Abstract. In this paper we present a simpler proof of that no inequality between $\text{cof}(\mathcal{SN})$ and $c$ can be decided in ZFC using techniques and results well known.

1. Introduction

Borel [Bor19] introduced the new class of Lebesgue measure zero subsets of the real line called strong measure zero sets, which we denote by $\mathcal{SN}$. The cardinal invariants associated with strong measure zero have been investigated. To summarize some of the results:

**Theorem A.** The following holds in ZFC

(i) (Carlson [Car93]) $\text{add}(\mathcal{N}) \leq \text{add}(\mathcal{SN})$,
(ii) $\text{cov}(\mathcal{N}) \leq \text{cov}(\mathcal{SN}) \leq \kappa$,
(iii) (Miller [Mil81]) $\text{cov}(\mathcal{M}) \leq \text{non}(\mathcal{SN}) \leq \text{cov}(\mathcal{N})$ and $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{non}(\mathcal{SN})\}$,
(iv) (Osuga [Osu08]) $\text{cof}(\mathcal{SN}) \leq 2^\kappa$.

Moreover, each of the following statements is consistent with ZFC

(v) (Goldstern, Judah and Shelah [GJS93]) $\text{cof}(\mathcal{M}) < \text{add}(\mathcal{SN})$,
(vi) (Pawlikowski [Paw90]) $\text{cov}(\mathcal{SN}) < \text{add}(\mathcal{M})$,
(vii) (Yorioka [Yor02]) $\kappa < \text{cof}(\mathcal{SN})$ (from CH),
(viii) (Yorioka [Yor02]) $\text{cof}(\mathcal{SN}) < c$,
(ix) (Laver [Lav76]) $\text{cof}(\mathcal{SN}) = c$.

To prove (vii) and (viii) Yorioka give a characterization of $\mathcal{SN}$, to do this he introduced the $\sigma$-ideals $\mathcal{I}_f$ parametrized by increasing functions $f \in \omega^\omega$, which we call *Yorioka ideals* (see Definition 2.1). These ideals are subideals of the null ideal $\mathcal{N}$ and they include $\mathcal{SN}$ and $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \in \omega^\omega \text{ increasing}\}$. Even more, he proved that $\text{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$ (see Definition 2.2) whenever $\text{add}(\mathcal{I}_f) = \text{cof}(\mathcal{I}_f) = \kappa$ for all increasing $f$. But Yorioka’s original proof assumes $\text{add}(\mathcal{I}_f) = \text{cof}(\mathcal{I}_f) = \mathfrak{d} = \text{cov}(\mathcal{M}) = \kappa$ for all increasing $f$, but $\mathfrak{d}$ and $\text{cof}(\mathcal{M})$ can be omitted since $\text{add}(\mathcal{N}) \leq \text{minadd} \leq \text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{M}) \leq \text{supcof} \leq \text{cof}(\mathcal{N})$ (see [Osu08, CM19]).

In this work, we provide a simpler proof of the result.

**Main Theorem** (Yorioka [Yor02]). Let $\kappa, \nu$ be an infinite cardinals such that $\mathfrak{r}_1 \leq \kappa = \kappa^{<\kappa} < \nu = \nu^\kappa$ and assume that $\lambda$ is a cardinal such that $\kappa \leq \lambda = \lambda^{<\lambda}$. Then there is some poset $Q$ such that $\Vdash_Q \text{add}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \kappa$, $\text{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa = \nu$ and $c = \lambda$.

This result give the consisteny that values value $\text{cof}(\mathcal{SN})$ may be less than $c$.
2. Proof the main theorem

We first start with basic definitions and facts:

Let $\kappa$ be an infinite cardinal. Let $f, g \in \kappa^\omega$. Set $f \leq^* g$ if $\exists \alpha < \kappa \forall \beta > \alpha (f(\beta) \leq g(\beta))$. Denote $\text{pow}_k : \omega \to \omega$ the function defined by $\text{pow}_k(i) := i^k$, and define the relation $\ll$ on $\omega^\omega$ as follows: $f \ll g$ iff $\forall k < \omega (f \circ \text{pow}_k \leq^* g)$.

**Definition 2.1.** For $\sigma \in (2^{<\omega})^\omega$ define

$$[\sigma]_{\infty} := \{x \in 2^\omega : \exists^\infty n < \omega(\sigma(n) \subseteq x)\} = \bigcap_{n < \omega} \bigcup_{m \geq n} [\sigma(m)]$$

and $h_\sigma \in \omega^\omega$ by $h_\sigma(i) := |\sigma(i)|$ for each $i < \omega$. Let $f \in \omega^\omega$ be a increasing function, set

$$I_f := \{X \subseteq 2^\omega : \exists \sigma \in (2^{<\omega})^\omega(\sigma(\omega) \subseteq [\sigma]_{\infty} \text{ and } h_\sigma \gg f)\}.$$ 

Any family of the form $I_f$ if $f$ increasing is called a Yorioka ideal, since Yorioka [Yor02] has proved that $I_f$ is a $\sigma$-ideal in this case, and $SN = \bigcap I_f$ is not increasing. Denote

$$\minadd = \min\{\text{add}(I_f) : f \text{ increasing}\}, \supcof = \sup\{\text{cof}(I_f) : f \text{ increasing}\}$$

**Definition 2.2.** Let $\kappa$ be a regular cardinals. Define the cardinal numbers $b_\kappa$ and $d_\kappa$ as follows:

$$b_\kappa = \min\{|F| : F \subseteq \kappa^\omega \text{ and } \forall g \in \kappa^\omega \exists f \in F(f \not\leq^* g)\} \text{ the (un)bounding number for } \kappa$$

and

$$d_\kappa = \min\{|D| : D \subseteq \kappa^\omega \text{ and } \forall g \in \kappa^\omega \exists f \in D(g \leq^* f)\} \text{ the dominating number for } \kappa$$

In particular, when $\kappa = \omega$, $b_\kappa$ and $d_\kappa$ are $b$ and $d$ respectively, well known as the (un)bounding number and the dominating number.

Set $\text{Fn}_{<\kappa}(I, J) := \{p \subseteq I \times J : |p| < \kappa \text{ and } p \text{ function}\}$ for sets $I, J$ and an infinite cardinal $\kappa$.

**Lemma 2.3.** Let $\nu, \kappa$ be uncountable cardinals such that $\kappa^{<\kappa} = \kappa$ and $\nu > \kappa$. Then $\text{Fn}_{<\kappa}(\nu \times \kappa, \kappa) \models b_\kappa \geq \nu$.

**Proof.** Let $\vartheta < \nu$ and let $\{\dot{x}_\alpha : \alpha < \vartheta\}$ be a set of $\text{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$-names of functions in $\kappa^\omega$. Since $\text{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$ is $(\kappa^{<\kappa})^+ = \kappa^+$-cc we can find a subset $S$ of $\nu$ of size $< \nu$ such that $\dot{x}_\alpha$ is a $\text{Fn}(S \times \kappa, \kappa)$-name for each $\alpha < \vartheta$.

**Claim 2.4.** $\text{Fn}_{<\kappa}(\kappa, \kappa)$ adds an unbounded function in $\kappa^\omega$ over the ground model.

**Proof.** Let $G$ be a $\text{Fn}_{<\kappa}(\kappa, \kappa)$-generic set over $V$. Let $c := c_G = \bigcup G \in \kappa^\omega$ be the real generic added by $\text{Fn}_{<\kappa}(\kappa, \kappa)$. Assume that $f \in \kappa^\omega \cap V$. We will prove that $f \not\leq^* c$. To see this, for $\alpha < \kappa$, define the sets $D_\alpha := \{p \in \text{Fn}_{<\kappa}(\kappa, \kappa) : \exists \beta > \alpha (p(\beta) > f(\beta))\}$ which are dense, so $G$ intersects all of these yielding $\forall \alpha < \kappa \exists \beta < \alpha (c(\beta) > f(\beta))$.\[\Box\]

By Claim 2.4, $\text{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$ forces that the $\kappa$-Cohen real at some $\xi \in \nu \setminus S$ is not dominated by any $\dot{x}_\alpha$.\[\Box\]

As mentioned in the introduction that $\text{add}(\mathcal{N}) \leq \minadd \leq \text{add}(\mathcal{M}) \leq \supcof \leq \text{cof}(\mathcal{N})$ (see [Osu08, CM19]) we can reformulate Yorioka’s characterization of $\text{cof}(SN)$ as follows.

**Theorem 2.5 (Yorioka [Yor02]).** Let $\kappa$ be a regular uncountable cardinal. Assume that $\kappa = \minadd = \supcof$. Then $\text{cof}(SN) = d_\kappa$.\[\Box\]
To prove our Main Theorem we need to preserve $d_\kappa$ for $\kappa$ regular. The following result show one condition under it can be preserved.

**Lemma 2.6.** Let $\kappa$ be a regular uncountable cardinal. Suppose that $P$ is a $\kappa$-cc. Then $\forces P \ V^\kappa = d_\kappa$.

**Proof.** It is enough to show that $P$ is $\kappa$-bounding because $\kappa$-bounding posets preserve $d_\kappa$. Let $\dot{x}$ be a $P$-name for a member of $\kappa$. We prove that $\forall \alpha < \kappa \exists x(\alpha) < \kappa (\forces P \dot{x}(\alpha) < z(\alpha))$. Fix any $\alpha < \kappa$. Towards a contradiction, assume that $\forall \beta < \kappa \exists p_\beta \in P (p_\beta \forces P \beta \leq \dot{x}(\alpha))$.

**Claim 2.7.** Assume that $P$ is $\kappa$-cc and $\{p_\alpha : \alpha < \kappa\} \subseteq P$. Then there is a $q \in P$ such that $q \forces |\{\alpha < \kappa : p_\alpha \in G\}| = \kappa$.

**Proof.** To reason by contradiction assume that $\forces P |\{\alpha < \kappa : p_\alpha \in G\}| < \kappa$. Let $\dot{\beta}$ be a $P$-name such that $\forces \dot{\beta} \in \kappa$ and $\{\alpha < \kappa : p_\alpha \in G\} \subseteq \dot{\beta}$. Fix a maximal antichain $A$ deciding $\dot{\beta}$ and a function $h : A \rightarrow \kappa$ such that $p \forces h(p) = \dot{\beta}$ for all $p \in A$. Set $\gamma := \sup_{p \in A} h(p) < \kappa$. since $\kappa$ is regular and $P$ is $\kappa$-cc, $\gamma < \kappa$, so $\forces P \{\alpha < \kappa : p_\alpha \in G\} \subseteq \gamma$. But $p_{\gamma+1} \forces \gamma + 1 \in \{\alpha < \kappa : p_\alpha \in G\} \subseteq \gamma$, which is a contradiction.

By Claim 2.7, we can find a condition $q \in P$ such that $q \forces |\{\beta < \kappa : p_\beta \in G\}| = \kappa$, so there are a $r \leq q$ and $\vartheta < \kappa$ such that $r \forces \dot{x}(\alpha) = \vartheta$, even more, we can find $s \leq r$ and $\varepsilon > \vartheta$ such that $s \forces p_\varepsilon \in G$. Hence $s \forces \dot{x}(\alpha) = \vartheta < \varepsilon \leq \dot{x}(\alpha)$ because $p_\varepsilon \forces \varepsilon \leq \dot{x}(\alpha)$ which is a contradiction.

For $\alpha < \kappa$ set $z \in \kappa^\kappa$ such that $q \forces \dot{x}(\alpha) < z(\alpha)$. This $z$ work. □

Now we are ready to prove the Main Theorem.

**Proof of the Main Theorem.** In $V$, we start with $P_0 := F_{\nu \times \kappa}(\nu \times \kappa, \kappa)$. Note that $P_0$ is $\kappa^+\text{-}cc$ and $< \kappa$-closed. Then $\forces P_0 \ V_\kappa = 2^\kappa = \nu$ by Lemma 2.3.

In $V^{P_0}$, let $P_1$ be the FS iteration of amoeba forcing of length $\lambda \kappa$. Then, $\forces P_1 \ add(\mathcal{N}) = \cof(\mathcal{N}) = \kappa$ and $\text{and} minadd = sup\cof = \kappa$. On the other hand, $\text{cov}(\mathcal{S}_\kappa) = \kappa$ because the length of the FS iteration has cofinality $\kappa$ (see e.g. [BJ95, Lemma 8.2.6]). Therefore, $\forces P_1 \ add(\mathcal{S}_\kappa) = \text{cov}(\mathcal{S}_\kappa) = \text{non}(\mathcal{S}_\kappa) = \kappa$ and $\cof(\mathcal{S}_\kappa) = d_\kappa = \nu$ by Theorem 2.5 and Lemma 2.6. □

### 3. Open Problems

Very quite recently, the author with Mejía and Rivera-Madrid [CMRM] constructed a poset forcing $\text{non}(\mathcal{S}_\kappa) < \text{cov}(\mathcal{S}_\kappa) < \text{cof}(\mathcal{S}_\kappa)$. This is first result where 3 cardinal invariants associated with $\mathcal{S}$ are pairwise different, but its still unknown for 4, so we ask.

**Question 3.1.** Is it consistent with ZFC that $\text{add}(\mathcal{S}_\kappa) < \text{non}(\mathcal{S}_\kappa) < \text{cov}(\mathcal{S}_\kappa) < \text{cof}(\mathcal{S}_\kappa)$?

In a work in progress, the author with Mejía and Yorioka have improved methods and results known from [Yor02] to prove the consistency of $\text{non}(\mathcal{S}_\kappa) < \text{non}(\mathcal{S}_\kappa) < \text{cof}(\mathcal{S}_\kappa)$. However its still unknown the following problem.

**Question 3.2.** Is it consistent with ZFC that $\text{add}(\mathcal{S}_\kappa) < \text{cof}(\mathcal{S}_\kappa) < \text{non}(\mathcal{S}_\kappa) < \text{cof}(\mathcal{S}_\kappa)$?

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1 A poset $P$ is $\kappa^\kappa$-bounding if for any $p \in P$ and any $P$-name $\dot{x}$ of a member for $\kappa^\kappa$, there are a function $z \in \kappa^\kappa$ and some $q \leq p$ that forces $\dot{x}(\alpha) \leq z(\alpha)$ for any $\alpha < \kappa$. 

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The method of \emph{k-uf-extendable matrix iterations}, introduced recently by the author with Brendle and Mejía [BCM], could be useful to answer the question above. For example they constructed a ccc poset forcing
\[ \text{add}(\mathcal{N}) > \text{add}(\mathcal{M}) < \text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) < \text{cof}(\mathcal{M}) = \text{cof}(\mathcal{N}). \]
In the same model, \( \text{cov}(\mathcal{S}\mathcal{N}) = \text{cov}(\mathcal{N}) < \text{non}(\mathcal{S}\mathcal{N}) = \text{non}(\mathcal{N}) < \text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) < \text{cof}(\mathcal{M}) = \text{cof}(\mathcal{N}) \).


\textit{E-mail address: miguel.montoya@tuwien.ac.at}

\textit{URL:} https://www.researchgate.net/profile/Miguel_Cardona_Montoya

\textbf{References}


