A Partition Relation Forced by Side Condition Method

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Abstract

We represent a consistency proof of a partition relation studied by S. Todorcevic. We make use of a so-called side condition method. We also report that a type of morass negates this partition relation.

Introduction

By [T], it is consistent that the partition relation $\omega_1 \longrightarrow \big(\omega_1, (\omega_1; \text{fin}\,\omega_1)\big)^2$ holds. This means that for any function $f : [\omega_1]^2 \longrightarrow \{0, 1\}$, either there exists a cofinal 0-homogeneous subset of ω_1 or there exists $(A, \mathcal{B})$ such that

- A is a cofinal subset of ω_1,
- $\mathcal{B}$ consists of finite subsets of ω_1 such that $\min[\mathcal{B}] = \{\min(b) \mid b \in \mathcal{B}\}$ is cofinal in ω_1,
- If (α, b) is such that $\alpha \in A$, $b \in \mathcal{B}$, and $\alpha < \min(b)$, then there exists $\beta \in b$ such that $f\big(\{\alpha, \beta\}\big) = 1$.

By [AM], a new type of iterated forcing with side condition is found. In this paper, we force the partition relation along the line of this new method of iterated forcing. By [M], a type of $(\omega, 1)$-morass is forced. We report that this type of morass negates the partition relation.

The length of the iterated forcing in this paper is restricted to ω_2. In particular, it is the case that $2^\omega = \omega_2$. Is it possible to construct any longer iteration in this context ?

§1. **0-Amalgable Coloring**

We find the following mathematical idea in [T], though it had no name.

Definition. Let $f : [\omega_1]^2 \longrightarrow 2$. We say f is 0-*amalgable*, if for any $(A, \mathcal{B})$ such that

- A is a cofinal subset of ω_1.
- $\mathcal{B} \subset [\omega_1]^{<\omega}$ such that $\min[\mathcal{B}] = \{\min(b) \mid b \in \mathcal{B}\}$ is cofinal in ω_1.

there exists $\delta = \delta_{A\mathcal{B}} < \omega_1$ such that for any $b \in \mathcal{B} \setminus \delta$, there exists $\alpha \in A \cap \delta$ such that for all $\beta \in b$, $f\big(\{\alpha, \beta\}\big) = 0$.

Let $f : [\omega_1]^2 \longrightarrow 2$ be 0-amalgable. Let $\mathcal{B} \subseteq [\omega_1]^{<\omega}$ be such that $\min[\mathcal{B}]$ is cofinal in ω_1. Let κ be a regular cardinal with $\kappa \geq (2^\omega)^+$. In particular, the power set $\mathcal{P}(\omega) \in H_{(2^\omega)^+} \subseteq H_\kappa$.

Lemma. Let $N_1 \in N_2 \in \cdots \in N_k$ be a finite $\in$-chain of countable elementary substructures of $(H_\kappa, \in)$. Let $\mathcal{B} \in N_1$ and $b = \{\beta_1 < \beta_2 < \cdots < \beta_k\} \in \mathcal{B}$ be such that

$$N_1 \cap \omega_1 \leq \beta_1 < N_2 \cap \omega_1 \leq \beta_2 < \cdots < N_k \cap \omega_1 \leq \beta_k.$$

Then there exists a tree $T \in N_1$ such that T consists of sequences of countable ordinals that are $<$-increasing, of length at most k, for each $t \in T$ with its length less than k, $\{\beta < \omega_1 \mid t^\frown\langle\beta\rangle \in T\}$ is cofinal in ω_1, and for each $t \in T$ such that its length equals k, $\text{rang}(t) \in \mathcal{B}$.

Proof. By induction on $k < \omega$.

□

Lemma. Let $N_1 \in N_2 \in \cdots \in N_k$, $\mathcal{B}$, b, and $T \in N_1$ be as above. Let us further assume that $f \in N_1$. Then there exists $\langle\zeta_1, \zeta_2, \cdots, \zeta_k\rangle \in T_k \cap N_1$ such that $f[\{\zeta_1, \zeta_2, \cdots, \zeta_k\} : \{\beta_1, \beta_2, \cdots, \beta_k\}] = \{0\}$ and so $\{\zeta_1, \zeta_2, \cdots, \zeta_k\} \in \mathcal{B} \cap N_1$, where $X : Y = \big\{\{x, y\} \mid x \in X, y \in Y\big\}$ for sets of ordinals $X < Y$.

Proof. We opt to be less formal so that we see better what is actually going on. Let $A_1 = \{\zeta_1 \mid \langle\zeta_1\rangle \in T_1\}$. Then $A_1 \in N_1$ is a cofinal subset of ω_1. Since f is 0-amalgable, there exists $\langle\zeta_1\rangle \in T_1 \cap N_1$ such that

$$f\big(\{\zeta_1, \beta_1\}\big) = 0, \quad f\big(\{\zeta_1, \beta_2\}\big) = 0, \quad \cdots, \quad f\big(\{\zeta_1, \beta_k\}\big) = 0.$$

Let $A_2 = \{\zeta_2 \mid \langle\zeta_1, \zeta_2\rangle \in T_2\}$. Then $A_2 \in N_1$ is a cofinal subset of ω_1. Since f is 0-amalgable, there exists ζ_2 such that $\langle\zeta_1, \zeta_2\rangle \in T_2 \cap N_1$ and

$$f\big(\{\zeta_2, \beta_1\}\big) = 0, \quad f\big(\{\zeta_2, \beta_2\}\big) = 0, \quad \cdots, \quad f\big(\{\zeta_2, \beta_k\}\big) = 0.$$

By repeating this argument k-times, we finally get $\langle\zeta_1, \zeta_2, \cdots, \zeta_k\rangle \in T_k \cap N_1$ such that for all $l = 1, 2, \cdots, k$, we have

$$f\big(\{\zeta_l, \beta_1\}\big) = 0, \quad f\big(\{\zeta_l, \beta_2\}\big) = 0, \quad \cdots, \quad f\big(\{\zeta_l, \beta_k\}\big) = 0.$$

Since $\{\zeta_1, \zeta_2, \cdots, \zeta_k\} \in \mathcal{B} \cap N_1$, we are done.

□

We formulate a second-order treatment of proper posets. Predense subsets are used to formulate generic conditions.

Definition. (Second-order) Let κ be an uncountable regular cardinal and P be a poset such that $P \subseteq H_\kappa$ and P has the κ-cc. Let N be a countable elementary substructure of a relational structure

$$\big(H_\kappa, \in, P, \leq_P, 1_P, H_\kappa \cap V^P, \{(p, \tau, \pi) \mid p \Vdash_P \text{“}\tau = \pi\text{”}\}\big).$$

We say $q \in P$ is (P, N)-generic, if for any predense subset D of P with $D \in N$, $D \cap N$ is predense below q.

Lemma. (Second-order) Let $q \in P$ and $N \prec (H_\kappa, \in, P, \cdots)$ be as above. The following are equivalent.

- q is (P, N)-generic.
- $q \Vdash_P$“$N[\dot{G}] \cap H_\kappa^V = N$”.
- $q \Vdash_P$“$N[\dot{G}] \cap \kappa = N \cap \kappa$”

Here, $N[\dot{G}] = \{\tau_{\dot{G}} \mid \tau \in N \cap V^P\}$.

If $P \in H_\kappa$, then $(P, \leq_P, 1_P) \in N \prec (H_\kappa, \in)$ iff $N \prec (H_\kappa, \in, P, \cdots)$. In this case, $q \in P$ is (P, N)-generic iff for any dense subset $D \in N$ of P, $D \cap N$ is predense below q.

Definition. (Second-order) Let κ be an uncountable regular cardinal and P be a poset such that $P \subseteq H_\kappa$ and P has the κ-cc. Then we say P is proper, if for any $p \in P$, $\{N \in [H_\kappa]^\omega \mid$ there exists $q \leq p$ such that q is (P, N)-generic$\}$ contains a club in $[H_\kappa]^\omega$.

We design a poset with a side condition that forces a generic cofinal 0-homogeneous subset of ω_1. To be proper, the side condition has to be structured. While giving up a preservation of ω_2, we may simply assume that they form an $\in$-chain. This formulation suffices for applying the Proper Forcing Axiom (PFA) or the Bounded Proper Forcing Axiom (BPFA). Here, we present a version of poset that satisfies a reasonable chain condition. To iteratively force later, we implicitly formulate similar, but not exactly the same, posets that are more dependent on objects in the intermediate stages.

Definition. (Two sorted version) Let f be 0-amalgable. Let $p = (\mathcal{M}^p, \mathcal{N}^p, A^p) \in P(f)$, if

- (cover) $\mathcal{M}^p$ is a finite set of countable elementary substructures of a relational structure $(H_\kappa, \in)$.
- (structured) $\mathcal{N}^p \subseteq \mathcal{M}^p$ is a finite f-symmetric system of countable elementary substructures of $(H_\kappa, \in)$. By this we mean
    - (el) If $N \in \mathcal{N}^p$, then $f \in N \prec (H_\kappa, \in)$.

- (ho) If $N_1, N_2 \in \mathcal{N}^p$ with $N_1 =_{\omega_1} N_2$, then there exists a (necessarily unique) isomorpism $\phi_{N_1 N_2}$ from $(N_1, \in)$ to $(N_2, \in)$ such that $\phi_{N_1 N_2}$ is the identity on the intersection $N_1 \cap N_2$, where $X =_{\omega_1} Y$ abbreviates $X \cap \omega_1 = Y \cap \omega_1$.
- (up) If $N_3, N_2 \in \mathcal{N}^p$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \mathcal{N}^p$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$, where $X <_{\omega_1} Y$ abbreviates $X \cap \omega_1 < Y \cap \omega_1$.
- (down) If $N_1, N_2, N_3 \in \mathcal{N}^p$ such that $N_1 =_{\omega_1} N_2$ and $N_3 \in N_1$, then $\phi_{N_1 N_2}(N_3) \in \mathcal{N}^p$.

- A^p is a finite 0-homogeneous subset of ω_1 w.r.t. f.
- (separation) If $N \in \mathcal{N}^p$ and $A^p \setminus N = \{\xi_1 < \xi_2 < \cdots < \xi_k\}$ with $k \geq 2$, then there exists an $\in$-chain $\{M_1 \in M_2 \in \cdots \in M_k\} \subseteq \mathcal{M}^p$ such that
  - $M_1 = N$.
  - $M_1 \cap \omega_1 \leq \xi_1 < M_2 \cap \omega_1 \leq \xi_2 < \cdots < M_k \cap \omega_1 \leq \xi_k$.

  We simply say that an $\in$-chain that starts with N and followed by elements of $\mathcal{M}^p$ separates $A^p \setminus N$.
- For $p, q \in P$, let $q \leq p$ in P, if $\mathcal{M}^q \supseteq \mathcal{M}^p$, $\mathcal{N}^q \supseteq \mathcal{N}^p$ and $A^q \supseteq A^p$.

Notation. Let P be a poset such that $P \subseteq H_\kappa$ and P has the κ-cc. For the sake of concise presentation, P's order relation $\leq$, a greatest element 1, P-names, and relevant forcing relations are omitted in relational structures. Hence we understand that a structure $(H_\kappa, \in, P)$ abbreviates a structure

$$\big(H_\kappa, \in, P, \leq, 1, H_\kappa \cap V^P, \{(p, \tau, \pi) \mid p \Vdash_P \text{“}\tau = \pi\text{”}\} \cap H_\kappa, \cdots\big).$$

Lemma. (1) $P(f) \subset H_\kappa$ has the $(2^\omega)^+$-cc.

(2) $P(f)$ is proper.

(3) There exists $p \in P(f)$ such that $p \Vdash_{P(f)}$“$\dot{A} = \bigcup\{A^p \mid p \in \dot{G}\}$ is a cofinal 0-homogeneous subset of ω_1”.

Proof. For (1): Let $\{p_i \mid i < (2^\omega)^+\}$ be an indexed family of conditions of $P(f)$. Let N_i be a countable elementary substructure of $(H_\kappa, \in)$ such that $f, p_i \in N_i$. By thinning, we may assume that $\{N_i \mid i < (2^\omega)^+\}$ forms a Δ-system and that for any $i < j < (2^\omega)^+$, $A^{p_i} = A^{p_j}$, $(N_i, \in, f, p_i)$ and $(N_j, \in, f, p_j)$ are isomorphic such that the isomorphism is the identity on the intersection $N_i \cap N_j$. Let

$$q = \big(\mathcal{M}^{p_i} \cup \mathcal{M}^{p_j}, \mathcal{N}^{p_i} \cup \mathcal{N}^{p_j}, A^{p_i} \cup A^{p_j}\big).$$

Then $q \in P(f)$ and $q \leq p_i, p_j$ in $P(f)$.

For (2): Let $p \in P(f)$ and $p \in N \prec \big(H_\kappa, \in, f, P(f)\big)$. Let

$$pN = (\mathcal{M}^p \cup \{N\}, \mathcal{N}^p \cup \{N\}, A^p).$$

We show that this pN is $(P(f), N)$-generic. To this end, let $D \in N$ be a predense subset of $P(f)$. Let $q \leq pN$, $d \in D$, $q \leq d$ in P. It suffices to find $h^+ \in P(f)$ and $d' \in D \cap N$ such that $h^+ \leq q, d'$. Let $\sigma_q = A^q \setminus N$.

Case 1. $\sigma_q = \emptyset$: Then $A^q \in N$. We have

$$\big(H_\kappa, \in, f, P(f)\big) \models \text{“There exists } (q', d') \text{ s.t. } q' \text{ in } P(f),\ d' \in D,\ q' \leq d',\ \mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'}, \text{ and } A^{q'} = A^q.\text{”}$$

Hence there exists $(q', d') \in N$ as such. Let

$$h^+ = \big(\mathcal{M}^q \cup \mathcal{M}^{q'} \cup \mathcal{M}^+, \mathcal{N}^q \cup \mathcal{N}^{q'} \cup \mathcal{N}^+, A^q \cup A^{q'}\big),$$

where

$$\mathcal{M}^+ = \big\{\phi_{NN'}(M) \mid M \in \mathcal{M}^{q'}, N' \in \mathcal{N}^q, N' =_{\omega_1} N\big\},$$

$$\mathcal{N}^+ = \{\phi_{NN'}(W) \mid W \in \mathcal{N}^{q'}, N' \in \mathcal{N}^q, N' =_{\omega_1} N\}.$$

Then $h^+ \in P(f)$, $\mathcal{M}^{h^+} = \mathcal{M}^q \cup \mathcal{M}^+$, $\mathcal{N}^{h^+} = \mathcal{N}^q \cup \mathcal{N}^+$, $A^{h^+} = A^q = A^{q'}$, and $h^+ \leq q, q'$.

Case 2. $\sigma_q \neq \emptyset$: Let us define $\mathcal{B}$ such that $\sigma \in \mathcal{B}$, if

- $\sigma \in [\omega_1]^{|\sigma_q|}$.
- There exists (q', d') such that $q' \in P(f)$, $d' \in D$, and
  - $\mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'}$.
  - $A^q \cap N$ gets end-extended by $A^{q'}$.
  - $\sigma = A^{q'} \setminus (A^q \cap N)$.

Then $\sigma_q \in \mathcal{B} \in N$ and $\min(\sigma_q) \geq N \cap \omega_1$. By lemma, there exists $\sigma' \in \mathcal{B} \cap N$ such that $f[\sigma' : \sigma_q] = \{0\}$. Since $\sigma' \in \mathcal{B} \cap N$, there exists $(q', d') \in N$ such that

- $q' \in P(f)$, $d' \in D$.
- $\mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'}$.
- $A^q \cap N$ gets end-extended by $A^{q'}$.
- $\sigma' = A^{q'} \setminus (A^q \cap N)$.

Let

$$h^+ = \big(\mathcal{M}^q \cup \mathcal{M}^{q'} \cup \mathcal{M}^+, \mathcal{N}^q \cup \mathcal{N}^{q'} \cup \mathcal{N}^+, A^q \cup A^{q'}\big),$$

where

$$\mathcal{M}^+ = \{\phi_{NN'}(M) \mid M \in \mathcal{M}^{q'}, N' \in \mathcal{N}^q, N' =_{\omega_1} N\},$$

$$\mathcal{N}^+ = \{\phi_{NN'}(W) \mid W \in \mathcal{N}^{q'}, N' \in \mathcal{N}^q, N' =_{\omega_1} N\}.$$

Then $h^+ \in P(f)$ and $h^+ \leq q, q'$. Hence $D \cap N$ is predense below pN in $P(f)$.

For (3): We make use of the properness of $P(f)$. Take a countable elementary substructure N such that $\dot{A} \in N \prec \big(H_\kappa, \in, f, P(f)\big)$. Let

$$p = (\{N\}, \{N\}, \{N \cap \omega_1\}) \leq (\{N\}, \{N\}, \emptyset) \leq (\emptyset, \emptyset, \emptyset).$$

Then $p \in P(f)$ and p is $(P(f), N)$-generic. Hence $p \Vdash_{P(f)}$ "$N \cap \omega_1 \in \dot{A} \in N[\dot{G}] =_{\omega_1} N$". Hence $p \Vdash_{P(f)}$ "$\dot{A}$ is cofinal below ω_1".

$\square$

We see, say under BPFA, that every 0-amalgable function has a cofinal 0-homogenous subset of ω_1. This expressed as follows.

Corollary. Let us assume BPFA, then $\omega_1 \longrightarrow \big(\omega_1,\ (\omega_1\ ;\ \mathrm{fin}\,\omega_1)\big)^2$ holds.

Proof. Let $f : [\omega_1]^2 \longrightarrow 2$. If f is 0-amalgable, then apply BPFA to $P(f)\lceil p$. We get a cofinal 0-homogeneous subset A of ω_1 w.r.t. f.

If f is not 0-amalgable, then there exists $(A, \mathcal{B})$ such that A is a cofinal subset of ω_1, $\mathcal{B} \subseteq [\omega_1]^{<\omega}$ such that $\min[\mathcal{B}]$ is cofinal in ω_1, and that for any $\delta < \omega_1$, there exists $\sigma_\delta \in \mathcal{B}$ such that $\min(\sigma_\delta) \geq \delta$ and for all $\alpha \in A \cap \delta$, there exists $\beta \in \sigma_\delta$ with $f\big(\{\alpha, \beta\}\big) = 1$.

Let $C = \{\gamma < \omega_1 \mid \gamma$ is a limit ordinal, and $\forall \delta < \gamma, \sigma_\delta < \gamma\}$. Then C is a closed cofinal subset of ω_1. Let A' be a cofinal subset of A such that for any distinct two elements $\alpha_1 < \alpha_2$ of A', there exists $\gamma \in C$ such that $\alpha_1 < \gamma < \alpha_2$ and so $\sigma_{\alpha_1} < \alpha_2 \leq \sigma_{\alpha_2}$. Let $\mathcal{B}' = \big\{\{\alpha\} \cup \sigma_\alpha \mid \alpha \in A'\big\}$. Then $\mathcal{B}' \subset [\omega_1]^{<\omega}$ is an uncountable disjoint finite subsets of ω_1 such that for any (α, σ) such that $\alpha \in A'$, $\sigma \in \mathcal{B}'$, and $\alpha < \min(\sigma)$, there exists $\beta \in \sigma$ such that $f\big(\{\alpha, \beta\}\big) = 1$.

$\square$

§2. **Iteration**

Let $\kappa = \omega_2$ in this section. We start with the ground model V where CH and $2^{\omega_1} = \omega_2$ hold. Let $\Phi : \omega_2 \longrightarrow H_{\omega_2}$ be a bookkeeping function. We force $\omega_1 \longrightarrow \big(\omega_1,\ (\omega_1; \operatorname{fin}\omega_1)\big)^2$ over V. We use Aspero-Mota type iteration such that for all $\alpha < \omega_2$ and for all $p \in P_{\alpha+1}$, we demand that

$$\mathcal{N}^p(\alpha) = \{N \in \mathcal{N}^p \mid NS^p\alpha\}$$

is $\mathcal{P}_{\leq\alpha}$-symmetric. This in turn limits lengths of iteration at the longest to ω_2.

Definition. Let $\alpha \leq \omega_2$. Let $\langle P_\eta \mid \eta < \alpha\rangle$ be a sequence of posets such that for each $\eta < \alpha$, $P_\eta \subseteq H_{\omega_2}$ and has the ω_2-cc. Let us form a relational structure

$$\mathcal{P}_{<\alpha} = \big(H_{\omega_2}, \in, \Phi, \langle\langle P_\eta \mid \eta < \alpha\rangle\rangle\big),$$

where we code

$$\langle\langle P_\eta \mid \eta < \alpha\rangle\rangle = \big\{(\eta, p) \mid \eta < \alpha, p \in P_\eta\big\} \subseteq H_{\omega_2}.$$

This structure includes the P_η's order relations, the greatest elements, the names $V^{P_\eta} \cap H_{\omega_2}$, relevant forcing relations, say, $\big\{(p, \tau, \pi) \mid p \Vdash_{P_\eta} \text{“}\tau = \pi\text{”}\big\} \cap H_{\omega_2}$, but omitted to mention for the sake of conciseness.

Definition. We recursively construct a sequence of posets $\langle P_\alpha \mid \alpha \leq \omega_2\rangle$. Let $\alpha \leq \omega_2$ and suppose we have constructed $\langle P_\eta \mid \eta < \alpha\rangle$ such that for each $\eta < \alpha$

- $P_\eta \subseteq H_{\omega_2}$ such that (ob), (symmetric), $(*)$, and (g) for η are satisfied.
- P_η has the ω_2-cc.
- P_η is proper in the following manner.

  (Lemma (main) for η) If $q \in P_\eta$ such that $S^q(N) = N \cap \eta$ and N gives a rise to a countable elementary substructure of $\mathcal{P}_{\leq\eta}$ that is written as

  $$N \prec \mathcal{P}_{\leq\eta} = \big(H_{\omega_2}, \in, \Phi, P_\eta, \langle\langle P_\zeta \mid \zeta < \eta\rangle\rangle\big),$$

  then q is (P_η, N)-generic.

  (Lemma (N-extension) for η) Let $p \in P_\eta$, $p \in N \prec \mathcal{P}_{<\eta}$, and

  $$q = (\mathcal{N}^p \cup \{N\}, S^p \cup \{(N, \zeta) \mid \zeta \in N \cap \eta\}, A^p),$$

  then $q \in P_\eta$, $q \leq p$ in P_η, and $S^q(N) = N \cap \eta$. Hence, if furthermore $N \prec \mathcal{P}_{\leq\eta}$, then q is (P_η, N)-generic.

Now we form P_α. Let $p = (\mathcal{N}^p, S^p, A^p) = (\mathcal{N}, S, A) \in P_\alpha$, if the following (ob), (symmetric), $(*)$, and (g) are satisfied.

(ob)

- $\mathcal{N}$ is a finite set of countable elementary substructures of a relational structure

$$(H_{\omega_2}, \in, \Phi)$$

such that the following (el), (ho), (up), and (down) are satisfied. We may refer $\mathcal{N}$ as a finite Φ-symmetric system.

- (el) If $N \in \mathcal{N}$, then $\big(N, \in \cap (N \times N), \Phi \cap N\big)$ (this simply denoted as either $(N, \in, \Phi)$ or N) is a countable elementary substructure of $(H_{\omega_2}, \in, \Phi)$. Denoted as

$$N \prec (H_{\omega_2}, \in, \Phi).$$

• (ho) If $N_1, N_2 \in \mathcal{N}$ with $N_1 \cap \omega_1 = N_2 \cap \omega_1$ (this denoted as $N_1 =_{\omega_1} N_2$), then there exists a necessarily unique isomorphism

$$\phi_{N_1 N_2} : (N_1, \in, \Phi) \longrightarrow (N_2, \in, \Phi)$$

such that $\phi_{N_1 N_2}$ is the identity on the intersection $N_1 \cap N_2$.

• (up) If $N_3, N_2 \in \mathcal{N}$ with $N_3 \cap \omega_1 < N_2 \cap \omega_1$ (this denoted as $N_3 <_{\omega_1} N_2$), then there exists $N_1 \in \mathcal{N}$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

• (down) If $N_3, N_2, N_1 \in \mathcal{N}$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$, then $\phi_{N_1 N_2}(N_3) \in \mathcal{N}$.

$$\begin{array}{ccc} N_1 & \sim & N_2 \\ | & & | \\ N_3 & \sim & \phi_{N_1 N_2}(N_3) \end{array}$$

• S is a relation from $\mathcal{N}$ to α (i.e., $S \subseteq \mathcal{N} \times \alpha$) such that for all $N \in \mathcal{N}$, $S(N) = \{\eta < \alpha \mid NS\eta\}$ is an initial segment of $\alpha \cap N$.

• A is finite relation from α to ω_1. For $\xi < \alpha$, write $A(\xi) = \{\zeta \mid \xi A \zeta\}$ (intended as a finite 0-homogeneous set w.r.t. a bookkept P_ξ-name $\Phi(\xi)$ s.t. $p\lceil\xi \Vdash_{P_\xi}$ "$\Phi(\xi) : [\omega_1]^2 \longrightarrow 2$ is 0-amalgable".)

(symmetric) For all $\eta < \alpha$, we demand $\mathcal{N}(\eta) = \{N \in \mathcal{N} \mid NS\eta\}$ is $\mathcal{P}_{\leq\eta}$-symmetric. By this we mean the following (el), (ho), (up), and (down).

(el) If $N \in \mathcal{N}(\eta)$ (this denoted as $NS\eta$), then

$$N \prec \mathcal{P}_{\leq\eta} = \big(H_{\omega_2}, \in, \Phi, P_\eta, \langle\langle P_\xi \mid \xi < \eta\rangle\rangle\big).$$

(ho) If $N_1 S\eta$, $N_2 S\eta$ s.t. $N_1 \cap \omega_1 = N_2 \cap \omega_1$ (this denoted as $N_1 S\eta =_{\omega_1} N_2 S\eta$), then

$$\big(N_1, \in, \Phi, P_\eta, \langle\langle P_\xi \mid \xi < \eta\rangle\rangle\big) \sim \big(N_2, \in, \Phi, P_\eta, \langle\langle P_\xi \mid \xi < \eta\rangle\rangle\big)$$

by the isomorphism $\phi_{N_1 N_2}$.

(up) If $N_3 S\eta$, $N_2 S\eta$ s.t. $N_3 \cap \omega_1 < N_2 \cap \omega_1$ (this denoted as $N_3 S\eta <_{\omega_1} N_2 S\eta$), there exists $N_1 \in \mathcal{N}(\eta)$ such that $N_3 \in N_1$ and $N_1 S\eta =_{\omega_1} N_2 S\eta$.

(down) If $N_1 S\eta =_{\omega_1} N_2 S\eta$, $N_3 S\eta$ and $N_3 \in N_1$, then $\phi_{N_1 N_2}(N_3) S\eta$.

$$\begin{array}{ccc} N_1 S\eta & \sim & N_2 S\eta \\ | & & | \\ N_3 S\eta & \sim & \phi_{N_1 N_2}(N_3) S\eta \end{array}$$

(∗) If $\xi \in \mathrm{dom}(A)$ and $p\lceil\xi \in P_\xi$, then $\Phi(\xi)$ is a P_ξ-name such that $p\lceil\xi \Vdash_{P_\xi}$ "$\Phi(\xi) : [\omega_1]^2 \longrightarrow 2$ is 0-amalgable and $A(\xi)$ is 0-homo w.r.t. $\Phi(\xi)$", where for any triple $p = (\mathcal{N}, S, A)$ and any ordinal ξ, we define

$$p\lceil\xi = \big(\mathcal{N}\lceil\xi, S\lceil\xi, A\lceil\xi\big),$$

$$\mathcal{N}\lceil\xi = \mathcal{N} \text{ (unchanged)},$$

$$S\lceil\xi = \big\{(N, \eta) \mid \eta < \xi, (N, \eta) \in S\big\},$$

$$A\lceil\xi = \big\{(\eta, \zeta) \mid \eta < \xi, (\eta, \zeta) \in A\big\}.$$

(g) If $\xi \in \mathrm{dom}(A)$, $N \in \mathcal{N}(\xi)$ and $|A(\xi) \setminus N| \geq 2$, then an $\in$-chain that starts with N and followed by elements of $\mathcal{M}(\xi) = \{M \in \mathcal{N} \mid MS\,[M \cap \xi], M \prec \mathcal{P}_{\leq\xi}\}$ separetes $A(\xi) \setminus N$, where $MS\,[M \cap \xi]$ abbreviate $M \cap \xi \subseteq \{\eta \mid MS\eta\}$. Notice that $\mathcal{M}(\xi) \supseteq \mathcal{N}(\xi)$.

For $p, q \in P_\alpha$, $q \leq p$ in P_α, if $\mathcal{N}^q \supseteq \mathcal{N}^p$, $S^q \supseteq S^p$, and $A^q \supseteq A^p$.

The following two lemmas confirm that we indeed have an iterated forcing of length $\alpha \leq \omega_2$.

Lemma. (Projection) Let $\rho < \alpha \leq \omega_2$. Then $p \in P_\alpha \mapsto p\lceil\rho \in P_\rho$ is a well-defined function.

(order-preserving) If $q \leq p$ in P_α, then $q\lceil\rho \leq p\lceil\rho$ in P_ρ.
(reduction) If $p \in P_\alpha$ and $h \leq p\lceil\rho$ in P_ρ, then $hp = (\mathcal{N}^h, S^h \cup S^p, A^h \cup A^p) \in P_\alpha$, $hp \leq p$ in P_α and $hp\lceil\rho = h$.

Proof. To show that $p\lceil\rho \in P_\rho$ and $hp \in P_\alpha$, we check the list of items (ob), (symmetric), (∗), and (g) corresponding to P_ρ and P_α, respectively. The checking is routine.

□

Lemma. (Complete suborder) Let $\rho < \alpha \leq \omega_2$. Then P_ρ is a complete suborder of P_α.

(suborder) $P_\rho \subset P_\alpha$ and for $p, q \in P_\rho$, $q \leq p$ in P_ρ iff $q \leq p$ in P_α.
(incompatibility) If $p, q \in P_\rho$, then p, q are incompatible in P_ρ iff p, q are incompatible in P_α.
(self-comparison) If $p \in P_\alpha$, then $p \leq p\lceil\rho$ in P_α.
(reduction) If $p \in P_\alpha$ and $h \leq p\lceil\rho$ in P_ρ, then h and p are compatible in P_α.
(Generic Objects) Let G_α be P_α-generic over V. Let $G_\alpha\lceil\rho = \{p\lceil\rho \mid p \in G_\alpha\}$. Then

$$G_\alpha \cap P_\rho = G_\alpha\lceil\rho$$

is P_ρ-generic over V.

Proof. To show that any $p \in P_\rho$ is in P_α, we check the list of items (ob), (symmetric), (∗), and (g) corresponding to P_α. The point is that any initial segment of $X \cap \rho$ is so in $X \cap \alpha$. The checking ought to be trivial.

□

We now establish the ω_2-cc that assures a second-order treatment of the poset P_α.

Lemma. $P_\alpha \subset H_{\omega_2}$ has the ω_2-cc.

Proof. Let $\langle p_i \mid i < \omega_2 \rangle$ be a sequence of conditions of P_α. For each $i < \omega_2$, pick a countable elementary substructure

$$N_i \prec \mathcal{P}_{<\alpha} = \big(H_{\omega_2}, \in, \Phi, \langle\langle P_\eta \mid \eta < \alpha \rangle\rangle\big)$$

such that $p_i \in N_i$, where we code

$$\langle\langle P_\eta \mid \eta < \alpha \rangle\rangle = \big\{(\eta, p) \mid \eta < \alpha, p \in P_\eta\big\}.$$

This structure includes the order relations, the greatest elements, the names, and relevant forcing relations, but omitted to mention by our convention.

By CH, we may assume that $\{N_i \mid i < \omega_2\}$ forms a Δ-system. We may also assume that N_i and N_j are isomorphic such that the isomorphism is the identity on the intersection $N_i \cap N_j$. Let

$$h^+ = \big(\mathcal{N}^{p_i} \cup \mathcal{N}^{p_j}, S^{p_i} \cup S^{p_j}, A^{p_i} \cup A^{p_j}\big).$$

Then $h^+ \in P_\alpha$ and $h^+ \leq p_i, p_j$ in P_α.

□

To further study P_α, we similarly form a relational structure with a distinguished predicate for P_α

$$\mathcal{P}_{\leq\alpha} = \big(H_{\omega_2}, \in, \Phi, P_\alpha, \langle\langle P_\eta \mid \eta < \alpha \rangle\rangle\big).$$

Here is a typical use of this structure. It turns out that it has a natural "expansion" in any generic extension. Notice that G_α is available as a predicate.

Lemma. Let us write $\omega_2 = \kappa$ for short. If $N \prec \mathcal{P}_{\leq\alpha}$ and G_α is P_α-generic over V, then in $V[G_\alpha]$

$$N[G_\alpha] \prec \big(H_\kappa^{V[G_\alpha]}, \in, H_\kappa^V, \Phi, G_\alpha, P_\alpha, \langle\langle P_\eta \mid \eta < \alpha\rangle\rangle\big).$$

Proof. (Out-line) For any formula $\phi(v_1, \cdots, v_k)$, find a formula $\phi^*(v, v_1, \cdots, v_k)$ such that for any $p \in P_\alpha$ and any $\tau_1 \cdots, \tau_k \in V^{P_\alpha}$,

$$p \Vdash_{P_\alpha} \text{“}\big(H_\kappa^{V[\dot{G}_\alpha]}, \in, H_\kappa^V, \Phi, \dot{G}_\alpha, P_\alpha, \langle\langle P_\eta \mid \eta < \alpha\rangle\rangle\big) \models \text{“}\phi(\tau_1, \cdots, \tau_k)\text{””}$$

iff

$$\mathcal{P}_{\leq\alpha} = \big(H_\kappa, \in, \Phi, P_\alpha, \langle\langle P_\eta \mid \eta < \alpha\rangle\rangle\big) \models \text{“}\phi^*(p, \tau_1, \cdots, \tau_k)\text{”}.$$

A crux of the matter is the maximal principle of P_α-names in H_κ and an observation

$$1 \Vdash_{P_\alpha} \text{“}\big(H_\kappa^{V[\dot{G}_\alpha]}, \in, \cdots\big) \models \text{“}\exists x \forall y\, \big(\phi(y, \tau_1, \cdots, \tau_k) \Longrightarrow \phi(x, \tau_1, \cdots, \tau_k)\big)\text{””}.$$

Hence no N-generic conditions are necessary. It is the same as in proper forcing.

□

Similarly, we have

Lemma. Let us write $\omega_2 = \kappa$ for short. Let $N \prec \mathcal{P}_{\leq\alpha}$ and $\rho \in N \cap \alpha$. Then

$$N \prec \big(H_\kappa, \in, \Phi, P_\rho, P_\alpha, \langle\langle P_\eta \mid \eta < \alpha\rangle\rangle\big).$$

Let G_ρ is P_ρ-generic over V. Then in $V[G_\rho]$

$$N[G_\rho] \prec \big(H_\kappa^{V[G_\rho]}, \in, H_\kappa^V, \Phi, G_\rho, P_\rho, P_\alpha, \langle\langle P_\eta \mid \eta < \alpha\rangle\rangle\big).$$

□

Lemma. (N-extension) Let $p \in P_\alpha$ and $p \in N \prec P_{<\alpha}$. Let

$$pN = \big(\mathcal{N}^p \cup \{N\}, S^p \cup \{(N, \eta) \mid \eta \in N \cap \alpha\}, A^p\big).$$

Then $pN \in P_\alpha$ such that $pN \leq p$ and $S^{pN}(N) = N \cap \alpha$.

Proof. Just check the list of items. Note that if $\eta < \alpha$ and $\eta \in N \prec \mathcal{P}_{<\alpha}$, then $\Phi(\eta) \in N \prec \mathcal{P}_{\leq\eta}$ holds.

□

Here is a crutial technical lemma to form an amalgamation of q and q' with a common head h in all cases (successor stages, all limit stages) in the proof of lemma (main). This is where the following functions, provided $\kappa = \omega_2$:

$$\underline{\text{If } N =_{\omega_1} N' \text{ and } \alpha \in N \cap N' \cap \omega_2, \text{ then } N \cap \alpha = N' \cap \alpha.}$$

The proof is basically a diagram-chase. However, it involves many cases to argue. There are, say, 3×3 cases for (up) and $3 \times 3 \times 3$ cases for (down). Some of them have 3 subcases to argue. In the chase, we take compositions of isomorphisms. And it is important to remember the isomorphisms are unique no matter how we compose them.

Lemma. (Technical) Let $q \in P_\alpha$, $S^q(N) = N \cap \alpha$, and $\rho \in N \cap \alpha$. Let $q' \in P_\alpha \cap N$. Let $h \leq q\lceil\rho, q'\lceil\rho$ in P_ρ. Let us denote the strong-sup of $N \cap \alpha$ by α_N. Namely, α_N is the least ordinal η such that $N \cap \alpha \subseteq \eta$. And so $\rho < \alpha_N \leq \alpha$. Define

$$S = S^h \cup S^q \cup S^{q'} \cup S^+,$$

where $S^+ = \{(\phi_{NN'}(W), \eta) \mid \rho \leq \eta < \alpha_N, N' \in \mathcal{N}^q(\eta), N' =_{\omega_1} N, W \in \mathcal{N}^{q'}(\eta)\}$.

$$\begin{array}{ccc} NS^q\eta & \sim & N'S^q\eta \\ | & & | \\ WS^{q'}\eta & \sim & \phi_{NN'}(W)S^+\eta \end{array}$$

Then

- $S(X) = \{\zeta \mid XS\zeta\}$ is an initial segment of $X \cap \alpha$ for each $X \in \mathcal{N}^h$.
- $S\lceil\rho = S^h$.
- For each $\eta < \alpha$, $\{X \in \mathcal{N}^h \mid XS\eta\}$ is $\mathcal{P}_{\leq\eta}$-symmetric. Namely, it satisfies (el), (ho), (up), and (down) with respect to
$$\big(H_{\omega_2}, \in, \Phi, P_\eta, \langle\langle P_\xi \mid \xi < \eta\rangle\rangle\big).$$

Proof. Since it is a lengthy diagram-chase, here we just observe that for any $X \in \mathcal{N}^h$, $S(X)$ is an initial segment of $X \cap \alpha$. To this end, let $XS\eta$ and $\zeta \in X \cap \eta$. We want to show $XS\zeta$. We specifically pick up the case $XS^+\eta$. If $\zeta < \rho$, then

$$\begin{array}{ccc} NS^h\zeta & \sim & N'S^h\zeta \\ | & & | \\ WS^h\zeta & \sim & X = \phi_{NN'}(W)S^h\zeta \end{array}$$

and so $XS\zeta$.

If $\rho \leq \zeta$, then

$$\begin{array}{ccc} NS^q\zeta & \sim & N'S^q\zeta \\ | & & | \\ WS^{q'}\zeta & \sim & X = \phi_{NN'}(W)S^+\zeta \end{array}$$

and so so $XS\zeta$.

□

Lemma. (MAIN) If $p \in P_\alpha$, $S^p(N) = N \cap \alpha$, and $N \prec P_{\leq\alpha}$, then p is (P_α, N)-generic.

Proof. Let us write $\omega_2 = \kappa$ for short. Let $D \subseteq P_\alpha$ be predense in P_α. We want to show that $D \cap N$ is predense below p. Let $q \leq p$ in P_α and $d \in D$ such that $q \leq d$. It suffices to find $q' \in P_\alpha \cap N$, $d' \in D \cap N$, and $h^+ \in P_\alpha$ such that $h^+ \leq q, q'$ and $q' \leq d'$. We argue by induction on α.

Case 1. $\alpha = 0$: Notice that $q = (\mathcal{N}^q, \emptyset, \emptyset) \in P_0$. We have

$$\mathcal{P}_{\leq 0} = \big(H_\kappa, \in, \cdots, P_0, \cdots\big) \models \text{“There exists } (q', d') \text{ s.t. } q' \text{ in } P_0, d' \in D, q' \leq d' \text{ in } P_0,\ \mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'}\text{”}.$$

Now
$$D, \mathcal{N}^q \cap N \in N \prec P_{\leq 0}.$$

Hence there exists $(q', d') \in N$ as such. Let

$$h^+ = \big(\mathcal{N}^q \cup \mathcal{N}^{q'} \cup \mathcal{N}^+, \emptyset, \emptyset\big),$$

where $\mathcal{N}^+ = \{\phi_{NN'}(M) \mid N' \in \mathcal{N}^q, N' =_{\omega_1} N,\ M \in \mathcal{N}^{q'}\}$. Then $h^+ \in P_0$, $\mathcal{N}^{h^+} = \mathcal{N}^q \cup \mathcal{N}^+$, and $h^+ \leq q, q'$.

Case 2. $\alpha = \alpha + 1$: We assume that $\alpha \in \mathrm{dom}(A^q)$, since case $\alpha \notin \mathrm{dom}(A^q)$ is similar and simpler. Let G_α be P_α-generic over V with $q\lceil\alpha \in G_\alpha$. We argue in $V[G_\alpha]$. Since $S^{q\lceil\alpha}(N) = N \cap \alpha$ and $N \prec P_{\leq\alpha}$, we have $N[G_\alpha] \cap H^V_\kappa = N$ by induction.

Subcase. $A^q(\alpha) \subset N$: Then $A^q(\alpha) \in N$. In $\big(H^{V[G_\alpha]}_\kappa, \in, \cdots, G_\alpha, P_\alpha, P_{\alpha+1}, \cdots\big)$, there exists (q', d') such that

$$q' \text{ in } P_{\alpha+1}, q'\lceil\alpha \text{ in } G_\alpha, d' \in D, q' \leq d' \text{ in } P_{\alpha+1}, \mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'},\ \alpha \in \mathrm{dom}(A^{q'}),\ A^{q'}(\alpha) = A^q(\alpha)\text{”}.$$

Since

$$\alpha, D, \mathcal{N}^q \cap N, A^q(\alpha) \in N[G_\alpha] \prec \big(H_\kappa^{V[G_\alpha]}, \in, \cdots, G_\alpha, P_\alpha, P_{\alpha+1}, \cdots\big),$$

there exists $(q', d') \in N$ as such. Let $h \in G_\alpha$ such that $h \leq q\lceil\alpha, q'\lceil\alpha$. Let

$$h^+ = \big(\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}\big),$$

where $S^+ = \big\{(\phi_{NN'}(W), \alpha) \mid N' \in \mathcal{N}^q(\alpha), N' =_{\omega_1} N, W \in \mathcal{N}^{q'}(\alpha)\big\}$.

Then $h^+ \in P_{\alpha+1}$, $h^+\lceil\alpha = h$, and $h^+ \leq q, q'$. Notice that the strong-sup of $(\alpha+1) \cap N$ satisfies $(\alpha+1)_N = \alpha+1$, as $\alpha \in N$. Hence if we set $\rho = \alpha \in N \cap (\alpha+1)$, then the interval of stages $\big[\rho, (\alpha+1)_N\big) \cap N = [\alpha, \alpha+1) = \{\alpha\}$ holds in lemma (technical).

Subcase. $A^q(\alpha) \not\subset N$: Let $\sigma_q = \{\xi_1 < \xi_2 < \cdots < \xi_k\} = A^q(\alpha) \setminus N$. Let us define $\mathcal{B}$ such that $\sigma \in \mathcal{B}$, if there exists (q', d') such that

- $q' \in P_{\alpha+1}$, $q'\lceil\alpha \in G_\alpha$, $d' \in D$, and $q' \leq d'$.
- $\mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'}$.
- $A^{q'}(\alpha)$ end-extends $A^q(\alpha) \cap N$.
- $\sigma = A^{q'(\alpha)} \setminus \big(A^q(\alpha) \cap N\big)$.

Then $\Phi(\alpha)$ is a P_α-name such that $\Phi(\alpha)_{G_\alpha} : [\omega_1]^2 \longrightarrow 2$ is 0-amalgable, $\Phi(\alpha)_{G_\alpha}, \mathcal{B} \in N[G_\alpha]$ and $\sigma_q \in \mathcal{B} \setminus N[G_\alpha]$. If $k \geq 2$, then σ_q is separated by an $\in$-chain starting with N and followed by elements of $\{M \mid M \in \mathcal{N}^q, MS^q[M \cap \alpha], M \prec \mathcal{P}_{\leq\alpha}\}$. Hence σ_q is separated by an $\in$-chain starting with $N[G_\alpha]$ and followed by elements of $\{M[G_\alpha] \mid M \in \mathcal{N}^q, MS^q[M \cap \alpha], M \prec \mathcal{P}_{\leq\alpha}\}$. Hence there exists $\sigma' = \{\zeta_1 < \zeta_2 < \cdots < \zeta_k\} \in \mathcal{B} \cap N$ such that $\Phi(\alpha)_{G_\alpha}[\sigma' : \sigma_q] = \{0\}$. Since $\sigma' \in \mathcal{B} \cap N$, there exists $(q', d') \in N$ as such. Let $h \in G_\alpha$ such that $h \leq q\lceil\alpha, q'\lceil\alpha$. Let

$$h^+ = \big(\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}\big),$$

where $S^+ = \big\{(\phi_{NN'}(W), \alpha) \mid N' \in \mathcal{N}^q(\alpha), N' =_{\omega_1} N,\ W \in \mathcal{N}^{q'}(\alpha)\big\}$.

Then $h^+ \in P_{\alpha+1}$,

$$A^{h^+} = A^h \cup \{(\alpha, i) \mid i \in A^q(\alpha) \cup A^{q'}(\alpha)\},$$

$h^+\lceil\alpha = h$, and $h^+ \leq q', q$.

Case 3. $\mathrm{cf}(\alpha) = \omega$: Let $\rho \in N \cap \alpha$ be such that $\mathrm{dom}(A^q) \subset \rho$. Then $q\lceil\rho \in P_\rho$, $S^{q\lceil\rho}(N) = N \cap \rho$, and $N \prec P_{\leq\rho}$. Let G_ρ be P_ρ-generic over V with $q\lceil\rho \in G_\rho$. We argue in $V[G_\rho]$. By induction, we have $N[G_\rho] \cap H_\kappa^V = N$. Since $\big(H_\kappa^{V[G_\rho]}, \in, \cdots, G_\rho, P_\rho, P_\alpha, \cdots\big)$ knows

$$\text{“}\exists\ (q', d') \text{ s.t. } q' \text{ in } P_\alpha, q'\lceil\rho \text{ in } G_\rho, d' \in D, q' \leq d', \mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'}, \mathrm{dom}(A^{q'}) \subset \rho\text{”},$$

and

$$\rho, D, \mathcal{N}^q \cap N \in N[G_\rho] \prec \big(H_\kappa^{V[G_\rho]}, \in, \cdots, G_\rho, P_\rho, P_\alpha, \cdots\big),$$

there exists $(q', d') \in N$ as such. Let $h \in G_\rho$ with $h \leq q'\lceil\rho, q\lceil\rho$. Let

$$h^+ = \big(\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}\big),$$

where $S^+ = \big\{(\phi_{NN'}(W), \eta) \mid \rho \leq \eta < \alpha, N' \in \mathcal{N}^q(\eta), N' =_{\omega_1} N, W \in \mathcal{N}^{q'}(\eta)\big\}$.

Then $h^+ \in P_\alpha$, $A^{h^+} = A^h$, $h^+\lceil\rho = h$, and $h^+ \leq q, q'$. Note that the strong-sup of $N \cap \alpha$ satisfies $\alpha_N = \alpha$.

Case 4. $\mathrm{cf}(\alpha) = \omega_1$: Then the strong-sup of $N \cap \alpha$ satisfies $\alpha_N = \sup(N \cap \alpha)$ and $\alpha_N < \alpha$. Let $\rho \in N \cap \alpha$ be such that

(1) If $Y <_{\omega_1} N$, $Y \in \mathcal{N}^q$, then $N \cap Y \cap \alpha < \rho$. (Reduces the number of subcases in case 2. Not essential in this paper.)
(2) $\mathrm{dom}(A^q) \cap \alpha_N \subset \rho$.

Then $q\lceil\rho \in P_\rho$, $S^{q\lceil\rho}(N) = N \cap \rho$, and $N \prec P_{\leq\rho}$. Let G_ρ be P_ρ-generic over V with $q\lceil\rho \in G_\rho$. We argue in $V[G_\rho]$. By induction, we have $N[G_\rho] \cap H_\kappa^V = N$. Since

$$\big(H_\kappa^{V[G_\rho]}, \in, \cdots, G_\rho, P_\rho, P_\alpha, \cdots\big) \models \text{"}\exists\,(q', d') \text{ s.t. } q' \text{ in } P_\alpha, q'\lceil\rho \text{ in } G_\rho, d' \in D, q' \leq d', \mathcal{N}^q \cap N \subseteq \mathcal{N}^{q'},\text{"}$$

$$\rho, D, \mathcal{N}^q \cap N \in N[G_\rho] \prec \big(H_\kappa^{V[G_\rho]}, \in, \cdots, G_\rho, P_\rho, P_\alpha, \cdots\big),$$

there exists $(q', d') \in N$ as such. Let $h \in G_\rho$ with $h \leq q\lceil\rho, q'\lceil\rho$. Let

$$h^+ = \big(\mathcal{N}^h, S^h \cup S^q \cup S^{q'} \cup S^+, A^h \cup A^q \cup A^{q'}\big),$$

where $S^+ = \big\{(\phi_{NN'}(W), \eta) \mid \rho \leq \eta < \alpha_N,\ N' \in \mathcal{N}^q(\eta),\ N' =_{\omega_1} N,\ W \in \mathcal{N}^{q'}(\eta)\big\}$. Then $h^+ \in P_\alpha$, $h^+\lceil\rho = h$, and $h^+ \leq q', q$.

We provide some details on (g) to check $h^+ \in P_\alpha$. Let $\xi \in \mathrm{dom}(A^{h^+})$, $XS^{h^+}\xi$, and $|A^{h^+}(\xi) \setminus X| \geq 2$. We want to show that $A^{h^+}(\xi) \setminus X$ gets separated by an $\in$-chain that starts with X and followed by elements of $\mathcal{M}^{h^+}(\xi) = \big\{M \mid MS^{h^+}[M \cap \xi], M \prec \mathcal{P}_{\leq\xi}\big\}$.

Case 1. $\xi < \rho$: Then $\xi \in \mathrm{dom}(A^h)$ and $XS^h\xi$. Hence an $\in$-chain that starts with X and followed by elements of $\mathcal{M}^h(\xi)$ separates $A^h(\xi) = A^{h^+}(\xi)$. But $\mathcal{M}^h(\xi) = \mathcal{M}^{h^+}(\xi)$.

Case 2. $\rho \leq \xi < \alpha_N$: Then $\xi \in \mathrm{dom}(A^{q'})$ and $A^{h^+}(\xi) = A^{q'}(\xi)$. Either $XS^{q'}\xi$ or $XS^+\xi$ holds.

Subcase. $XS^{q'}\xi$: Then an $\in$-chain that starts with X and followed by elements of $\mathcal{M}^{q'}(\xi)$ separates $A^{q'}(\xi)$. But $A^{h^+}(\xi) = A^{q'}(\xi)$ and $\mathcal{M}^{h^+}(\xi) \supseteq \mathcal{M}^{q'}(\xi)$.

Subcase. $XS^+\xi$: Let $N' =_{\omega_1} N$, $WS^{q'}\xi$, and $\phi_{NN'}(W) = X$. Then an $\in$-chain that starts with W and followed by elements of $\mathcal{M}^{q'}(\xi)$ separates $A^{q'}(\xi)$. Map this $\in$-chain by $\phi_{NN'}$. Then we have an $\in$-chain that starts with X and followed by elements of $\mathcal{M}^{h^+}(\xi)$.

Case 3. $\alpha_N \leq \xi < \alpha$: Then $\xi \in \mathrm{dom}(A^q)$ and an $\in$-chain that starts with X and followed by elements of $\mathcal{M}^q(\xi)$ separates $A^q(\xi)$. But $A^{h^+}(\xi) = A^q(\xi)$ and $\mathcal{M}^{h^+}(\xi) \supseteq \mathcal{M}^q(\xi)$.

□

To show that for any 0-amalgable f, there exists an uncountable 0-homogeneous set in the final model $V[G_{\omega_2}]$, we prepare the following.

Lemma. Let $p \in P_{\omega_2}$ and $\dot{f}$ be a P_{ω_2}-name such that $p \Vdash_{P_{\omega_2}}$ "$\dot{f} : [\omega_1]^2 \longrightarrow 2$ is 0-amalgable". Then there exists (α, q) such that $p, q \in P_{\alpha+1}$, $q \leq p$ in $P_{\alpha+1}$, $\Phi(\alpha)$ is a P_α-name, and $q \Vdash_{P_{\omega_2}}$ "$(\dot{A}_\alpha)_{(\dot{G}_{\omega_2} \cap P_{\alpha+1})}$ is an uncountable 0-homogeneous set w.r.t. $\dot{f} = \Phi(\alpha)_{(\dot{G}_{\omega_2} \cap P_\alpha)} : [\omega_1]^2 \longrightarrow 2$ that is 0-amalgable", where $\Vdash_{P_{\alpha+1}}$ "$\dot{A}_\alpha = \bigcup\{A^r(\alpha) \mid r \in \dot{G}_{\alpha+1}\}$".

Proof. Since H_{ω_2} is book-kept by $\Phi : \omega_2 \longrightarrow H_{\omega_2}$, we have $\alpha < \omega_2$ such that $p \in P_\alpha$, $\Phi(\alpha)$ is a P_α-name, and $p \Vdash_{P_{\omega_2}}$ "$\dot{f} = \Phi(\alpha)_{(\dot{G}_{\omega_2} \cap P_\alpha)}$". Since $p \Vdash_{P_{\omega_2}}$ "$\dot{f} : [\omega_1]^2 \longrightarrow 2$ is 0-amalgable", by going down-ward, we have $p \Vdash_{P_\alpha}$ "$\Phi(\alpha) : [\omega_1]^2 \longrightarrow 2$ is 0-amalgable". Let $p, \dot{A}_\alpha \in M \prec \mathcal{P}_{\leq\alpha+1}$. Let

$$q = \big(\mathcal{N}^p \cup \{M\}, S^p \cup \{(M, \eta) \mid \eta \in M \cap (\alpha+1)\}, A^p \cup \{(\alpha, M \cap \omega_1)\}\big).$$

Then $q \in P_{\alpha+1}$ and $q \leq p$ in $P_{\alpha+1}$. We observe this q works. Since $MS^q\big[M \cap (\alpha+1)\big]$ and $M \prec \mathcal{P}_{\leq\alpha+1}$, we know that q is $(P_{\alpha+1}, M)$-generic. Hence $q \Vdash_{P_{\alpha+1}}$ "$M[\dot{G}_{\alpha+1}] \cap \omega_1 = M \cap \omega_1 \in \dot{A}_\alpha \in M[\dot{G}_{\alpha+1}]$". Hence

$q \Vdash_{P_{\alpha+1}}$ "$\dot{A}_\alpha$ is an uncountable 0-homogeneous set w.r.t. $\Phi(\alpha)$". By going up-ward, $q \Vdash_{P_{\omega_2}}$ "$(\dot{A}_\alpha)_{(\dot{G}_{\omega_2} \cap P_{\alpha+1})}$ is an uncountable 0-homogeneous set w.r.t. $\dot{f}$".

□

For an $(\omega, 1)$-morass that exists in ZFC, see [V]. For a weakly nice $(\omega, 1)$-morass that can be forced to exist, see [M]. We report the following.

Theorem. If a weakly nice $(\omega, 1)$-morass exists, then the partition relation $\omega_1 \longrightarrow (\omega_1,\ (\omega_1\,;\,\text{fin}\,\omega_1))^2$ fails.

Inspecting a proof of a theorem of Hajnal, say, a proof on page 141 in [HL], we also report the following.

Theorem. (CH) The partition relation $\omega_1 \longrightarrow (\omega_1,\ (\omega_1\,;\,\text{fin}\,\omega_1))^2$ fails.

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