8.\end{aligned}$$

Then the maps $\hat{J}_{z_j}$, $\hat{J}_{\bar{z}_\alpha}$ are representations of an orthonormal basis for $\mathbb{R}^{r+8,s}$ as it was shown in [4]. Therefore if the minimal admissible modules $\mathfrak{u}^{r,s}_{min}$ and $\mathfrak{u}^{p,q}_{min}$, $r+s=p+q$, respectively, have bases with equal lexicographical order, then the bases of the modules

$$\mathfrak{v}^{r,s}=\mathfrak{u}^{r,s}\otimes\mathfrak{w}^{8,0}_{min}, \quad \text{and} \quad \mathfrak{v}^{p,q}=\mathfrak{u}^{p,q}\otimes\mathfrak{w}^{8,0}_{min}$$

will also have equal lexicographical order. Moreover, the collections $\mathcal{I}^{\mathbb{C}}_{r+8,s}$ with the fixed value $n=r+s+8$ will coincide, since we only add new involutions

$$\hat{F}_1=\hat{J}_{\bar{z}_1}\hat{J}_{\bar{z}_2}\hat{J}_{\bar{z}_3}\hat{J}_{\bar{z}_4},\ \hat{F}_2=\hat{J}_{\bar{z}_1}\hat{J}_{\bar{z}_2}\hat{J}_{\bar{z}_5}\hat{J}_{\bar{z}_6},\ \hat{F}_3=\hat{J}_{\bar{z}_1}\hat{J}_{\bar{z}_2}\hat{J}_{\bar{z}_7}\hat{J}_{\bar{z}_8},\ \hat{F}_4=\hat{J}_{\bar{z}_1}\hat{J}_{\bar{z}_3}\hat{J}_{\bar{z}_5}\hat{J}_{\bar{z}_7}.$$

Note that if $F_a=J_{z_i}J_{z_j}J_{z_k}J_{z_l}\in\mathcal{I}_{r,s}$, then $\hat{F}_a=\hat{J}_{z_i}\hat{J}_{z_j}\hat{J}_{z_k}\hat{J}_{z_l}\in\mathcal{I}_{r+8,s}$.

Analogous considerations can be made for $\mathfrak{n}_{r,s+8}$ and $\mathfrak{n}_{r+4,s+4}$ starting from $\mathfrak{n}_{r,s}$.

The arguments of [6, Section 4.2] can be extended to the existence of the isomorphism for the complexified Lie algebras based on arbitrary admissible module, since the arguments does not depend on the field of real or complex numbers.

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