TODORČEVIĆ'S FRAGMENTS OF MARTIN'S AXIOM AND VARIATIONS OF UNIFORMIZATIONS OF LADDER SYSTEM COLORINGS

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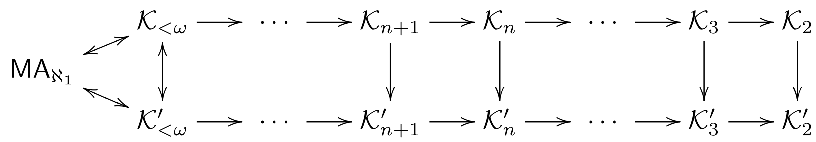
Introduction

In this article, we introduce parametrized versions of Devlin-Shelah's assertion about uniformizations of ladder system colorings. For a subset $\mathcal{S}$ of the power set of $\omega_1 \cap \mathsf{Lim}$, $\mathsf{U}(\mathcal{S})$ is the assertion that, for any coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$ of the ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$, there exist $S \in \mathcal{S}$ and a function from ω_1 into ω which uniformizes the restricted coloring $\langle f_\alpha : \alpha \in S \rangle$. Devlin-Shelah's original assertion is the assertion $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$. This follows from $\mathsf{MA}_{\aleph_1}$, and is equivalent to the existence of non-free Whitehead group. The axiom $\mathcal{K}'_2$, which is one of Todorčević's fragments of Martin's Axiom, implies the assertion $\mathsf{U}([\omega_1 \cap \mathsf{Lim}]^{\aleph_1})$. Todorčević-Veličković pointed out that $\mathcal{K}'_4$ implies $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$. We show that the axiom $\mathcal{K}_3$ implies the assertion $\mathsf{U}(\mathsf{stat})$, and similarly, $\mathcal{K}'_4$ implies $\mathsf{U}(\mathsf{club})$. By Larson-Todorčević's result, it is shown that it is consistent that $\mathsf{U}([\omega_1 \cap \mathsf{Lim}]^{\aleph_1})$ holds and $\mathsf{U}(\mathsf{club})$ fails

1. Background and preliminaries

1.1. **Todorčević's fragments of Martin's Axiom.** In 1980s, Todorčević investigated Martin's Axiom from the view point of Ramsey theory, and introduced the following fragments of Martin's Axiom: $\mathcal{K}_{<\omega}$ denotes the assertion that every ccc forcing notion has precaliber $\aleph_1$; $\mathcal{K}_n$ denotes the assertion that every ccc forcing notion has the property K_n; $\mathcal{K}'_{<\omega}$ denotes the assertion that every ccc partition $K_0 \cup K_1 = [\omega_1]^{<\aleph_0}$ has an uncountable K_0-homogeneous set; $\mathcal{K}'_n$ denotes the assertion that every ccc partition $K_0 \cup K_1 = [\omega_1]^n$ has an uncountable K_0-homogeneous set.[*1] The following diagram is a summary of implications of these fragments of $\mathsf{MA}_{\aleph_1}$. The triangle on the left side of the diagram is Todorčević-Veličković theorem [11, Corollary 2.7].



It is not known whether any other implications in this diagram hold under ZFC.

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[*1] They are defined by Todorčević in several papers. In [5, Definition 4.9] and [11, §2], $\mathcal{K}_n$'s are defined as assertions for ccc forcing notions, however in [6, §4] and [8, §7], $\mathcal{K}_n$'s are defined as assertions for ccc partitions. To separate them, we use the notations as above. These notations are same to ones in [12].

Larson-Todorčević introduced a property of ccc partitions on $[\omega_1]^2$, called the rectangle refining property, and introduced the assertion $\mathcal{K}'_2(\text{rec})$ that every partition on $[\omega_1]^2$ with the rectangle refining property has an uncountable homogeneous set. Larson-Todorčević proved that it is consistent that a Suslin tree can force $\mathcal{K}'_2(\text{rec})$ [6]. More precisely, they introduced the assertion $\mathsf{MA}_{\aleph_1}(S)$ which asserts that there exists a coherent Suslin tree S such that the forcing axiom for all ccc forcing notions which preserves S to be Suslin holds, and showed that, under $\mathsf{MA}_{\aleph_1}(S)$, S forces $\mathcal{K}'_2(\text{rec})$. In [13], the author developed their result to $\mathcal{K}_{<\omega}(\text{rec})$ in some sense, that is, under $\mathsf{MA}_{\aleph_1}(S)$, S forces $\mathcal{K}_{<\omega}(\text{rec})$ in some sense.

1.2. **Uniformizations of ladder system colorings.** The notion of uniformization of a ladder system coloring was introduced by Devlin-Shelah, in order to study the non-free Whitehead groups [2]. The following (4) is a parametrized version of their assertion introduced in [2, 5.2 Theorem].

Definition 1.1. (1) A ladder system on ω_1 is a sequence $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ such that, for each $\alpha \in \omega_1 \cap \mathsf{Lim}$, C_α is an unbounded subset of α and the order type of C_α is ω.

(2) A coloring of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ is a sequence $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ such that, for each $\alpha \in \omega_1 \cap \mathsf{Lim}$, f_α is a function from C_α into ω.

(3) For each coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ and a subset S of ω_1, a function φ from ω_1 into ω uniformizes the restricted coloring $\langle f_\alpha : \alpha \in S\rangle$ if for every $\alpha \in S$, f_α and $\varphi \restriction C_\alpha$ are almost equal, that is, the set

$$\{\xi \in C_\alpha : f_\alpha(\xi) \neq \varphi(\xi)\}$$

is finite.

(4) For a subset $\mathcal{S}$ of the power set of $\omega_1 \cap \mathsf{Lim}$, $\mathsf{U}(\mathcal{S})$ is the assertion that, for any coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ of a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$, there exist $S \in \mathcal{S}$ and a function from ω_1 into ω which uniformizes the restricted coloring $\langle f_\alpha : \alpha \in S\rangle$.

Devlin-Shelah introduced the assertion $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$ in [2, 5.2 Theorem]. They pointed out that $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$ is a sufficient condition of the existence of a non-free Whitehead group [2, §6]. Moreover, Eklof-Shelah showed that $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$ is equivalent to the existence of a non-free Whitehead group [4], [3, Ch. XIII].

For any nonstationary subset N of $\omega_1 \cap \mathsf{Lim}$, one can prove the assertion $\mathsf{U}(\{N\})$ from ZFC [3, Ch. II Exercise 20 (a)]. Devlin-Shelah showed that $\mathsf{MA}_{\aleph_1}$ implies $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$ [2, 5.2 Theorem]. It follows from their proof that $\mathcal{K}'_2$ implies $\mathsf{U}([\omega_1 \cap \mathsf{Lim}]^{\aleph_1})$. In [13], the author proved that $\mathcal{K}'_2(\text{rec})$ implies the assertion $\mathsf{U}([\omega_1 \cap \mathsf{Lim}]^{\aleph_1})$. Todorčević-Veličković pointed out that $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$ is followed from $\mathcal{K}'_4$ [11, §2].

On the other hand, Larson-Todorčević essentially proved that a Suslin tree forces the negation of the assertion $\mathsf{U}(\mathsf{club})$, where club stands for the set of all club subsets of $\omega_1 \cap \mathsf{Lim}$ [5, Theorem 6.2]. Therefore, it is proved that under $\mathsf{MA}_{\aleph_1}(S)$, S forces that $\mathsf{U}([\omega_1 \cap \mathsf{Lim}]^{\aleph_1})$ holds and $\mathsf{U}(\mathsf{club})$ fails. The author does not know whether $\mathsf{U}(\mathsf{club})$ implies $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim}\})$.

In the next section, it is proved that $\mathcal{K}_3'$ implies $\mathsf{U}(\mathsf{stat})$, where stat stands for the set of all stationary subsets of $\omega_1 \cap \mathsf{Lim}$. By a similar argument, it is proved that $\mathcal{K}_4'$ implies $\mathsf{U}(\mathsf{stat})$.

2. $\mathcal{K}_3'$ implies $\mathsf{U}(\mathsf{stat})$

In this section, we prove the title of the section. Here, for ordinals α, β and γ, we write $\{\alpha, \beta\}_<$, or $\{\alpha, \beta, \gamma\}_<$, when $\alpha < \beta$, or $\alpha < \beta < \gamma$.

Let $\langle e_\alpha : \alpha \in \omega_1 \rangle$ be a sequence such that

- each e_α is an injective function from α into ω, and
- $\langle e_\alpha : \alpha \in \omega_1 \rangle$ is a coherent sequence, that is, for each $\alpha, \beta \in \omega_1$ with $\alpha < \beta$, the set
$$\{\xi \in \alpha : e_\beta(\xi) \neq e_\alpha(\xi)\}$$
is finite [7, 9, 10].

Let $\langle r_\alpha : \alpha \in \omega_1 \rangle$ be an injective sequence of members of the set ${}^\omega 2$. For each $\alpha, \beta \in \omega_1$ with $\alpha < \beta$, and each $n \in \omega$, define

$$\sigma(\alpha, \beta) := \min \{n \in \omega : r_\alpha(n) \neq r_\beta(n)\},$$

$$F_n(\beta) := \{\xi \in \beta : e_\beta(\xi) \leq n\} \cup \{\beta\},$$

and

$$b(\alpha, \beta) := \min \left(F_{\sigma(\alpha,\beta)}(\beta) \setminus \alpha\right).$$

(See e.g. [7, §6].) Then, similar to [11, Theorem 2.1], the following is proved.

Lemma 2.1. *Let X be an uncountable subset of ω_1 and M a countable elementary submodel of $H_{\aleph_2}$ such that the set*

$$\{\langle e_\alpha, : \alpha \in \omega_1 \rangle, \langle r_\alpha : \alpha \in \omega_1 \rangle, X\}$$

belongs to the model M. Then, for any $\beta \in X \setminus M$, there exists $\alpha \in X \cap M$ such that $b(\alpha, \beta) = \omega_1 \cap M$ and, for any $\xi \in \beta \setminus \alpha$, $e_\beta(\xi) = e_{\omega_1 \cap M}(\xi)$.

Proof. Let $\beta \in X \cap M$. Since the sequence $\langle r_\alpha : \alpha \in \omega_1 \rangle$ is injective and X is uncountable, we can find $\alpha' \in X \setminus (\{\beta\} \cup M)$ such that

$$r_{\alpha'} \restriction e_\beta(\omega_1 \cap M) = r_\beta \restriction e_\beta(\omega_1 \cap M).$$

(Here, we do not mind whether α' is less than β or not.) We should notice that there exist uncountably many such α'. Define $n := \sigma(\beta, \alpha')$ (or $\sigma(\alpha', \beta)$). Then we notice that $e_\beta(\omega_1 \cap M) \leq n$. Since the function e_β is injective and the sequence $\langle e_\alpha : \alpha \in \omega_1 \rangle$ is coherent, we can find $\gamma \in \omega_1 \cap M$ such that for any $\xi \in \beta \setminus \gamma$,

$$e_{\omega_1 \cap M}(\xi) = e_\beta(\xi) > n.$$

By elementarity of M, we can find $\alpha \in (X \cap M) \setminus \gamma$ that is a copy of α', which means here that $\alpha \geq \gamma$ and $r_\alpha \restriction (n+1) = r_{\alpha'} \restriction (n+1)$. Then

$$\sigma(\alpha, \beta) = \sigma(\beta, \alpha') = n.$$

Therefore $b(\alpha, \beta) = \omega_1 \cap M$. □

The following is the main preliminary lemma of the proof.

Lemma 2.2. *Let* $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ *be a ladder system,* $\{\eta^\alpha_n : n \in \omega\}$ *the increasing enumeration of* C_α *for each* $\alpha \in \omega_1 \cap \mathsf{Lim}$*,* I *an uncountable subset of the set* $[\omega_1]^{<\aleph_0}$*,* κ *an enough large regular cardinal,* M *a countable elementary submodel of* H_κ *that contains the set*

$$\{\langle e_\alpha : \alpha \in \omega_1\rangle, \langle r_\alpha : \alpha \in \omega_1\rangle, \langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle, I, H_{\aleph_2}\},$$

and $\tau \in I \setminus M$*. Then there exists* $J \in [I]^{\aleph_1} \cap M$ *such that, for every* $\nu \in J \cap M$*,*

(1) ν *is an end-extension of* $\tau \cap M$*, that is,* $\tau \cap M \subseteq \nu$ *and* $\min(\nu \setminus (\tau \cap M)) > \max(\tau \cap M)$*,*

(2) *for any* $\{\alpha, \beta\}_<$ *and* $\{\gamma, \delta\}_<$ *in the set* $[\nu \cup \tau]^2$*, if* $\{\alpha, \beta, \gamma, \delta\} \not\subseteq \nu$ *and* $\{\alpha, \beta, \gamma, \delta\} \not\subseteq \tau$ *and* $\{b(\alpha, \beta), b(\gamma, \delta)\} \subseteq \mathsf{Lim}$*, then*

$$\max\Big(\Big\{\eta^{b(\alpha,\beta)}_n : n \geq e_\beta(\alpha)\Big\} \cap \Big\{\eta^{b(\gamma,\delta)}_n : n \geq e_\delta(\gamma)\Big\}\Big) < \max\Big(\bigcup\{C_{b(\alpha',\beta')} \cap M : \{\alpha', \beta'\}_< \in [\tau]^2, b(\alpha', \beta') \geq \omega_1 \cap M\}\Big).$$

Proof. By simplifying the argument, for each $\gamma \in \omega_1 \setminus \mathsf{Lim}$, we define $C_\gamma := \{\gamma - 1\}$ and $\eta^\gamma_n := \gamma - 1$ for every $n \in \omega$. Define

$$L_0 := \big\{b(\alpha, \beta) : \{\alpha, \beta\}_< \in [\tau \setminus M]^2\big\}$$

and

$$L_1 := \Big\{\min\big(F_{\sigma(\alpha,\beta)}(\beta) \cap \big[(\omega_1 \cap M) + 1, \beta\big]\big) : \beta \in \tau \setminus M, \alpha \in \tau \cap \beta\Big\},$$

where

$$\big[(\omega_1 \cap M) + 1, \beta\big] := \{\xi \in \beta + 1 : (\omega_1 \cap M) + 1 \leq \xi\}.$$

We notice that

$$(L_0 \cup L_1) \cap M = \emptyset.$$

Take a number $\overline{m} \in \omega$ such that

- for any $\delta \in L_0 \cup L_1$, $\{\eta^\delta_n : n \geq \overline{m}\} \cap M = \emptyset$,
- the set $\big\{\{\eta^\delta_n : n \geq \overline{m}\} : \delta \in \{\omega_1 \cap M\} \cup L_0 \cup L_1\big\}$ is pairwise disjoint,
- for any $\delta \in \{\omega_1 \cap M\} \cup L_0 \cup L_1$ and any $\{\alpha, \beta\}_< \in [\tau]^2$, if $b(\alpha, \beta) \in \mathsf{Lim} \setminus \{\delta\}$, then
$$\Big\{\eta^{b(\alpha,\beta)}_n : n \geq e_\beta(\alpha)\Big\} \cap \{\eta^\delta_n : n \geq \overline{m}\} = \emptyset,$$
- $\overline{m} > \max\{e_\beta(\alpha), e_\beta(\omega_1 \cap M) : \beta \in \tau \setminus M, \alpha \in \tau \cap \beta\}$, and
- $\overline{m} > \max\big\{\sigma(\alpha, \beta) : \{\alpha, \beta\}_< \in [\tau]^2\big\}$.

Next, take an ordinal $\overline{\xi} \in \omega_1 \cap M$ such that

- $\tau \cap M \subseteq \overline{\xi}$,
- for any $\{\alpha, \beta\}_< \in [\tau]^2$,
  if $b(\alpha, \beta) < \omega_1 \cap M$, then $b(\alpha, \beta) < \overline{\xi}$, and
  if $b(\alpha, \beta) \in \mathsf{Lim} \setminus M$, then
$$C_{b(\alpha,\beta)} \cap M = C_{b(\alpha,\beta)} \cap \overline{\xi},$$
  and
- for any $\beta \in \tau \setminus M$ and any $\zeta \in (\omega_1 \cap M) \setminus \overline{\xi}$, $e_\beta(\zeta) > \overline{m}$.

Then we notice that

$$\max\Big(\bigcup\{C_{b(\alpha,\beta)} \cap M : \{\alpha, \beta\}_< \in [\tau]^2, b(\alpha, \beta) \geq \omega_1 \cap M\}\Big) < \overline{\xi}.$$

Let $\{\beta^\tau_i : i \in n\}$ be the increasing enumeration of the set $\tau \setminus M$. Define

$J := \Big\{\nu \in I : \quad \bullet\ \nu \cap \overline{\xi} = \tau \cap M,$

- $\left|\nu \setminus \overline{\xi}\right| = n$, and let $\{\beta_i^\nu : i \in n\}$ be the increasing enumeration of the set $\nu \setminus \overline{\xi}$,
- for each $\alpha \in \tau \cap M$ and $i \in n$,
$$C_{b(\alpha,\beta_i^\nu)} \cap \overline{\xi} = C_{b(\alpha,\beta_i^\tau)} \cap \overline{\xi},$$
- for each $\{i,j\}_< \in [n]^2$, $C_{b(\beta_i^\nu,\beta_j^\nu)} \cap \overline{\xi} = C_{b(\beta_i^\tau,\beta_j^\tau)} \cap \overline{\xi}$, and
- for each $i \in n$, $r_{\beta_i^\nu} \restriction \overline{m} = r_{\beta_i^\tau} \restriction \overline{m} \Big\}$.

Since $\tau \in J \in M$ and $\tau \notin M$, J is uncountable. Moreover, we notice that, for any $\nu \in J$,

$$\overline{m} > \max\left\{\sigma(\alpha,\beta) : \{\alpha,\beta\}_< \in [\nu \cup \tau]^2\right\}.$$

Let $\nu \in J$. Show that ν satisfies the conditon (2) of the lemma.

Let $\alpha \in \tau \cap M$ $(= \nu \cap M)$ and $i \in n$. Then

$$C_{b(\alpha,\beta_i^\tau)} \cap \overline{\xi} = C_{b(\alpha,\beta_i^\nu)} \cap \overline{\xi}.$$

Moreover,

either both $b(\alpha,\beta_i^\tau) < \omega_1 \cap M$ and $b(\alpha,\beta_i^\tau) < \overline{\xi}$ hold,
or both $b(\alpha,\beta_i^\tau) \in \mathsf{Lim} \setminus M$ and $C_{b(\alpha,\beta_i^\tau)} \cap M = C_{b(\alpha,\beta_i^\tau)} \cap \overline{\xi}$ hold.

Therefore, for any α, $\alpha' \in \tau \cap M$ and any $i, i' \in n$ with $\langle \alpha, i\rangle \neq \langle \alpha', i'\rangle$, the pair of the sets $\{\alpha, \beta_i^\nu\}$ and $\{\alpha', \beta_{i'}^\tau\}$ satisfies the condition (2).

Let $\{i,j\} \in [n]^2$. Then

$$C_{b(\beta_i^\nu,\beta_j^\nu)} \cap \overline{\xi} = C_{b(\beta_i^\tau,\beta_j^\tau)} \cap \overline{\xi}.$$

Therefore, by a similar observation in the previous paragraph, for any $\{i,j\} \in [n]^2$, the pair of the sets $\left\{\beta_i^\nu, \beta_j^\nu\right\}$ and $\left\{\beta_i^\tau, \beta_j^\tau\right\}$ satisfies the condition (2).

Let $i \in n$. Then, for any $\zeta \in [\beta_i^\nu, \omega_1 \cap M)$,

$$e_{\beta_i^\tau}(\zeta) > \overline{m} \geq \sigma(\beta_i^\nu, \beta_i^\tau).$$

Moreover, in this case,

$$\sigma(\beta_i^\nu, \beta_i^\tau) \geq e_{\beta_i^\tau}(\omega_1 \cap M).$$

Hence then, $b(\beta_i^\nu, \beta_i^\tau) = \omega_1 \cap M$ and

$$\left\{\eta_n^{b(\beta_i^\nu,\beta_i^\tau)} : n \geq e_{\beta_i^\tau}(\beta_i^\nu)\right\} \subseteq \left\{\eta_n^{\omega_1 \cap M} : n \geq \overline{m}\right\}.$$

Therefore, by the third condition of the number $\overline{m}$, for any $\{\alpha,\beta\}_< \in [\tau]^2$ and any $i \in n$, the pair of the sets $\{\alpha,\beta\}$ and $\{\beta_i^\nu, \beta_i^\tau\}$ satisfies the condition (2).

Let $\{i,j\} \in [n]^2$. Then, by the previous observation,

$$b(\beta_i^\nu, \beta_j^\tau) \begin{cases} = \omega_1 \cap M & \text{if } \sigma(\beta_i^\nu, \beta_j^\tau) \geq e_{\beta_j^\tau}(\omega_1 \cap M), \\ \in L_1 & \text{otherwise.} \end{cases}$$

Therefore, for any $\{\alpha,\beta\}_< \in [\tau]^2$ and any $\{i,j\} \in [n]^2$, the pair of the sets $\{\alpha,\beta\}$ and $\left\{\beta_i^\nu, \beta_j^\tau\right\}$ satisfies the condition (2). $\square$

In the following proof, for each $\tau \in [\omega_1]^{<\aleph_0}$, define

$$L(\tau) := \left\{b(\alpha,\beta) : \{\alpha,\beta\}_< \in [\tau]^2\right\} \cap \mathsf{Lim}.$$

$L(\tau)$ is a finite set of ordinals. For each $\tau \in [\omega_1]^{<\aleph_0}$, m_τ denotes the size of τ, and let $\{\beta_i^\tau : i \in m_\tau\}$ be the increasing enumeration of τ.

Theorem 2.3. $\mathcal{K}_3'$ *implies* $\mathsf{U(stat)}$.

Proof. Let $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ be a ladder system, and $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ a coloring of the ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$. Define the set K_0 that consists of all sets $\{\alpha, \beta, \gamma\}_<$ in the set $[\omega_1]^3$ with the property that the set

$$\left(f_{b(\alpha,\beta)} \restriction \left\{\eta_n^{b(\alpha,\beta)} : n \geq e_\beta(\alpha)\right\}\right) \cup \left(f_{b(\alpha,\gamma)} \restriction \left\{\eta_n^{b(\alpha,\gamma)} : n \geq e_\gamma(\alpha)\right\}\right)$$

forms a function. We show that K_0 is a ccc partition.

Let $I \in \left[[\omega_1]^{<\aleph_0}\right]^{\aleph_1}$ be an uncountable set of finite K_0-homogeneous sets, and M a countable elementary submodel of H_κ that contains the set

$$\{\langle e_\alpha : \alpha \in \omega_1\rangle, \langle r_\alpha : \alpha \in \omega_1\rangle, \langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle, \langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle, I, H_{\aleph_2}\}.$$

By elementarity of M, we can take $\tau \in I \setminus M$ such that $\omega_1 \cap M \notin L(\tau)$, and can take $\overline{\eta} \in \omega_1 \cap M$ such that

$$\bigcup_{\delta \in L(\tau)} (C_\delta \cap M) \subseteq \overline{\eta}.$$

Define the subset I' of the set I that consists of all sets ν in I such that

- $m_\nu = m_\tau$,
- for any $\{i,j\}_< \in [m_\tau]^2$,
$$b(\beta_i^\nu, \beta_j^\nu) \in \mathsf{Lim} \iff b(\beta_i^\tau, \beta_j^\tau) \in \mathsf{Lim},$$
and
- for any $\{i,j\}_< \in [m_\tau]^2$, whenever $b(\beta_i^\sigma, \beta_j^\sigma) \in \mathsf{Lim} \setminus M$,
$$f_{b(\beta_i^\sigma, \beta_j^\sigma)} \restriction \overline{\eta} = f_{b(\beta_i^\tau, \beta_j^\tau)} \restriction \overline{\eta}.$$

Then, since $\tau \in I' \in M$ and $\tau \notin M$, I' is uncountable. By applying I' and τ to **Lemma 2.2**, we obtain $J \in [I']^{\aleph_1} \cap M$ that satisfies the condition in the lemma. Then we can conclude that, for each $\nu \in J$, $\nu \cup \tau$ is K_0-homogeneous.

By $\mathcal{K}_3'$, there exists an uncountable K_0homogeneous subset X of ω_1. Take a continuous $\in$-chain $\langle M_\xi : \xi \in \omega_1\rangle$ of countable elementary submodels of $H_{\aleph_2}$ such that M_0 contains the set

$$\{\langle e_\alpha : \alpha \in \omega_1\rangle, \langle r_\alpha : \alpha \in \omega_1\rangle, \langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle, X\}.$$

Then the set $D := \{\omega_1 \cap M_\xi : \xi \in \omega_1\}$ is club in ω_1. For each $\xi \in \omega_1$, by **Lemma 2.1**, there are $\beta_\xi \in X \setminus M_\xi$ and $\alpha_\xi \in X \cap M_\xi$ such that $b(\alpha_\xi, \beta_\xi) = \omega_1 \cap M_\xi$. Then there are $\overline{\alpha} \in \omega_1$ and a subset Γ of ω_1 such that

- for every $\xi \in \Gamma$, $\alpha_\xi = \overline{\alpha}$, and
- the set $S := \{b(\overline{\alpha}, b_\xi) : \xi \in \Gamma\}$ is a stationary subset of D.

Then, since X is K_0-homogeneous, the set

$$\bigcup_{\xi \in \Gamma} \left(f_{b(\overline{\alpha},\beta_\xi)} \restriction \left\{\eta_n^{b(\overline{\alpha},\beta_\xi)} : n \geq e_{\beta_\xi}(\overline{\alpha})\right\}\right)$$

forms a function, and uniformizes the restricted coloring $\langle f_\delta : \delta \in S\rangle$. □

Remark 2.4. It follows from the proof of the previous theorem that the forcing notion of finite K_0-homogeneous sets in the proof satisfies the property $\mathsf{R}_{1,\aleph_1}$ [12, 14], and so satisfies Chodounsky-Zapletal's Y-cc [1].

Remark 2.5. For a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$ and a coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$ of the ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$, define the set K_0 that consists of all sets τ in the set $[\omega_1]^4$ with the property that, for any $\{\{\alpha, \beta\}_<, \{\gamma, \delta\}_<\} \in \left[[\tau]^2\right]^2$, the set

$$\left(f_{b(\alpha,\beta)} \restriction \left\{\eta_n^{b(\alpha,\beta)} : n \geq e_\beta(\alpha)\right\}\right) \cup \left(f_{b(\gamma,\delta)} \restriction \left\{\eta_n^{b(\gamma,\delta)} : n \geq e_\delta(\gamma)\right\}\right)$$

forms a function. As in the previous proof, it also follows from **Lemma 2.2** that K_0 is a ccc partition. By the previous proof, we notice that an uncountable K_0-homogeneous set produces a club subset D of ω_1 as above and a function that uniformizes the restricted coloring $\langle f_\delta : \delta \in D \rangle$. It concludes that $\mathcal{K}'_4$ implies $\mathsf{U}(\mathsf{club})$.

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