YORIOKA'S CHARACTERIZATION OF THE COFINALITY OF THE STRONG MEASURE ZERO IDEAL AND ITS INDEPENDENCY FROM OF CONTINUUM

MIGUEL A. CARDONA

Abstract. In this paper we present a simpler proof of that no inequality between $\mathrm{cof}(\mathcal{SN})$ and $\mathfrak{c}$ can be decided in ZFC using tecniques and results well known.

1. Introduction

Borel [Bor19] introduced the new class of Lebesgue measure zero subsets of the real line called *strong measure zero* sets, which we denote by $\mathcal{SN}$. The cardinal invariants associated with strong measure zero have been investigated. To summarize some of the results:

Theorem A. The following holds in ZFC

(i) (Carlson [Car93]) $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN})$,
(ii) $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{SN}) \leq \mathfrak{c}$,
(iii) (Miller [Mil81]) $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathrm{cov}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{SN})\}$,
(iv) (Osuga [Osu08]) $\mathrm{cof}(\mathcal{SN}) \leq 2^{\mathfrak{d}}$.

Moreover, each of the following staments is consistent with ZFC

(v) (Goldstern, Judah and Shelah [GJS93]) $\mathrm{cof}(\mathcal{M}) < \mathrm{add}(\mathcal{SN})$,
(vi) (Pawlikowski [Paw90]) $\mathrm{cov}(\mathcal{SN}) < \mathrm{add}(\mathcal{M})$,
(vii) (Yorioka [Yor02]) $\mathfrak{c} < \mathrm{cof}(\mathcal{SN})$ (from CH),
(viii) (Yorioka [Yor02]) $\mathrm{cof}(\mathcal{SN}) < \mathfrak{c}$,
(ix) (Laver [Lav76]) $\mathrm{cof}(\mathcal{SN}) = \mathfrak{c}$.

To prove (vii) and (viii) Yorioka give a characterization of $\mathcal{SN}$, to do this he introduced the σ-ideals $\mathcal{I}_f$ parametrized by increasing functions $f \in \omega^\omega$, which we call *Yorioka ideals* (see Definition 2.1). These ideals are subideals of the null ideal $\mathcal{N}$ and they include $\mathcal{SN}$ and $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \in \omega^\omega \text{ increasing}\}$. Even more, he proved that $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$ (see Definition 2.2) whenever $\mathrm{add}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_f) = \kappa$ for all increasing f. But Yorioka's original proof assumes $\mathrm{add}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_f) = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \kappa$ for all increasing f, but $\mathfrak{d}$ and $\mathrm{cov}(\mathcal{M})$ can be omitted since $\mathrm{add}(\mathcal{N}) \leq \mathrm{minadd} \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{supcof} \leq \mathrm{cof}(\mathcal{N})$ (see [Osu08, CM19]).

In this work, we provide a simpler proof of the result.

Main Theorem (Yorioka [Yor02])**.** Let κ, ν be an infinite cardinals such that $\aleph_1 \leq \kappa = \kappa^{<\kappa} < \nu = \nu^\kappa$ and assume that λ is a cardinal such that $\kappa \leq \lambda = \lambda^{\aleph_0}$. Then there is some poset $\mathbb{Q}$ such that $\Vdash_{\mathbb{Q}} \mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \kappa$, $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa = \nu$ and $\mathfrak{c} = \lambda$.

This result give the consisteny that values value $\mathrm{cof}(\mathcal{SN})$ may be less than $\mathfrak{c}$.

2010 *Mathematics Subject Classification.* 03E17, 03E35, 03E40.
Key words and phrases. Strong measure zero sets, cardinal invariants, Yorioka ideals.
The author was partially supported by the Austrian Science Fund (FWF) P30666.

2. Proof the main theorem

We first start with basic definitions and facts:

Let κ be an infinite cardinal. Let $f, g \in \kappa^\kappa$. Set $f \leq^* g$ if $\exists \alpha < \kappa \forall \beta > \alpha (f(\beta) \leq g(\beta))$. Denote $\mathrm{pow}_k : \omega \to \omega$ the function defined by $\mathrm{pow}_k(i) := i^k$, and define the relation $\ll$ on ω^ω as follows: $f \ll g$ iff $\forall k < \omega (f \circ \mathrm{pow}_k \leq^* g)$.

Definition 2.1. For $\sigma \in (2^{<\omega})^\omega$ define

$$[\sigma]_\infty := \{x \in 2^\omega : \exists^\infty n < \omega (\sigma(n) \subseteq x)\} = \bigcap_{n<\omega} \bigcup_{m \geqslant n} [\sigma(m)]$$

and $\mathrm{ht}_\sigma \in \omega^\omega$ by $\mathrm{ht}_\sigma(i) := |\sigma(i)|$ for each $i < \omega$. Let $f \in \omega^\omega$ be a increasing function, set

$$\mathcal{I}_f := \{X \subseteq 2^\omega : \exists \sigma \in (2^{<\omega})^\omega (X \subseteq [\sigma]_\infty \text{ and } h_\sigma \gg f)\}.$$

Any family of the form $\mathcal{I}_f$ if f increasing is called a *Yorioka ideal*, since Yorioka [Yor02] has proved that $\mathcal{I}_f$ is a σ-ideal in this case, and $\mathcal{SN} = \bigcap \{\mathcal{I}_f : f \text{ increasing}\}$. Denote

$$\mathrm{minadd} = \min\{\mathrm{add}(\mathcal{I}_f) : f \text{ increasing}\}, \ \ \mathrm{supcof} = \sup\{\mathrm{cof}(\mathcal{I}_f) : f \text{ increasing}\}$$

Definition 2.2. Let κ be a regular cardinals. Define the cardinal numbers $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ as follows:

$$\mathfrak{b}_\kappa = \min\{|F| : F \subseteq \kappa^\kappa \ \& \ \forall g \in \kappa^\kappa \exists f \in F (f \not\leq^* g)\} \text{ *the (un)bounding number for* } \kappa^\kappa$$

and

$$\mathfrak{d}_\kappa = \min\{|D| : D \subseteq \kappa^\kappa \ \& \ \forall g \in \kappa^\kappa \exists f \in D (g \leq^* f)\} \text{ *the dominating number for* } \kappa^\kappa$$

In particular, when $\kappa = \omega$, $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ are $\mathfrak{b}$ and $\mathfrak{d}$ respectively, well known as the *(un)bounding number* and *the dominating number*.

Set $\mathrm{Fn}_{<\kappa}(I, J) := \{p \subseteq I \times J : |p| < \kappa \text{ and } p \text{ function}\}$ for sets I, J and an infinite cardinal κ.

Lemma 2.3. *Let ν, κ be uncountable cardinals such that $\kappa^{<\kappa} = \kappa$ and $\nu > \kappa$. Then* $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa) \Vdash \mathfrak{d}_\kappa \geq \nu$.

Proof. Let $\vartheta < \nu$ and let $\{\dot{x}_\alpha : \alpha < \vartheta\}$ be a set of $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$-names of functions in κ^κ. Since $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$ is $(\kappa^{<\kappa})^+ = \kappa^+$-cc we can find a subset S of ν of size $< \nu$ such that $\dot{x}_\alpha$ is a $\mathrm{Fn}(S \times \kappa, \kappa)$-name for each $\alpha < \vartheta$.

Claim 2.4. $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$ *adds an unbounded function in κ^κ over the ground model.*

Proof. Let G be a $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$-generic set over V. Let $c := c_G = \bigcup G \in \kappa^\kappa$ be the real generic added by $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$. Assume that $f \in \kappa^\kappa \cap V$. We will prove that $f \not\leq^* c$. To see this, for $\alpha < \kappa$, define the sets $D_\alpha := \{p \in \mathrm{Fn}_{<\kappa}(\kappa, \kappa) : \exists \beta > \alpha (p(\beta) > f(\beta))\}$ which are dense, so G intersects all of these yielding $\forall \alpha < \kappa \exists \beta < \alpha (c(\beta) > f(\beta))$. □

By Claim 2.4, $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$ forces that the κ-Cohen real at some $\xi \in \nu \smallsetminus S$ is not dominated by any $\dot{x}_\alpha$. □

As mentioned in the introduction that $\mathrm{add}(\mathcal{N}) \leq \mathrm{minadd} \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{supcof} \leq \mathrm{cof}(\mathcal{N})$ (see [Osu08, CM19]) we can reformulate Yorioka's characterization of $\mathrm{cof}(\mathcal{SN})$ as follows.

Theorem 2.5 (Yorioka [Yor02])**.** *Let κ be a regular uncountable cardinal. Assume that* $\kappa = \mathrm{minadd} = \mathrm{supcof}$. *Then* $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$.

To prove our Main Thereom we need to preserve $\mathfrak{d}_\kappa$ for κ regular. The following result show one condition under it can be preserved.

Lemma 2.6. *Let κ be a regular uncountable cardinal. Suppose that $\mathbb{P}$ is a κ-cc. Then $\Vdash_{\mathbb{P}} \mathfrak{d}_\kappa^V = \mathfrak{d}_\kappa$.*

Proof. It is enough to show that $\mathbb{P}$ is κ^κ-bounding[1] because κ^κ-bounding posets preserve $\mathfrak{d}_\kappa$. Let $\dot{x}$ be a $\mathbb{P}$-name for a member of κ^κ. We prove that $\forall\alpha < \kappa \exists z(\alpha) < \kappa(\Vdash_{\mathbb{P}} \dot{x}(\alpha) < z(\alpha))$. Fix any $\alpha < \kappa$. Towards a contradiction, assume that $\forall\beta < \kappa \exists p_\beta \in \mathbb{P}(p_\beta \Vdash_{\mathbb{P}} \beta \leq \dot{x}(\alpha))$.

Claim 2.7. *Assume that $\mathbb{P}$ is κ-cc and $\{p_\alpha : \alpha < \kappa\} \subseteq \mathbb{P}$. Then there is a $q \in \mathbb{P}$ such that $q \Vdash |\{\alpha < \kappa : p_\alpha \in \dot{G}\}| = \kappa$.*

Proof. To reason by contradiction assume that $\Vdash_{\mathbb{P}} |\{\alpha < \kappa : p_\alpha \in \dot{G}\}| < \kappa$. Let $\dot{\beta}$ be a $\mathbb{P}$-name such that $\Vdash \dot{\beta} \in \kappa$ and $\{\alpha < \kappa : p_\alpha \in \dot{G}\} \subseteq \dot{\beta}$. Fix a maximal antichain A deciding $\dot{\beta}$ and a function $h : A \to \kappa$ such that $p \Vdash h(p) = \dot{\beta}$ for all $p \in A$. Set $\gamma := \sup_{p \in A} h(p) < \kappa$. since κ is regular and $\mathbb{P}$ is κ-cc, $\gamma < \kappa$, so $\Vdash_{\mathbb{P}} \{\alpha < \kappa : p_\alpha \in \dot{G}\} \subseteq \gamma$. But $p_{\gamma+1} \Vdash \gamma + 1 \in \{\alpha < \kappa : p_\alpha \in \dot{G}\} \subseteq \gamma$, which is a contradiction. □

By Claim 2.7, we can find a condition $q \in \mathbb{P}$ such that $q \Vdash |\{\beta < \kappa : p_\beta \in \dot{G}\}| = \kappa$, so there are a $r \leq q$ and $\vartheta < \kappa$ such that $r \Vdash \dot{x}(\alpha) = \vartheta$, even more, we can find $s \leq r$ and $\varepsilon > \vartheta$ such that $s \Vdash p_\varepsilon \in \dot{G}$. Hence $s \Vdash \dot{x}(\alpha) = \vartheta < \varepsilon \leq \dot{x}(\alpha)$ because $p_\varepsilon \Vdash \varepsilon \leq \dot{x}(\alpha)$ which is a contradiction.

For $\alpha < \kappa$ set $z \in \kappa^\kappa$ such that $\Vdash_{\mathbb{P}} \dot{x}(\alpha) < z(\alpha)$. This z work. □

Now we are ready to prove the Main Theorem.

Proof of the Main Theorem. In V, we start with $\mathbb{P}_0 := \mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$. Note that $\mathbb{P}_0$ is κ^+-cc and $<\kappa$-closed. Then $\Vdash_{\mathbb{P}_0} \mathfrak{d}_\kappa = 2^\kappa = \nu$ by Lemma 2.3.

In $V^{\mathbb{P}_0}$, let $\mathbb{P}_1$ be the FS iteration of amoeba forcing of length $\lambda\kappa$. Then, $\Vdash_{\mathbb{P}_1} \mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \kappa$ and $\mathfrak{c} = \lambda$. In particular, $\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \kappa$ and minadd $=$ supcof $= \kappa$. On the other hand, $\mathrm{cov}(\mathcal{SN}) = \kappa$ because the length of the FS iteration has cofinality κ (see e.g. [BJ95, Lemma 8.2.6]). Therefore, $\Vdash_{\mathbb{P}_1} \mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \kappa$ and $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa = \nu$ by Theorem 2.5 and Lemma 2.6. □

3. Open problems

Very quite recently, the author with Mejía and Rivera-Madrid [CMRM] constructed a poset forcing $\mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$. This is first result where 3 cardianl invariants associated with $\mathcal{SN}$ are pairwise different, but its still unknown for 4, so we ask.

Question 3.1. Is it consistent with ZFC that $\mathrm{add}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

In a work in progress, the author with Mejía and Yorioka have improved methods and results known from [Yor02] to prove the consistency of $\mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$. However its still unknown the following problem.

Question 3.2. Is it consistent with ZFC that $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

[1]A poset $\mathbb{P}$ is *κ^κ-bounding* if for any $p \in \mathbb{P}$ and any $\mathbb{P}$-name $\dot{x}$ of a member for κ^κ, there are a function $z \in \kappa^\kappa$ and some $q \leq p$ that forces $\dot{x}(\alpha) \leq z(\alpha)$ for any $\alpha < \kappa$.

The method of κ-*uf-extendable matrix iterations*, introduced recently by the author with Brendle and Mejía [BCM], could be useful to answer the question above. For example they constructed a ccc poset forcing

$$\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}).$$

In the same model, $\mathrm{cov}(\mathcal{SN}) = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{SN}) = \mathrm{non}(\mathcal{N})$ by Theorem A and because this model is obtained by a FS iteration of length with cofinality ν (where ν is the desired value for $\mathrm{non}(\mathcal{M})$), and it is well known that such cofinality becomes an upper bound of $\mathrm{cov}(\mathcal{SN})$ (see e.g. [BJ95, Lemma 8.2.6]). But it is unknown how to deal with $\mathrm{add}(\mathcal{SN})$ and $\mathrm{cof}(\mathcal{SN})$ in this context.

References

[BCM] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. Filter-linkedness and its effect on the preservation of cardinal characteristics. arXiv:1809.05004.

[BJ95] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. A K Peters, Wellesley, Massachusetts, 1995.

[Bor19] Émile Borel. Sur la classification des ensembles de mesure nulle. *Bulletin de la Société Mathématique de France*, 47:97–125, 1919.

[Car93] Timothy J. Carlson. Strong measure zero and strongly meager sets. *Proc. Amer. Math. Soc.*, 118(2):577–586, 1993.

[CM19] Miguel A. Cardona and Diego A. Mejía. On cardinal characteristics of Yorioka ideals. *MLQ*, 2019. In press. arXiv:1703.08634.

[CMRM] Miguel A. Cardona, Diego A. Mejía, and Ismael E. Rivera-Madrid. The covering number of the strong measure zero ideal can be above almost everything else. arXiv:1902.01508v1.

[GJS93] Martin Goldstern, Haim Judah, and Saharon Shelah. Strong measure zero sets without Cohen reals. *J. Symbolic Logic*, 58(4):1323–1341, 1993.

[Lav76] Richard Laver. On the consistency of Borel's conjecture. *Acta Math.*, 137(3-4):151–169, 1976.

[Mil81] Arnold W. Miller. Some properties of measure and category. *Trans. Amer. Math. Soc.*, 266(1):93–114, 1981.

[Osu08] Noboru Osuga. The cardinal invariants of certain ideals related to the strong measure zero ideal. *Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku*, 1619:83–90, 2008.

[Paw90] Janusz Pawlikowski. Finite support iteration and strong measure zero sets. *J. Symbolic Logic*, 55(2):674–677, 1990.

[Yor02] Teruyuki Yorioka. The cofinality of the strong measure zero ideal. *J. Symbolic Logic*, 67(4):1373–1384, 2002.

Institute of Discrete Mathematics and Geometry, TU Wien, Wiedner Hauptstrasse 8–10/104 A–1040 Wien, Austria.

E-mail address: miguel.montoya@tuwien.ac.at

URL: https://www.researchgate.net/profile/Miguel_Cardona_Montoya