Fourier transform for prehomogeneous vector spaces over finite field

Kazuki Ishimoto
Graduate School of Mathematics, Kobe University

Abstract

Let (G, V) be a prehomogeneous vector space over a finite field of odd characteristic. Taniguchi and Thorne [2] developed a method to calculate explicit formulas of the Fourier transforms of any G-invariant functions over V. By means of their method, we calculate the Fourier transform of any G-invariant function for several prehomogeneous vector spaces. .

1 Introduction

Let K be a field and $\overline{K}$ be the algebraic closure. Let V be a finite dimensional representation of a reductive algebraic group G defined over K. When there exists a $G(\overline{K})$-orbit of $V(\overline{K})$ which is Zariski open, we refer to the pair (G, V) as a prehomogeneous vector space. Taniguchi and Thorne [2] developed a general method to compute the Fourier transform and applied it to obtain explicit formulas for the prehomogeneous vector spaces $1\otimes\mathrm{Sym}^2(\mathbb{F}_q^2)$, $\mathrm{Sym}^3(\mathbb{F}_q^2)$, $1\otimes\mathrm{Sym}^2(\mathbb{F}_q^3)$, $2\otimes\mathrm{Sym}^2(\mathbb{F}_q^2)$, $2\otimes\mathrm{Sym}^2(\mathbb{F}_q^3)$, where $\mathbb{F}_q$ is the finite field of order a prime power q. There are many prehomogeneous vector spaces for which the explicit formula of the Fourier transform is not yet calculated. The speaker calculated the explicit formula of the Fourier transform for 9 more prehomogeneous vector spaces $\mathbb{F}_q^2\otimes\mathbb{F}_q^2\otimes\mathbb{F}_q^2$, $\mathbb{F}_q^2\otimes\mathbb{F}_q^2\otimes\mathbb{F}_q^3$, $\mathbb{F}_q^2\otimes\mathbb{F}_q^2\otimes\mathbb{F}_q^4$, $\mathbb{F}_q^2\otimes\mathrm{H}_2(\mathbb{F}_{q^2})$, $\mathbb{F}_q^2\otimes\wedge^2(\mathbb{F}_q^4)$, the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$, $\mathbb{F}_q^2\otimes\mathbb{F}_q^3\otimes\mathbb{F}_q^3$, $\mathbb{F}_q^2\otimes\mathrm{H}_3(\mathbb{F}_{q^2})$, $\mathbb{F}_q^2\otimes\wedge^2(\mathbb{F}_q^6)$ by using the method developed by Taniguchi and Thorne. In this paper, we see the calculation method and the results for some spaces.

2 Fourier transform

Let p is an odd prime and $\mathbb{F}_p$ is the finite field of order p. Let q is any power of p and $\mathbb{F}_q$ is the finite field of order q. Let V be a finite dimensional vector space over $\mathbb{F}_q$ with a finite group G linearly acting on V. Suppose the pair (G, V) satisfies the following Assumption 2.

Assumption 1. *There exist an automorphism $\iota : G \ni g \mapsto g^\iota \in G$ of order 2 and a non-degenerate bilinear form $\beta : V \times V \to \mathbb{F}_q$ such that*

$$\beta(gx, g^\iota y) = \beta(x, y) \quad (x, y \in V, g \in G).$$

Then we can identify the dual space V^* with V by the linear isomorphism $V \ni x \mapsto \beta(x, \cdot) \in V^*$ (see [2] for detail). We reformulate the definition of the Fourier transform only in terms of V. For $\phi : V \to \mathbb{C}$, we define its Fourier transform $\widehat{\phi} : V \to \mathbb{C}$ as follows:

$$\widehat{\phi}(y) := |V|^{-1} \sum_{x\in V} \phi(x)\exp\left(\frac{2\pi i \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\beta(x, y))}{p}\right). \tag{1}$$

Here $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map. Let $\mathcal{F}_V^G$ be the set of all G-invariant maps from V to $\mathbb{C}$, i.e.,

$$\mathcal{F}_V^G := \{\phi : V \to \mathbb{C} \mid \phi(gx) = \phi(x) \quad (g \in G, x \in V)\}.$$

Note that $\mathcal{F}_V^G$ is a finite dimensional vector space over $\mathbb{C}$. We can easily see that if ϕ is a G-invariant function, $\widehat{\phi}$ is also G-invariant. In fact, the Fourier transform map $\mathcal{F}_V^G \ni \phi \mapsto \widehat{\phi} \in \mathcal{F}_V^G$ is a linear isomorphism. Let $\mathcal{O}_i (1 \leq i \leq r)$ be all the distinct G-orbits in V, and for each i let e_i be the indicator function of $\mathcal{O}_i$. The functions $e_1, \dots, e_r$ are clearly G-invariant, and they form a basis of $\mathcal{F}_V^G$. Thus we only have to calculate the Fourier transform of $e_1, \dots, e_r$ to calculate that of all $\phi \in \mathcal{F}_V^G$. We use the following proposition for our calculation of $\widehat{e}_i$.

Proposition 2. *[2, Proposition 6] Let W be a subspace of V, and let $W^{\perp} := \{y \in V \mid \forall x \in W, \beta(x,y) = 0\}$. Then*

$$\sum_{i=1}^{r} \frac{|\mathcal{O}_i \cap W|}{|\mathcal{O}_i|} \cdot \widehat{e_i} = \frac{|W|}{|V|} \sum_{j=1}^{r} \frac{|\mathcal{O}_j \cap W^{\perp}|}{|\mathcal{O}_j|} \cdot e_j.$$

In this paper, we call $W^{\perp}$ the orthogonal complement of W. By Proposition 3, when we choose one subspace of V, we obtain one equation of linear combinations of $\widehat{e_i}$ and e_j. Therefore if we choose r different subspaces and the corresponding equations are linearly independent, we obtain an expression of each $\widehat{e_i}$ in terms of $e_1, ..., e_r$. In other words, we can determine the following r-by-r matrix M explicitly:

$$(\widehat{e_1}, ..., \widehat{e_n}) = (e_1, ..., e_n)M.$$

We calculate the matrix M with this approach.

3 Main result

The speaker calculated the Fourier transforms for the following prehomogeneous vector spaces over a finite filed:

- $V = 2 \otimes 2 \otimes 2$, the space of pairs of 2-by-2matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$,
- $V = 2 \otimes 2 \otimes 3$, the space of triplets of 2-by-2matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_3$,
- $V = 2 \otimes 2 \otimes 4$, the space of quadruples of 2-by-2matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_4$,
- $V = 2 \otimes \mathrm{H}_2(\mathbb{F}_{q^2})$, the space of pairs of Hermitian matrices of order 2; $G = \mathrm{GL}_2 \times \mathrm{GL}_2(\mathbb{F}_{q^2})$,
- $V = 2 \otimes \wedge^2(4)$, the space of pairs of alternating matrices of order 4; $G = \mathrm{GL}_2 \times \mathrm{GL}_4$,
- V is the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$; $G = \mathrm{GL}_1 \times \mathrm{GL}_2(\mathbb{F}_{q^3})$,
- $V = 2 \otimes 3 \otimes 3, G = \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_3$,
- $V = 2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2}), G = \mathrm{GL}_2 \times \mathrm{GL}_3(\mathbb{F}_{q^2})$,
- $V = 2 \otimes \wedge^2(6), G = \mathrm{GL}_2 \times \mathrm{GL}_6$.

Here, we write about the complicated cases $V = 2 \otimes 3 \otimes 3, 2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2}), 2 \otimes \wedge^2(6)$.

3.1 $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

Let $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ and $G = G_1 \times G_2 \times G_3 = \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_3$. We write $x \in V$ as $x = (A, B)$ where A and B are 3-by-3 matrices, and write $g \in G$ as $g = (g_1, g_2, g_3)$ where $g_1 \in \mathrm{GL}_2$ and $g_2, g_3 \in \mathrm{GL}_3$. G acts on V by

$$gx = (g_2 A g_3^T, g_2 B g_3^T) g_1^T.$$

V consists of 21 G-orbits in all. The following elements $x_1, ..., x_{21}$ are representatives:

$$x_1 = (0,0),\ x_2 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, 0),\ x_3 = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, 0),\ x_4 = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, 0),$$

$$x_5 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 0 & & \\ & & 0 \end{bmatrix}),\ x_6 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 0 & \\ 1 & & \\ & & 0 \end{bmatrix}),\ x_7 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & \\ & & 0 \end{bmatrix}),$$

$$x_8 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}),\ x_9 = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & -1 & \\ \mu_0 & \mu_1 & \\ & & 0 \end{bmatrix}),\ x_{10} = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 0 & \\ 0 & & \end{bmatrix}),$$

$$x_{11} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix} & & 1\\ 1 & & \\ & 0 & \end{bmatrix}),\ x_{12} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix} & & 0\\ & 0 & \\ 1 & & \end{bmatrix}),\ x_{13} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix} & 1 & \\ & & 0\\ 1 & & \end{bmatrix}),$$

$$x_{14} = (\begin{bmatrix} & 1 & \\ 1 & & \\ & & 0\end{bmatrix}, \begin{bmatrix} & & 1\\ & 0 & \\ 1 & & \end{bmatrix}),\ x_{15} = (\begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix}0 & & \\ & 0 & \\ & & 1\end{bmatrix}),\ x_{16} = (\begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix}0 & & \\ & & 1\\ & 1 & \end{bmatrix}),$$

$$x_{17} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix}0 & & \\ & 0 & \\ & & 1\end{bmatrix}),\ x_{18} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix}0 & & \\ & & 1\\ & 1 & \end{bmatrix}),\ x_{19} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix}0 & & \\ & 1 & \\ & & 1\end{bmatrix}),$$

$$x_{20} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}, \begin{bmatrix} & -1 & \\ \mu_0 & \mu_1 & \\ & & 1\end{bmatrix}),\ x_{21} = (\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}, \begin{bmatrix} & -1 & \\ & \nu_2 & -1\\ \nu_0 & \nu_1 & \end{bmatrix}).$$

Here, μ_1, μ_0, ν_2, ν_1, $\nu_0 \in \mathbb{F}_q$ are elements such that $X^2 + \mu_1 X + \mu_0, X^3 + \nu_2 X^2 + \nu_1 X + \nu_0 \in \mathbb{F}_q[X]$ is irreducible.

The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}), W_2 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\*&*&*\end{bmatrix}), W_3 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\*&*&*\\*&*&*\end{bmatrix}),$$

$$W_4 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_5 = (\begin{bmatrix}0&0&0\\0&0&0\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\*&*&*\end{bmatrix}), W_6 = (\begin{bmatrix}0&0&*\\0&0&*\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\0&0&*\end{bmatrix}),$$

$$W_7 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}), W_8 = (\begin{bmatrix}0&0&0\\*&*&*\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\*&*&*\end{bmatrix}), W_9 = (\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}),$$

$$W_{10} = (\begin{bmatrix}0&0&0\\0&0&0\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\*&*&*\\*&*&*\end{bmatrix}), W_{11} = (\begin{bmatrix}0&0&0\\*&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\*&*&*\\*&*&*\end{bmatrix}), W_{12} = (\begin{bmatrix}0&0&*\\0&0&*\\0&0&*\end{bmatrix}, \begin{bmatrix}0&*&*\\0&*&*\\0&*&*\end{bmatrix}),$$

$$W_{13} = (\begin{bmatrix}0&*&*\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\0&*&*\\0&*&*\end{bmatrix}), W_{14} = (\begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}), W_{15} = (\begin{bmatrix}0&0&0\\0&0&*\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\*&*&*\\*&*&*\end{bmatrix}),$$

$$W_{16} = (\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}), W_{17} = (\begin{bmatrix}0&0&0\\0&0&0\\*&*&*\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_{18} = (\begin{bmatrix}*&*&*\\*&*&*\\*&*&0\end{bmatrix}, \begin{bmatrix}*&*&0\\*&*&0\\0&0&0\end{bmatrix}),$$

$$W_{19} = (\begin{bmatrix}0&0&0\\*&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}*&*&*\\0&0&0\\*&*&*\end{bmatrix}), W_{20} = (\begin{bmatrix}0&0&0\\*&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_{21} = (\begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}).$$

The orthogonal complements of these subspaces are as follows:
$W_1^\perp = W_{21}$, $W_2^\perp = W_{20}$, $W_3^\perp = W_{17}$, $W_4^\perp = W_4$, $W_5^\perp = W_{11}$, $W_6^\perp = W_{13}$, $W_7^\perp = W_{18}$, $W_8^\perp = W_{19}$, $W_9^\perp = W_{14}$, $W_{10}^\perp = W_{10}$, $W_{12}^\perp = W_{12}$, $W_{15}^\perp = (\begin{bmatrix}*&*&*\\*&*&0\\*&*&0\end{bmatrix}, \begin{bmatrix}*&*&0\\0&0&0\\0&0&0\end{bmatrix})$, $W_{16}^\perp = W_{16}$. (See Remark **??** for the convention for some of these equalities).

Theorem 3. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbits $\mathcal{O}_i := Gx_i$ and the subspaces W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	W_{11}	W_{12}
$\mathcal{O}_1$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1001]$	$[1011]$	$[1002]$	$[1011]$	$[1011]$	$[1010]a_1$	$2[1001]$	$[1030]$	$[1001]a_1$	$[1021]$	$[1001]a_1$
$\mathcal{O}_3$	0	0	$[2111]$	$[2112]$	0	0	$[2120]$	0	$[2120]$	$[2111]$	$[2121]$	$[2111]$
$\mathcal{O}_4$	0	0	0	$[3311]$	0	0	0	0	0	0	0	0
$\mathcal{O}_5$	0	0	0	0	$[2111]$	0	$[2110]$	0	$[2120]$	$[2111]$	$[2121]$	$[2101]$
$\mathcal{O}_6$	0	0	0	0	0	$[2111]$	$[2110]$	$[2001]$	$[2120]$	$[2101]$	$[2111]$	$[2111]$
$\mathcal{O}_7$	0	0	0	0	0	0	$[3120]$	0	$[3130]$	$[3111]$	$[3131]$	$[3111]$
$\mathcal{O}_8$	0	0	0	0	0	0	0	$[2111]$	$\frac{1}{2}[2330]$	$[2311]$	$\frac{1}{2}[2331]$	$[2311]$
$\mathcal{O}_9$	0	0	0	0	0	0	0	0	$\frac{1}{2}[4310]$	0	$\frac{1}{2}[4311]$	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0	0	0	$[3311]$	$[3331]$	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0	0	0	0	$[4421]$	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0	0	0	0	0	$[3311]$
$\mathcal{O}_{13}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{15}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{16}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{17}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{18}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{19}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{20}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{21}$	0	0	0	0	0	0	0	0	0	0	0	0

	W_{13}	W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	W_{19}	W_{20}	W_{21}	$W_{15}^{\perp}$
$\mathcal{O}_1$	1	1	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	$[1021]$	$[1010]b_1$	$[1000]c_1$	$[1000]b_2$	$[1021]$	$[1010]b_5$	$[1001]a_2$	$[1001]b_1$	$[1012]$	$[1000]c_1$
$\mathcal{O}_3$	$[2121]$	$[2130]$	$[2110]b_1$	$[2110]a_2$	$[2112]$	$[2140]$	$2[2111]$	$[2131]$	$[2122]$	$[2110]b_1$
$\mathcal{O}_4$	0	0	$[3310]$	$[3300]$	$[3311]$	$[3320]$	0	$[3311]$	$[3321]$	$[3310]$
$\mathcal{O}_5$	$[2111]$	$[2111]$	$[2120]$	$[2100]b_3$	$[2111]$	$[2130]$	$[2111]$	$[2121]$	$[2112]$	$[2120]$
$\mathcal{O}_6$	$[2121]$	$[2111]$	$[2120]$	$[2100]b_3$	$[2111]$	$[2130]$	$[2101]a_3$	$[2121]$	$[2112]$	$[2120]$
$\mathcal{O}_7$	$[3131]$	$[3130]$	$[3130]$	$[3110]a_4$	$[3121]$	$[3120]b_3$	$2[3111]$	$[3121]a_1$	$[3132]$	$[3130]$
$\mathcal{O}_8$	$\frac{1}{2}[2331]$	$[2330]$	$[2320]$	$\frac{1}{2}[2300]b_4$	$[2321]$	$\frac{1}{2}[2320]a_5$	$3[2311]$	$\frac{1}{2}[2321]a_2$	$\frac{1}{2}[2332]$	$[2320]$
$\mathcal{O}_9$	$\frac{1}{2}[4311]$	0	0	$\frac{1}{2}[4300]$	0	$\frac{1}{2}[4310]$	0	$\frac{1}{2}[4311]$	$\frac{1}{2}[4312]$	0
$\mathcal{O}_{10}$	0	$[3330]$	$[3320]$	$[3300]a_4$	$[3321]$	$[3320]a_2$	$2[3311]$	$[3321]a_1$	$[3332]$	$[3320]$
$\mathcal{O}_{11}$	0	0	0	$[4400]$	0	$[4420]$	0	$[4421]$	$[4422]$	0
$\mathcal{O}_{12}$	$[3331]$	$[3330]$	$[3320]$	$[3300]a_4$	$[3321]$	$[3320]a_2$	$[3211]a_2$	$[3321]a_1$	$[3332]$	$[3320]$
$\mathcal{O}_{13}$	$[4421]$	0	0	$[4400]$	0	$[4420]$	$[4311]$	$[4421]$	$[4422]$	0
$\mathcal{O}_{14}$	0	$[4420]$	0	$[4400]$	0	$[4420]$	$[4311]$	$[4421]$	$[4422]$	0
$\mathcal{O}_{15}$	0	0	$[4320]$	$[4300]a_1$	$[4321]$	$[4320]a_1$	0	$[4331]$	$[4332]$	$[4320]$
$\mathcal{O}_{16}$	0	0	0	$[5400]$	0	$[5420]$	0	$[5421]$	$[5432]$	0
$\mathcal{O}_{17}$	0	0	0	0	$[3611]$	$[3620]$	$2[3411]$	$[3611]a_3$	$[3622]$	0
$\mathcal{O}_{18}$	0	0	0	0	0	$[4620]$	$2[4411]$	$2[4621]$	$[4632]$	0
$\mathcal{O}_{19}$	0	0	0	0	0	0	$[4511]$	$\frac{1}{2}[4721]$	$\frac{1}{6}[4732]$	0
$\mathcal{O}_{20}$	0	0	0	0	0	0	0	$\frac{1}{2}[5711]$	$\frac{1}{2}[5722]$	0
$\mathcal{O}_{21}$	0	0	0	0	0	0	0	0	$\frac{1}{3}[6731]$	0

Here, let $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ *and*

$$
\begin{array}{lll}
a_1 = 2q+1 & b_1 = 2q^2+2q+1 & c_1 = q^3+4q^2+3q+1 \\
a_2 = 3q+1 & b_2 = 5q^2+3q+1 & \\
a_3 = q+2 & b_3 = q^2+3q+1 & \\
a_4 = 4q+1 & b_4 = b_7 = q^2+8q+1 & \\
a_5 = 5q+1 & b_5 = 3q^2+2q+1 &
\end{array}
$$

By this result and Proposition 3, the representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_{21}$ is given as follows:

$$q^{-18}\left[\begin{array}{ccccccccccc}
1 & [1012] & [2122] & [3321] & [2112] & [2112] & [3132] & \frac{1}{2}[2332] & \frac{1}{2}[4312] & [3332] & [4422] \\
1 & e_1 & [1120]d_1 & [2310]c_1 & [1110]d_1 & [1110]d_1 & [2120]d_2 & \frac{1}{2}[1320]d_3 & \frac{1}{2}[3310]c_2 & [2320]d_2 & [3420]c_2 \\
1 & [0010]d_1 & qg_1 & [1300]e_2 & [1110]c_2 & [1110]c_2 & [1110]e_3 & \frac{1}{2}[1310]d_4 & -\frac{1}{2}[2300]c_1 & [1310]e_3 & -[2410]c_1 \\
1 & [0001]c_1 & [0101]e_2 & q^3f_1 & -[1111] & -[1111] & -[1111]b_1 & -\frac{1}{2}[1311]a_1 & \frac{1}{2}[2301] & -[1311]b_1 & [2411] \\
1 & [0010]d_1 & [1120]c_2 & -[2320] & qf_2 & [1110]c_2 & [1120]e_4 & \frac{1}{2}[1320]d_5 & \frac{1}{2}[2310]d_6 & [1320]e_4 & [2420]d_6 \\
1 & [1000]d_1 & [1120]c_2 & -[2320] & [1110]c_2 & qf_2 & [1120]e_4 & \frac{1}{2}[1320]d_5 & \frac{1}{2}[2310]d_6 & -[2320]b_2 & -[2410]c_1 \\
1 & d_2 & qe_3 & -[1300]b_1 & qe_3 & qe_3 & qg_2 & \frac{1}{2}[1300]d_7 & -\frac{1}{2}[1300]d_8 & [1300]e_5 & -[1400]d_9 \\
1 & d_3 & [1100]d_4 & -[2300]a_1 & [1100]d_5 & [1100]d_5 & [2100]d_7 & \frac{1}{2}q^3e_6 & -\frac{1}{2}[3320] & [1300]e_7 & -[2400]c_3 \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & [0110]d_6 & [0110]d_6 & -[0120]d_8 & -\frac{1}{2}[1340] & \frac{1}{2}q^3e_8 & [1330] & -[1410] \\
1 & d_2 & qe_3 & -[1300]b_1 & qe_4 & -[1100]b_2 & [1100]e_5 & \frac{1}{2}q^3e_7 & \frac{1}{2}[2310] & q^3f_3 & -[1400]d_9 \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & [0110]d_6 & -qc_1 & -[0110]d_9 & -\frac{1}{2}[0310]c_3 & -\frac{1}{2}[1300] & -[0310]d_9 & q^4d_8 \\
1 & d_2 & qe_3 & -[1300]b_1 & -[1100]b_2 & qe_4 & [1100]e_5 & \frac{1}{2}q^3e_7 & \frac{1}{2}[2310] & -[1310]c_4 & [2420] \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & -qc_1 & [0110]d_6 & -[0110]d_9 & -\frac{1}{2}[0310]c_3 & -\frac{1}{2}[1300] & [1330] & -[1410] \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & -qc_1 & -qc_1 & [1110]d_{10} & \frac{1}{2}0310]d_{11} & \frac{1}{2}[2310]b_3 & -[0310]d_9 & -[1410] \\
1 & [0010]c_2 & qe_9 & -q^3c_5 & -qc_1 & -qc_1 & qf_4 & -\frac{1}{2}[0310]c_3 & -\frac{1}{2}[1300] & q^3c_3 & -[1410] \\
1 & -b_2 & [0120] & -q^3 & [0110] & [0110] & qb_1c_6 & -\frac{1}{2}[0310]b_4 & -\frac{1}{2}[1300]b_3 & -q^3a_1 & q^4 \\
1 & d_5 & qb_1c_1 & -[2310] & -[1120] & -[1120] & -[1110]c_7 & -\frac{1}{2}[0310]b_5 & \frac{1}{2}[2300] & -[1310]b_4 & [2410] \\
1 & c_8 & -[0110]b_1 & [1300] & -qb_1 & -qb_1 & qe_{10} & -\frac{1}{2}q^3c_9 & -\frac{1}{2}[1300]b_6 & q^3b_7 & -[1400] \\
1 & c_{10} & -[1110]a_1 & q^3a_2 & -[1100]a_1 & -[1100]a_1 & -[2110]a_3 & \frac{1}{2}q^3c_{11} & -\frac{1}{2}[3300] & -[1300]b_8 & [2400] \\
1 & -b_2 & [0120] & -q^3 & [0110] & [0110] & -[0110]c_{12} & -\frac{1}{2}[0310]b_4 & \frac{1}{2}q^3c_{13} & [0310]b_1 & -[1410] \\
1 & -[0011] & [0111] & -[0310] & [0101] & [0101] & -[0111] & \frac{1}{2}[0311] & -\frac{1}{2}[1301] & -[0311] & [0401]
\end{array}\right.$$

$$\left.\begin{array}{cccccccccc}
[3332] & [4422] & [4422] & [4332] & [5432] & [3622] & [4632] & \frac{1}{6}[4732] & \frac{1}{2}[5722] & \frac{1}{3}[6731] \\
[2320]d_2 & [3420]c_2 & [3420]c_2 & [3330]c_2 & -[4420]b_2 & [2610]d_5 & [3620]c_8 & \frac{1}{6}[3720]c_{10} & -\frac{1}{2}[4710]b_2 & -\frac{1}{3}[5730] \\
[1310]e_3 & -[2410]c_1 & -[2410]c_1 & [2310]e_9 & [3430] & [1600]b_1c_1 & -[2620]b_1 & -\frac{1}{6}[3720]a_1 & \frac{1}{2}[3720] & \frac{1}{3}[4720] \\
-[1311]b_1 & [2411] & [2411] & -[1311]c_5 & -[2411] & -[2611] & [2611] & \frac{1}{6}[1711]a_2 & -\frac{1}{2}[2701] & -\frac{1}{3}[3720] \\
-[2320]b_2 & -[2410]c_1 & -[2410]c_1 & -[2320]c_1 & [3430] & -[2630] & -[2620]b_1 & -\frac{1}{6}[3720]a_1 & \frac{1}{2}[3720] & \frac{1}{3}[4720] \\
[1320]e_4 & [2420]d_6 & -[2410]c_1 & -[2320]c_1 & [3430] & -[2630] & -[2620]b_1 & -\frac{1}{6}[3720]a_1 & \frac{1}{2}[3720] & \frac{1}{3}[4720] \\
[1300]e_5 & -[1400]d_9 & [2400]d_{10} & [1300]f_4 & [2400]b_1c_6 & -[1600]c_7 & [1600]e_{10} & -\frac{1}{6}[3710]a_3 & -\frac{1}{2}[2700]c_{12} & -\frac{1}{3}[3710] \\
[1300]e_7 & -[2400]c_3 & [2400]d_{11} & -[2310]c_3 & -[3410]b_4 & -[1600]b_5 & -[2600]c_9 & \frac{1}{6}[2700]c_{11} & -\frac{1}{2}[3700]b_4 & \frac{1}{3}[4710] \\
[1330] & -[1410] & [2420]b_3 & -[1320] & -[2420]b_3 & [1610] & -[1620]b_6 & -\frac{1}{6}[3720] & \frac{1}{2}[1710]c_{13} & -\frac{1}{3}[3720] \\
-[1310]c_4 & [2420] & -[1400]d_9 & [1300]c_3 & -[2400]a_1 & -[1600]b_4 & [1600]b_7 & -\frac{1}{6}[2700]b_8 & \frac{1}{2}[2700]b_1 & -\frac{1}{3}[3710] \\
[1330] & -[1410] & -[1410] & -[1320] & [1410] & [1610] & -[1610] & \frac{1}{6}[2710] & -\frac{1}{2}[2710] & \frac{1}{3}[2710] \\
q^3f_3 & -[1400]d_9 & -[1400]d_9 & [1300]c_3 & -[2400]a_1 & -[1600]b_4 & [1600]b_7 & -\frac{1}{6}[2700]b_8 & \frac{1}{2}[2700]b_1 & -\frac{1}{3}[3710] \\
-[0310]d_9 & q^4d_8 & -[1410] & -[1320] & [1410] & [1610] & -[1610] & \frac{1}{6}[2710] & -\frac{1}{2}[2710] & \frac{1}{3}[2710] \\
-[0310]d_9 & -[1410] & q^4d_8 & -[1320] & [1410] & [1610] & -[1610] & \frac{1}{6}[2710] & -\frac{1}{2}[2710] & \frac{1}{3}[2710] \\
q^3c_3 & -[1410] & -[1410] & q^3f_5 & -[1400]b_1b_3 & -[1600]b_1 & -[1600]c_{12} & \frac{1}{6}[1700]a_1a_2 & \frac{1}{2}[1700] & \frac{1}{3}[2710] \\
-q^3a_1 & q^4 & q^4 & -q^3b_1b_3 & q^4e_{11} & [1600] & -[1600]b_3 & \frac{1}{6}[1700]a_2 & \frac{1}{2}[1700] & -\frac{1}{3}[1710] \\
-[1310]b_4 & [2410] & [2410] & -[2310]b_1 & [3410] & q^6b_9 & [1610]a_4 & -\frac{1}{2}[1710] & -\frac{1}{2}[2700] & 0 \\
q^3b_7 & -[1400] & -[1400] & -[1300]c_{12} & -[2400]b_3 & q^6a_4 & q^6b_{10} & -\frac{1}{2}[1700] & \frac{1}{2}[1700] & 0 \\
-[1300]b_8 & [2400] & [2400] & [1300]a_1a_2 & [2400]a_2 & -3q^6 & -3[1600] & q^7 & 0 & 0 \\
[0310]b_1 & -[1410] & -[1410] & [0310] & [1410] & -q^6 & [0610] & 0 & -q^7 & 0 \\
-[0311] & [0401] & [0401] & [0311] & -[0411] & 0 & 0 & 0 & 0 & q^7
\end{array}\right]$$

Here, let $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ and

$$
\begin{array}{lll}
a_1 = 2q+1 & c_1 = q^3-q-1 & e_1 = 2q^5+2q^4-2q^2-2q-1 \\
a_2 = 2q-1 & c_2 = q^3-q^2-q-1 & e_2 = q^5-q^3-q^2+q+1 \\
a_3 = 3q+1 & c_3 = 2q^3-2q-1 & e_3 = q^5-2q^4-2q^3+q^2+2q+1 \\
a_4 = q-2 & c_4 = 2q^3-q^2-2q-1 & e_4 = q^5-2q^3+q+1 \\
b_1 = q^2-q-1 & c_5 = q^3-q^2+1 & e_5 = q^5-2q^4-q^3+3q^2+3q+1 \\
b_2 = 2q^2+2q+1 & c_6 = q^3+q+1 & e_6 = 5q^5-7q^4-4q^3+4q^2+3q+1 \\
b_3 = q^2+1 & c_7 = q^3+q^2-2q-1 & e_7 = 2q^5-4q^4-3q^3+3q^2+3q+1 \\
b_4 = q^2-2q-1 & c_8 = q^3-2q^2-2q-1 & e_8 = q^5+q^4-q+1 \\
b_5 = 2q^2-3q-1 & c_9 = q^3-4q-1 & e_9 = q^5-q^4-q^3+q^2+2q+1 \\
b_6 = q^2-q+1 & c_{10} = 2q^3-2q^2-2q-1 & e_{10} = q^5-2q^4+q^3+2q^2-2q-1 \\
b_7 = 2q^2-2q-1 & c_{11} = q^3-q^2+5q+1 & e_{11} = q^5-q^4+q^3-q^2-1 \\
b_8 = q^2-3q-1 & c_{12} = q^3-q^2+q+1 & f_1 = q^6-q^5-q^4+q^2-1 \\
b_9 = q^2-2 & c_{13} = q^3+q^2-q+1 & f_2 = q^6+q^5-q^4-2q^3+q+1 \\
b_{10} = q^2-2q+2 & d_1 = q^4+q^3-q^2-q-1 & f_3 = q^6-3q^5+4q^3-2q-1 \\
 & d_2 = 2q^4-2q^2-2q-1 & f_4 = q^6-q^5+2q^3-2q-1 \\
 & d_3 = 3q^4-2q^2-2q-1 & f_5 = q^6-2q^5+q^4-q^2+q+1 \\
 & d_4 = 2q^4-q^3-4q^2-3q-1 & g_1 = q^7+q^6-3q^4-2q^3+q^2+2q+1 \\
 & d_5 = q^4+q^3-2q^2-2q-1 & g_2 = q^7-4q^5+q^4+4q^3-2q-1 \\
 & d_6 = q^4-q^3+1 & \\
 & d_7 = q^4-4q^3-7q^2-4q-1 & \\
 & d_8 = q^4-q^2+1 & \\
 & d_9 = q^4-q^3-q^2+q+1 & \\
 & d_{10} = q^4+q^2+2q+1 & \\
 & d_{11} = q^4-2q^3+2q+1 &
\end{array}
$$

3.2 $\mathbb{F}_q^2 \otimes \mathrm{H}_3(\mathbb{F}_q)$

For $a \in \mathbb{F}_{q^2}$, let $\overline{a}$ be the conjugate of a over $\mathbb{F}_q$. Define the norm map as follows:

$$\mathrm{N}_2 : \mathbb{F}_{q^2} \ni z \mapsto z\overline{z} \in \mathbb{F}_q.$$

N_2 is surjective and $\mathrm{N}_2|_{\mathbb{F}_{q^2}^\times} : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ is a surjective group homomorphism. Let $\mathrm{H}_n(\mathbb{F}_{q^2})$ be the set of Hermitian matrices of order n. We consider $\mathrm{H}_3(\mathbb{F}_{q^2})$, i.e.,

$$\mathrm{H}_3(\mathbb{F}_{q^2}) := \left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix} \in \mathrm{M}_3(\mathbb{F}_{q^2}) \;\middle|\; a_{ii} \in \mathbb{F}_q, a_{ij} \in \mathbb{F}_{q^2} (1 \le i < j \le 3) \right\}.$$

Let $V = \mathbb{F}_q^2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2})$ and $G = G_1 \times G_2 = \mathrm{GL}_2(\mathbb{F}_q) \times \mathrm{GL}_3(\mathbb{F}_{q^2})$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \mathrm{H}_3(\mathbb{F}_{q^2})$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_2(\mathbb{F}_q)$ and $g_2 \in \mathrm{GL}_3(\mathbb{F}_{q^2})$. The action of G on V is defined by

$$gx = (g_2 A \overline{g_2^T}, g_2 B \overline{g_2^T}) g_1^T.$$

Here, for a matrix h, $\overline{h}$ is the matrix whose (i,j)-entry is the conjugate over $\mathbb{F}_q$ of the (i,j)-entry of h.

V consists of 15 G-orbits in all. The following elements $x_1, ..., x_{15}$ are representatives:

$$x_1 = (0,0),\ x_2 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, 0),\ x_3 = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, 0),\ x_4 = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, 0),$$

$$x_5 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & \\ & & 0 \end{bmatrix}),\ x_6 = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}),\ x_7 = (\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & & -1 \\ \mu_0 & \mu_1 & \\ & & 0 \end{bmatrix}),$$

$$x_8 = (\begin{bmatrix} & 1 & \\ 1 & & \\ & & 0 \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix}),\ x_9 = (\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}),\ x_{10} = (\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix} 0 & & \\ & & 1 \\ & 1 & \end{bmatrix}),$$

$$x_{11} = \left(\begin{bmatrix}1&&\\&1&\\&&0\end{bmatrix}, \begin{bmatrix}0&&\\&0&\\&&1\end{bmatrix}\right),\ x_{12} = \left(\begin{bmatrix}1&&\\&1&\\&&0\end{bmatrix}, \begin{bmatrix}0&&\\&&1\\&1&\end{bmatrix}\right),\ x_{13} = \left(\begin{bmatrix}1&&\\&1&\\&&0\end{bmatrix}, \begin{bmatrix}0&&\\&1&\\&&1\end{bmatrix}\right),$$

$$x_{14} = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&\mu_0\\0&\overline{\mu_0}&\mu_1\end{bmatrix}\right),\ x_{15} = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&\nu_0&\nu_1\\\overline{\nu_0}&0&0\\\overline{\nu_1}&0&\nu_2\end{bmatrix}\right).$$

Here, $\mu_1, \nu_2 \in \mathbb{F}_q$ and $\mu_0, \nu_1, \nu_0 \in \mathbb{F}_{q^2}$ are elements such that $X^2 + \mu_1 X - N(\mu_0), X^3 + \nu_2 X^2 - (N(\nu_0) + N(\nu_1))X + \nu_2 N(\nu_0) \in \mathbb{F}_q[X]$ are irreducible. Since N_2 is surjective, there exist such $\mu_1, \mu_0, \nu_2, \nu_1, \nu_0$.

The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right), W_2 = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}\right), W_3 = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}\right),$$

$$W_4 = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}\right), W_5 = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}\right), W_6 = \left(\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}\right),$$

$$W_7 = \left(\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}\right), W_8 = \left(\begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}\right), W_9 = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}\right),$$

$$W_{10} = \left(\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}\right), W_{11} = \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}\right), W_{12} = \left(\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}\right),$$

$$W_{13} = \left(\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}\right), W_{14} = \left(\begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}\right) \text{ and } W_{15} = V.$$

The orthogonal complements of these subspaces are as follows:
$W_1^\perp = W_{15}$, $W_2^\perp = \left(\begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}\right)$, $W_3^\perp = W_{13}$, $W_4^\perp = W_4$, $W_5^\perp = \left(\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}\right)$,
$W_7^\perp = W_{14}$, $W_6^\perp = W_8$, $W_9^\perp = W_{12}$, $W_{10}^\perp = W_{10}$ and $W_{11}^\perp = W_{11}$.

Theorem 4. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbits $\mathcal{O}_i := Gx_i$ and the subspaces W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9
$\mathcal{O}_1$	1	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	[101000]	[100000]	[100101]	[101000]	[101010]	[101000]	[101000]	[100001]
$\mathcal{O}_3$	0	0	[111010]	[110111]	[111010]	[111010]	[112000]	[112010]	[110000]c_1
$\mathcal{O}_4$	0	0	0	[230011]	0	0	0	0	[231000]
$\mathcal{O}_5$	0	0	0	0	[211010]	[212010]	[212000]	[212010]	[211010]
$\mathcal{O}_6$	0	0	0	0	0	$\frac{1}{2}$[231010]	0	0	[230000]
$\mathcal{O}_7$	0	0	0	0	0	$\frac{1}{2}$[231010]	[231000]	[231010]	0
$\mathcal{O}_8$	0	0	0	0	0	0	0	[242010]	0
$\mathcal{O}_9$	0	0	0	0	0	0	0	0	[331000]
$\mathcal{O}_{10}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{13}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{15}$	0	0	0	0	0	0	0	0	0

W_{10}	W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	$W_2^{\perp}$	$W_5^{\perp}$
1	1	1	1	1	1	1	1
[100001]	[100001]	[10001]	$[100000]d_1$	[101010]	[101101]	[101010]	[101010]
$[110000]c_2$	$[110010]c_3$	$[110000]f_1$	$[110010]d_2$	$[111000]c_1$	[111111]	$[111010]c_3$	[111020]
[231000]	[231010]	[231010]	[230011]	[232000]	[231011]	[232010]	[231010]
$[211000]b_2$	[211010]	$[211000]b_2$	[211011]	$[212000]b_3$	[212111]	[212020]	[211011]
$\frac{1}{2}[230010]$	[230000]	$\frac{1}{2}[230010]$	$\frac{1}{2}[230010]b_2$	$\frac{1}{2}[231010]$	$\frac{1}{2}[231111]$	$\frac{1}{2}[231010]$	$\frac{1}{2}[231010]$
$\frac{1}{2}[232000]$	0	$\frac{1}{2}[232000]$	$\frac{1}{2}[231010]$	$\frac{1}{2}[231000]b_4$	$\frac{1}{2}[231111]$	$\frac{1}{2}[231010]b_3$	$\frac{1}{2}[231010]$
[242000]	0	[243000]	[242010]	[242010]	[242111]	$[242010]b_3$	[242010]
[331000]	[331000]	[331000]	[331020]	[332000]	[332111]	[332010]	[331010]
[342000]	0	[343000]	[342010]	[343000]	[343111]	[343020]	[342010]
0	[261000]	[261000]	$[260010]b_1$	[262000]	[261111]	[262010]	[261010]
0	0	[362000]	2[361010]	[362000]	[362111]	[362011]	[361010]
0	0	0	$\frac{1}{2}[470010]$	0	$\frac{1}{6}[471111]$	$\frac{1}{6}[473010]$	0
0	0	0	$\frac{1}{2}[371010]$	[372000]	$\frac{1}{2}[372111]$	$\frac{1}{2}[372020]$	0
0	0	0	0	0	$\frac{1}{3}[473011]$	$\frac{1}{3}[473010]$	0

Here, let $[abcdef] = (q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f$ *and*

$$\begin{array}{lll} b_1 = q^2+2 & c_1 = q^3+2q^2+1 & d_1 = q^4+q^3+q^2+q+1 \\ b_2 = 2q^2+q+1 & c_2 = q^3+3q^2+q+1 & d_2 = q^4+q^2+q+1 \\ b_3 = 2q^2+1 & c_3 = q^3+q^2+1 & f_1 = q^5+q^4+q^3+3q^2+q+1 \\ b_4 = 3q^2+1 & & \end{array}$$

By this result and Proposition 3, the representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_{21}$ can be calculated.

3.3 $\mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$

Let $\wedge^2(\mathbb{F}_q^6)$ be the set of all alternating matrices of order 6 over $\mathbb{F}_q$. We write $A \in \wedge^2(6)$ as

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}_q.$$

Let $V = \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$ and $G = G_1 \times G_2 = \mathrm{GL}_2 \times \mathrm{GL}_6$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \wedge^2(6)$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_2$ and $g_2 \in \mathrm{GL}_6$. The action of G on V is defined by

$$gx = (g_2 A g_2^T, g_2 B g_2^T) g_1^T.$$

Let $u_{lmn} (1 \le l \le 2, 1 \le n < m \le 6)$ be the element of V that the (n,m)-entry and (m,n)-entry of lth matrix is 1 and -1 respectively and the rest are all 0. For example,

$$u_{112} = (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}).$$

The set $\{u_{lmn} \mid 1 \le l \le 2, 1 \le n < m \le 6\}$ is a $\mathbb{F}_q$-basis of V.

V consists of 18 G-orbits in all. The following elements $x_1, ..., x_{18}$ are representatives:
$x_1 = 0$,
$x_2 = u_{112}$,
$x_3 = u_{112} + u_{134}$,
$x_4 = u_{112} + u_{134} + u_{156}$,

$x_5 = u_{112} + u_{213}$,
$x_6 = u_{112} + u_{214} + u_{223}$,
$x_7 = u_{112} + u_{234}$,
$x_8 = u_{112} + u_{134} + u_{214} + \mu_0 u_{223} + \mu_1 u_{234}$,
$x_9 = u_{112} + u_{215} + u_{234}$,
$x_{10} = u_{112} + u_{134} + u_{215} + u_{223}$,
$x_{11} = u_{114} + u_{123} + u_{216} + u_{225}$,
$x_{12} = u_{112} + u_{216} + u_{225} + u_{234}$,
$x_{13} = u_{114} + u_{123} + u_{216} + u_{225} + u_{234}$,
$x_{14} = u_{112} + u_{236} + u_{245}$,
$x_{15} = u_{112} + u_{134} + u_{236} + u_{245}$,
$x_{16} = u_{112} + u_{134} + u_{234} + u_{256}$,
$x_{17} = u_{112} + u_{134} + u_{214} + \mu_0 u_{223} + \mu_1 u_{234} + u_{256}$,
$x_{18} = u_{112} + u_{134} + u_{156} + \nu_2 u_{212} + u_{216} + u_{223} + \nu_1 u_{225} + \nu_0 u_{245}$.
Here, μ_1, μ_0, ν_2, ν_1, $\nu_0 \in \mathbb{F}_q$ are elements such that $X^2 + \mu_1 X + \mu_0, X^3 + \nu_2 X^2 + \nu_1 X + \nu_0 \in \mathbb{F}_q[X]$ is irreducible.

The subspaces we choose to calculate the Fourier transform are as follows:

$$
\begin{aligned}
W_1 =& \{0\},\\
W_2 =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116} \rangle_{\mathbb{F}_q},\\
W_3 =& \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156} \rangle_{\mathbb{F}_q},\\
W_4 =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156} \rangle_{\mathbb{F}_q},\\
W_5 =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216} \rangle_{\mathbb{F}_q},\\
W_6 =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{212} \rangle_{\mathbb{F}_q},\\
W_7 =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q},\\
W_8 =& \langle u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_9 =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q},\\
W_{10} =& \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_{11} =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q},\\
W_{12} =& \langle u_{114}, u_{115}, u_{116}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_{13} =& \langle u_{112}, u_{113}, u_{114}, u_{123}, u_{124}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234} \rangle_{\mathbb{F}_q},\\
W_{14} =& \langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_{15} =& \langle u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_{16} =& \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{213}, u_{214}, u_{215}, u_{216}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_{17} =& \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}\\
& , u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q},\\
W_{18} =& V.
\end{aligned}
$$

The orthogonal complements of these subspaces are as follows:
$W_1^\perp = W_{18}$, $W_2^\perp = W_{17}$, $W_3^\perp = W_{14}$, $W_4^\perp = W_4$, $W_5^\perp = W_{10}$, $W_6^\perp = W_{15}$, $W_7^\perp = W_{16}$, $W_8^\perp = W_{11}$, $W_{12}^\perp = W_{12}$, $W_{13}^\perp = W_{13}$ and

$$W_9^\perp = \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256} \rangle_{\mathbb{F}_q}.$$

Theorem 5. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbits $\mathcal{O}_i := Gx_i$ and the subspaces W_j are given as* *follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7
$\mathcal{O}_1$	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,0,0,0,1,0]$	$[1,0,0,0,1,1,0]$	$[1,0,0,1,0,1,1]$	$[1,0,1,0,0,1,0]$	$[1,0,1,0,0,1,0]$	$[1,0,1,0,0,0,0]c_1$
$\mathcal{O}_3$	0	0	$[2,2,0,1,0,1,0]$	$[2,2,0,2,0,1,1]$	0	$[2,2,1,1,1,0,0]$	0
$\mathcal{O}_4$	0	0	0	$[3,6,0,1,0,1,0]$	0	0	0
$\mathcal{O}_5$	0	0	0	0	$[2,1,1,0,1,1,0]$	$[2,1,2,0,1,0,0]$	$[2,1,1,0,1,0,0]a_1$
$\mathcal{O}_6$	0	0	0	0	0	$[3,2,1,1,1,0,0]$	0
$\mathcal{O}_7$	0	0	0	0	0	0	$[2,3,1,1,1,0,0]$
$\mathcal{O}_8$	0	0	0	0	0	0	0
$\mathcal{O}_9$	0	0	0	0	0	0	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0
$\mathcal{O}_{13}$	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0
$\mathcal{O}_{16}$	0	0	0	0	0	0	0
$\mathcal{O}_{17}$	0	0	0	0	0	0	0
$\mathcal{O}_{18}$	0	0	0	0	0	0	0

W_8	W_9	W_{10}	W_{11}	W_{12}	W_{13}
1	1	1	1	1	1
$[1,0,1,1,1,0,0]$	$[1,0,1,0,0,0,0]d_1$	$[1,1,0,0,1,1,0]$	$[1,0,1,1,0,0,0]c_2$	$[1,0,0,1,0,1,0]$	$[1,0,0,0,0,0,0]e_1$
$[2,2,1,1,0,0,0]$	$[2,2,1,1,1,0,0]$	$[2,2,1,1,0,1,0]$	$[2,2,2,1,1,0,0]$	$[2,2,0,2,0,0,0]c_2$	$[2,2,0,0,0,0,0]e_2$
0	0	0	0	$[3,6,1,1,0,0,0]$	$[3,6,1,0,0,0,0]$
$[2,1,2,1,1,0,0]$	$[2,1,1,0,1,0,0]d_2$	$[2,1,1,1,1,1,0]$	$[2,1,2,1,2,0,0]$	$[2,1,2,1,1,0,0]$	$[2,1,2,0,0,0,0]d_3$
$[3,2,2,1,1,0,0]$	$[3,2,2,1,1,0,0]$	$[3,2,2,1,1,1,0]$	$[3,2,2,2,1,0,0]$	$[3,2,1,2,1,0,0]$	$[3,2,1,0,0,0,0]e_3$
$\frac{1}{2}[2,5,1,1,1,0,0]$	$[2,5,1,1,1,0,0]$	$\frac{1}{2}[2,5,1,1,1,1,0]$	$\frac{1}{2}[2,5,3,1,1,0,0]$	$[2,5,0,2,0,0,0]$	$\frac{1}{2}[2,5,0,0,0,0,0]d_4$
$\frac{1}{2}[4,5,1,1,0,0,0]$	0	$\frac{1}{2}[4,5,1,1,0,1,0]$	$\frac{1}{2}[4,5,1,1,1,0,0]$	0	$\frac{1}{2}[4,5,2,0,0,0,0]$
0	$[3,5,2,1,1,0,0]$	$[3,5,2,1,1,1,0]$	$[3,5,4,1,1,0,0]$	$[3,5,2,2,0,0,0]$	$[3,5,2,0,0,0,0]c_3$
0	0	$[4,6,2,1,1,1,0]$	$[4,6,3,1,1,0,0]$	0	$[4,6,4,0,0,0,0]$
0	0	0	$[4,8,2,1,1,0,0]$	0	$[4,8,2,0,0,0,0]$
0	0	0	0	$[4,6,1,2,0,0,0]$	$[4,6,1,0,0,0,0]c_4$
0	0	0	0	0	$[5,8,2,0,0,0,0]$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	$W_9^{\perp}$
1	1	1	1	1	1
$[1,0,0,0,0,1,0]d_5$	$[1,0,1,0,1,1,0]$	$[1,0,1,0,1,0,0]c_5$	$[1,0,0,0,0,2,0]$	$[1,0,1,1,0,1,1]$	$[1,0,2,0,2,0,0]$
$[2,2,0,2,0,1,1]$	$[2,2,1,1,0,0,1]d_2$	$[2,2,1,1,0,0,0]c_1$	$[2,2,0,0,1,1,0]d_5$	$[2,2,1,2,0,1,1]$	$[2,2,1,1,0,0,0]c_6$
$[3,6,0,1,0,1,0]$	$[3,6,1,1,1,0,0]$	0	$[3,6,0,1,0,1,0]$	$[3,6,1,1,0,1,0]$	0
$[2,1,2,0,1,1,0]$	$[2,1,2,1,2,0,0]$	$[2,1,1,0,1,0,0]e_4$	$[2,1,2,0,2,1,0]$	$[2,1,2,1,1,1,1]$	$[2,1,1,2,1,0,0]$
$[3,2,1,1,1,1,0]$	$[3,2,1,1,1,0,0]d_2$	$[3,2,2,1,1,0,0]a_2$	$[3,2,1,2,1,1,0]$	$[3,2,2,2,1,1,1]$	$[3,2,3,1,1,0,0]$
$[2,5,0,1,1,1,0]$	$\frac{1}{2}[2,5,1,1,1,0,0]b_1$	$\frac{1}{2}[2,5,1,1,1,0,0]b_2$	$\frac{1}{2}[2,5,0,1,1,1,0]b_4$	$\frac{1}{2}[2,5,1,2,1,1,1]$	$\frac{1}{2}[2,5,1,1,1,0,0]a_2$
0	$\frac{1}{2}[4,5,1,1,0,0,0]$	$\frac{1}{2}[4,5,1,1,0,0,0]$	$\frac{1}{2}[4,5,1,1,0,1,0]$	$\frac{1}{2}[4,5,1,2,0,1,1]$	$\frac{1}{2}[4,5,1,1,0,0,0]$
$[3,5,2,1,1,1,0]$	$[3,5,3,2,1,0,0]$	$[3,5,2,1,1,0,0]b_3$	$[3,5,2,2,1,1,0]$	$[3,5,3,2,1,1,1]$	$[3,5,3,1,1,0,0]$
0	$[4,6,3,1,1,0,0]$	$[4,6,2,1,1,0,0]a_1$	$[4,6,3,1,1,1,0]$	$[4,6,3,2,1,1,1]$	$[4,6,2,1,1,0,0]$
0	$[4,8,2,1,1,0,0]$	$[4,7,3,1,1,0,0]$	$[4,8,2,1,1,1,0]$	$[4,8,2,2,1,1,1]$	0
$[4,6,1,1,1,1,0]$	$[4,6,1,1,1,0,0]c_4$	0	$[4,6,1,1,2,1,0]$	$[4,6,2,2,1,1,1]$	0
0	$[5,8,2,1,1,0,0]$	0	$[5,8,2,1,1,1,0]$	$[5,8,3,2,1,1,1]$	0
$[3,11,0,1,0,1,0]$	$[3,11,1,1,1,0,0]$	$2[3,9,1,1,1,0,0]$	$[3,11,0,1,0,1,0]b_5$	$[3,11,1,2,0,1,1]$	0
0	$[4,11,1,1,1,0,0]$	$2[4,9,2,1,1,0,0]$	$2[4,11,1,1,1,1,0]$	$[4,11,2,2,1,1,1]$	0
0	0	$[4,11,1,1,1,0,0]$	$\frac{1}{2}[4,13,0,1,1,1,0]$	$\frac{1}{6}[4,13,1,2,1,1,1]$	0
0	0	0	$\frac{1}{2}[5,13,1,1,0,1,0]$	$\frac{1}{2}[5,13,2,2,0,1,1]$	0
0	0	0	0	$\frac{1}{3}[6,13,3,1,1,1,0]$	0

Here, let $[a,b,c,d,e,f,g]=(q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f (q^2-q+1)^g$ *and*

$$\begin{array}{lll}
a_1 = q+2, & c_1 = 2q^3+2q+1, & d_1 = 2q^4+q^3+2q^2+q+1, \\
a_2 = 2q+1, & c_2 = q^3+q^2+1, & d_2 = q^4+q^3+2q^2+2q+1, \\
b_1 = 2q^2+2q+1, & c_3 = 2q^3+5q^2+3q+1, & d_3 = q^4+2q^3+3q^2+q+1, \\
b_2 = 2q^2+4q+1, & c_4 = q^3+2q^2+q+1, & d_4 = 3q^4+8q^3+10q^2+4q+1, \\
b_3 = 2q^2+3q+2, & c_5 = 2q^3+q^2+q+1, & d_5 = q^4+q^2+q+1, \\
b_4 = 2q^2+q+1, & c_6 = q^3+q+1, & e_1 = q^5+4q^4+4q^3+3q^2+2q+1, \\
b_5 = q^2+2, & & e_2 = 2q^5+4q^4+4q^3+5q^2+3q+1, \\
 & & e_3 = q^5+5q^4+6q^3+6q^2+3q+1, \\
 & & e_4 = q^5+2q^4+3q^3+4q^2+2q+1
\end{array}$$

By this result and Proposition 3, the representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_{21}$ can be calculated.

References

[1] Taniguchi Takashi, and Frank Thorne."Secondary terms in counting functions for cubic fields." Duke Mathematical Journal 162.13 (2013): 2451-2508.

[2] Taniguchi, Takashi, and Frank Thorne."Orbital exponential sums for prehomogeneous vector spaces." preprint arXiv:1607.07827 (2016).

[3] D.J. Wright and A. Yukie. Prehomogeneous vector spaces and field extensions. Invent. Math., 110:283-314, 1992.

Graduate School of Mathematics
Kobe University
1-1 Rokkodai-cho, Nada-ku, Kobe, 657-8501
JAPAN
E-mail address: `k.ishimoto1024@gmail.com`