Besov 空間の基礎と積公式について[1]
Navier-Stokes 方程式の定常問題への応用[2]
Besov 空間における熱半群の $L^p - L^q$ 型評価について[3]

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1 Introduction to the Besov space and Leibniz rule

Let us recall homogeneous and inhomogeneous Besov spaces. For that purpose, we first introduce the Littlewood-Paley decomposition of functions defined on $\mathbb{R}^n$ in terms with the partition $\{\varphi_j\}_{j=-\infty}^{\infty}$ of unity in the Fourier variables. We take $\phi \in C_0^\infty(\mathbb{R}^n)$ in such a way that supp $\phi = \{\xi \in \mathbb{R}^n; \frac{1}{2} \le |\xi| \le 2\}$ satisfying $\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1$ for all $\xi \neq 0$. The functions φ_j are defined as $\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi), j \in \mathbb{Z}$, where $\mathcal{F}$ denotes the Fourier transform. Let ψ be as $\mathcal{F}\psi(\xi) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j}\xi)$. For $1 \le p \le \infty$ and $s \in \mathbb{R}$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by $\dot{B}^s_{p,q} \equiv \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ with the seminorm

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left\{\sum_{j=-\infty}^{\infty} (2^{sj}\|\varphi_j * f\|_{L^p})^q\right\}^{\frac{1}{q}} & \text{for } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\varphi_j * f\|_{L^p} & \text{for } q = \infty, \end{cases}$$

where $\mathcal{P}$ is the set of polynomials in $\mathbb{R}^n$. We also define the corresponding inhomogeneous Besov space $B^s_{p,q}$ by $B^s_{p,q} \equiv \{f \in \mathcal{S}'; \|f\|_{B^s_{p,q}} < \infty\}$ with the norm

$$\|f\|_{B^s_{p,q}} = \begin{cases} \|\psi * f\|_{L^p} + \left\{\sum_{j=0}^{\infty} (2^{sj}\|\varphi_j * f\|_{L^p})^q\right\}^{\frac{1}{q}} & \text{for } 1 \le q < \infty, \\ \|\psi * f\|_{L^p} + \sup_{j\in\mathbb{N}} 2^{sj}\|\varphi_j * f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

For more precise, see e.g., Bergh–Löfström [2]. The following lemma is a fundamental property of Besov spaces.

Proposition 1.1 (i) *If $q_1 \le q_2$, then it holds that $\dot{B}^s_{p,q_1} \subset B^s_{p,q_2}$ for all $1 \le p \le \infty$ and $s \in \mathbb{R}$.*
(ii) *It holds the continuous continuous embedding*

$$\dot{B}^s_{p,1} \subset \dot{H}^s_p \subset \dot{B}^s_{p,\infty}$$

for all $1 \le p \le \infty$ and $s \in \mathbb{R}$, where

$$\dot{H}^s_p \equiv \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{H}^s_p} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < \infty\}.$$

[1]清水扇丈氏（京都大）と金子健太氏（早稲田大）との共同研究 [8].
[2]清水扇丈氏（京都大）と金子健太氏（早稲田大）との共同研究 [8].
[3]清水扇丈氏（京都大）と岡田晃氏（京都大）との共同研究 [10], [11], [12].

If $s_0 \neq s_1$, *we have*

$$(\dot{H}_p^{s_0}, \dot{H}_p^{s_1})_{\theta,q} = \dot{B}^s_{p,q} \quad \textit{for } 1 \le p, q \le \infty \textit{ and } 0 < \theta < 1,$$

where $s = (1-\theta)s_0 + s_1\theta$.

(iii) *If* $s > 0$, *then it we have that*

$$B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}$$

for all $1 \le p, q \le \infty$.

We next consider the embedding theorem.

Proposition 1.2 *Let* $1 \le p \le p_1 \le \infty$, *and* $s_1, s_2 \in \mathbb{R}$ *satisfy*

$$\frac{n}{p} - s = \frac{n}{p_1} - s_1.$$

Let $1 \le q \le q_1 \le \infty$. *Then it holds that*

$$B^s_{p,q} \subset B^{s_1}_{p_1,q_1}, \quad \dot{B}^s_{p,q} \subset \dot{B}^{s_1}_{p_1,q_1}.$$

Finally, we consider the Leibnitz rule in the homogeneous Besov space.

Lemma 1.1 *([8, Proposition 2.2])* (i) *Let* $1 \le p, q \le \infty$, $s > 0$, $\alpha > 0$ *and* $\beta > 0$. *Assume that* $1 \le p_1, p_2, \tilde{p_1}, \tilde{p_2} \le \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p_1}} + \frac{1}{\tilde{p_2}}$. *If* $f \in \dot{B}^{s+\alpha}_{p_1,q} \cap \dot{B}^{-\beta}_{\tilde{p_1},\infty}$ *and* $g \in \dot{B}^{-\alpha}_{p_2,\infty} \cap \dot{B}^{s+\beta}_{\tilde{p_2},q}$, *then we have* $fg \in \dot{B}^s_{p,q}$ *with the estimate*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C(\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{B}^{-\beta}_{\tilde{p_1},\infty}} \|g\|_{\dot{B}^{s+\beta}_{\tilde{p_2},q}}) \tag{1.1}$$

where $C = C(p, p_1, p_2, \tilde{p_1}, \tilde{p_2}, q, s, \alpha, \beta)$.

(ii) *Let* $1 \le p, q \le \infty$ *and* $s > 0$. *Assume that* $1 \le p_1, p_2, \tilde{p_1}, \tilde{p_2} \le \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p_1}} + \frac{1}{\tilde{p_2}}$. *If* $f \in \dot{B}^s_{p_1,q} \cap L^{\tilde{p_1}}$ *and* $g \in L^{p_2} \cap \dot{B}^s_{\tilde{p_2},q}$, *then we have* $fg \in \dot{B}^s_{p,q}$ *with the estimate*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C(\|f\|_{\dot{B}^s_{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{\tilde{p_1}}} \|g\|_{\dot{B}^s_{\tilde{p_2},q}}) \tag{1.2}$$

where $C = C(p, p_1, p_2, \tilde{p_1}, \tilde{p_2}, q, s)$.

Proof. (i) We make use of the following paraproduct formula of fg due to Bony [3]. Our method is related to Christ-Weinstein [4, Proposition 3.3] and Kozono-Shimada [7, Lemma 2.1].

$$\begin{aligned} f \cdot g &= \sum_{k=-\infty}^{\infty} (\varphi_k * f)(P_k g) + \sum_{k=-\infty}^{\infty} (P_k f)(\varphi_k * g) + \sum_{k=-\infty}^{\infty} \sum_{|l-k| \le 2} (\varphi_k * f)(\varphi_l * g) \\ &=: h_1 + h_2 + h_3, \end{aligned} \tag{1.3}$$

where $P_k g = \sum_{l=-\infty}^{k-3} \varphi_l * g$. We first consider the case $1 \le q < \infty$. Since

$$\begin{aligned} &\text{supp } \mathcal{F}((\varphi_k * f)(P_k g)) \subset \{\xi \in \mathbb{R}^n; \in 2^{k-2} \le |\xi| \le 2^{k+2}\}, \\ &\text{supp } \mathcal{F}\varphi_j = \{\xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1}\}, \end{aligned}$$

we have that

$$
\begin{aligned}
\|h_1\|_{\dot{B}^s_{p,q}} &= \left\{ \sum_{j=-\infty}^{\infty} (2^{sj}\|\varphi_j * h_1\|_{L^p})^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{k=-\infty}^{\infty} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{|k-j|\le 2} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}}.
\end{aligned}
$$

Since $\varphi_j(x) = 2^{jn}(\mathcal{F}^{-1}\phi)(2^j x)$ for all $j \in \mathbb{Z}$, it holds by the Hausdorff-Young and the Hölder inequalities that

$$
\|\varphi_j * ((\varphi_k * f)(P_k g))\|_{L^p} \le \|\varphi_j\|_{L^1}\|(\varphi_k * f)(P_k g)\|_{L^p} \le \|\mathcal{F}^{-1}\phi\|_{L^1}\|\varphi_k * f\|_{L^{p_1}}\|P_k g\|_{L^{p_2}}
$$

for all $j, k \in \mathbb{Z}$. Hence it follows from the Minkowski inequality that

$$
\begin{aligned}
\|h_1\|_{\dot{B}^s_{p,q}} &\le C \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|k-j|\le 2} \|\varphi_k * f\|_{L^{p_1}}\|P_k g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&= C \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|l|\le 2} \|\varphi_{j+l} * f\|_{L^{p_1}}\|P_{j+l} g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \sum_{|l|\le 2} \left\{ \sum_{j=-\infty}^{\infty} (2^{sj}\|(\varphi_{j+l} * f\|_{L^{p_1}}\|P_{j+l} g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\
&= C \sum_{|l|\le 2} \left\{ \sum_{i=-\infty}^{\infty} (2^{si}2^{-sl}\|\varphi_i * f\|_{L^{p_1}}\|P_i g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\
&= C \sum_{|l|\le 2} 2^{-sl} \left\{ \sum_{i=-\infty}^{\infty} \left(2^{(s+\alpha)i}\|\varphi_i * f\|_{L^{p_1}} 2^{-\alpha i} \left\| \sum_{k=-\infty}^{i-3} \varphi_k * g_k \right\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \left\{ \sum_{i=-\infty}^{\infty} \left(2^{(s+\alpha)i}\|\varphi_i * f\|_{L^{p_1}} \sum_{k=-\infty}^{i-3} 2^{-\alpha k}\|\varphi_k * g\|_{L^{p_2}} 2^{-\alpha(i-k)} \right)^q \right\}^{\frac{1}{q}} \\
&\le C \sup_{k\in\mathbb{Z}} 2^{-\alpha k}\|\varphi_k * g\|_{L^{p_2}} \left\{ \sum_{i=-\infty} \left(2^{(s+\alpha)i}\|\varphi_i * f\|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} \right)^q \right\}^{\frac{1}{q}} \\
&= C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}}, \qquad (1.4)
\end{aligned}
$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. In the above estimate it should be noted that $\sum_{l=3}^{\infty} 2^{-\alpha l} < \infty$ since $\alpha > 0$. In the case $q = \infty$, we see similarly to (1.4) that

$$\|h_1\|_{\dot{B}^s_{p,\infty}} \le C \sup_{k\in\mathbb{Z}} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} \sup_{i\in\mathbb{Z}} 2^{(s+\alpha)i} \|\varphi_i * f\|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} = C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} \|f\|_{\dot{B}^{s+\alpha}_{p_1,\infty}},$$

with $C = C(n, p, p_1, p_2, s, \alpha)$, from which and (1.4) it follows that

$$\|h_1\|_{\dot{B}^s_{p,q}} \le C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} \|f\|_{\dot{B}^{s+\alpha}_{p_1,q}} \quad \text{for all } 1 \le q \le \infty, \tag{1.5}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$.

Replacing the role of f by g, we obtain similarly to (1.4) and (1.5) that

$$\|h_2\|_{\dot{B}^s_{p,q}} \le C\|f\|_{\dot{B}^{-\beta}_{\tilde{p}_1,\infty}} \|g\|_{\dot{B}^{s+\beta}_{\tilde{p}_2}} \quad \text{for all } 1 \le q \le \infty, \tag{1.6}$$

where $C = C(n, p, \tilde{p}_1, \tilde{p}_2, q, s, \beta)$.

Next we treat h_3 in $\dot{B}^s_{p,q}$. Let us consider the case $1 \le q < \infty$. Since

$$\operatorname{supp} \mathcal{F}((\varphi_k * f)(\varphi_l * g)) \subset \{\xi \in \mathbb{R}^n; |\xi| \le 2^{\max\{k,l\}+2}\},$$

we have that

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,q}} &= \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * h_3\|_{L^p})^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{k=-\infty}^{\infty} \sum_{|l-k|\le 2} \varphi_j * (\varphi_k * f)(\varphi_l * g) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{\max\{k,l\}\ge j-2} \sum_{|l-k|\le 2} \varphi_j * (\varphi_k * f)(\varphi_l * g) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{r\ge -4} \sum_{|t|\le 2} \|\varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\
&\le \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{r\ge -4} \sum_{|t|\le 2} \|\varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} \right)^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

By the Hausdorff-Young and the Hölder inequalities, it holds that

$$\begin{aligned}
\|\varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} &\le \|\varphi_j\|_{L^1} \|(\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} \\
&\le \|\mathcal{F}^{-1}\phi\|_{L^1} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}}
\end{aligned}$$

for all $j, r, t \in \mathbb{Z}$. Hence it follows from the Minkowski inequality that

$$
\begin{aligned}
& \|h_3\|_{\dot{B}^s_{p,q}} \\
\le\ & C\left\{\sum_{j=-\infty}^{\infty}\left(2^{sj}\sum_{r\ge -4}\sum_{|t|\le 2}\|\varphi_{j+r}*f\|_{L^{p_1}}\|\varphi_{j+r+t}*g\|_{L^{p_2}}\right)^q\right\}^{\frac{1}{q}} \\
\le\ & C\sum_{r\ge -4}\sum_{|t|\le 2}\left\{\sum_{j=-\infty}^{\infty}(2^{sj}\|\varphi_{j+r}*f\|_{L^{p_1}}\|\varphi_{j+r+t}*g\|_{L^{p_2}})^q\right\}^{\frac{1}{q}} \\
=\ & C\sum_{r\ge -4}2^{-sr}\sum_{|t|\le 2}2^{\alpha t}\left\{\sum_{j=-\infty}^{\infty}\left(2^{(s+\alpha)(j+r)}\|\varphi_{j+r}*f\|_{L^{p_1}}2^{-\alpha(j+r+t)\|}\|\varphi_{j+r+t}*g\|_{L^{p_2}}\right)^q\right\}^{\frac{1}{q}} \\
\le\ & C\sup_{l\in\mathbb{Z}}2^{-\alpha l}\|\varphi_l*g\|_{L^{p_2}}\sum_{r\ge -4}2^{-sr}\sum_{|t|\le 2}2^{\alpha t}\left\{\sum_{k=-\infty}^{\infty}(2^{(s+\alpha)k}\|\varphi_k*f\|_{L^{p_1}})^q\right\}^{\frac{1}{q}} \\
=\ & C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}}, \qquad (1.7)
\end{aligned}
$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. In the above estimate it should be noted that $\sum_{r\ge -4} 2^{-sr} < \infty$ since $s > 0$. In case $q = \infty$, similarly to (1.7), we have that

$$
\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,\infty}} &\le C\sup_{l\in\mathbb{Z}}2^{-\alpha l}\|\varphi_l*g\|_{L^{p_2}}\sum_{r\ge -4}2^{-sr}\sum_{|t|\le 2}2^{\alpha t}\sup_{k\in\mathbb{Z}}2^{(s+\alpha)k}\|\varphi_k*f\|_{L^{p_1}} \\
&= C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,\infty}}
\end{aligned}
$$

with $C = C(n, p, p_1, p_2, s, \alpha)$, from which and (1.7) it follows that

$$
\|h_3\|_{\dot{B}^s_{p,q}} \le C\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}}\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}} \quad \text{for all } 1\le q\le\infty, \qquad (1.8)
$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. Now the desired estimate (1.1) is a consequence of (1.5), (1.6) and (1.8).

(ii) We also make use of the paraproduct formula (1.3). Let us first consider the case $1 \le q < \infty$. In the same way as in (1.4), we have

$$
\begin{aligned}
\|h_1\|_{\dot{B}^s_{p,q}} &\le C\sum_{|l|\le 2}2^{-sl}\left\{\sum_{i=-\infty}^{\infty}(2^{si}\|\varphi_i*f\|_{L^{p_1}}\|P_ig\|_{L^{p_2}})^q\right\}^{\frac{1}{q}} \\
&\le C\sup_{i\in\mathbb{Z}}\|P_ig\|_{L^{p_2}}\left\{\sum_{i=-\infty}^{\infty}(2^{si}\|\varphi_i*f\|_{L^{p_1}})^q\right\}^{\frac{1}{q}}. \qquad (1.9)
\end{aligned}
$$

It should be noticed that

$$
\sum_{l=-\infty}^{k}\varphi_l(x) = 2^{kn}\psi(2^kx) = \psi_{2^{-k}}(x), \quad \forall k\in\mathbb{Z},
$$

where $f_\varepsilon(x) = \varepsilon^{-n} f(x/\varepsilon)$ for $\varepsilon > 0$. Hence we have $\| \sum_{l=-\infty}^{k} \varphi_l \|_{L^1} = \|\psi\|_{L^1}$ for all $k \in \mathbb{Z}$, and it holds that

$$\|P_i g\|_{L^{p_2}} = \left\| \sum_{l=-\infty}^{i-3} \varphi_l * g \right\|_{L^{p_2}} = \|\psi_{2^{i-3}} * g\|_{L^{p_2}} \le \|\psi\|_{L^1} \|g\|_{L^{p_2}} \quad \text{for all } i \in \mathbb{Z},$$

from which and (1.9) it follows that

$$\|h_1\|_{\dot{B}^s_{p,q}} \le C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^s_{p_1,q}}, \tag{1.10}$$

where $C = C(n, p, p_1, p_2, q, s)$.

In case $q = \infty$, we have that

$$\|h_1\|_{\dot{B}^s_{p,\infty}} \le C \sup_{i \in \mathbb{Z}} \|P_i g\|_{L^{p_2}} \sup_{i \in \mathbb{Z}} 2^{si} \|\varphi_i * f\|_{L^{p_1}} \le C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^s_{p_1,\infty}},$$

from which and (1.10) it follows that

$$\|h_1\|_{\dot{B}^s_{p,q}} \le C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^s_{p_1,q}} \quad \text{for all } 1 \le q \le \infty, \tag{1.11}$$

where $C = C(n, p, p_1, p_2, q, s)$.

Replacing the role of f by g, we have similarly to (1.11) that

$$\|h_2\|_{\dot{B}^s_{p,q}} \le C \|f\|_{L^{\tilde{p}_1}} \|g\|_{\dot{B}^s_{\tilde{p}_1,q}} \quad \text{for all } 1 \le q \le \infty, \tag{1.12}$$

where $C = C(n, p, \tilde{p}_1, \tilde{p}_2, q, s)$.

Concerning the estimate of h_3 in $\dot{B}^s_{p,q}$ for $1 \le q < \infty$, we have similarly to (1.7) that

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,q}} &\le C \sum_{r \ge -4} 2^{-sr} \sum_{|t| \le 2} \left\{ \sum_{j=-\infty}^{\infty} (2^{s(j+r)} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\
&\le C \sup_{i \in \mathbb{Z}} \|\varphi_i * g\|_{L^{p_2}} \sum_{r \ge -4} 2^{-sr} \left\{ \sum_{i=-\infty}^{\infty} (2^{si} \|\varphi_i * f\|_{L^{p_1}})^q \right\}^{\frac{1}{q}} \\
&\le C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^s_{p_1,q}},
\end{aligned} \tag{1.13}$$

where $C = C(n, p, p_1, p_2, q, s)$.

In case $q = \infty$, we have similarly to the above that

$$\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,\infty}} &\le C \sum_{r \ge -4} 2^{-sr} \sum_{|t| \le 2} \sup_{j \in \mathbb{Z}} 2^{s(j+r)} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}} \\
&\le C \sup_{i \in \mathbb{Z}} \|\varphi_i * g\|_{L^{p_2}} \sup_{l \in \mathbb{Z}} 2^{sl} \|\varphi_l * f\|_{L^{p_1}} \sum_{r \ge -4} 2^{-sr} \\
&\le C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^s_{p_1,\infty}},
\end{aligned}$$

from which and (1.13) it follows that

$$\|h_3\|_{\dot{B}^s_{p,q}} \le C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^s_{p_1,q}} \quad \text{for all } 1 \le q \le \infty, \tag{1.14}$$

where $C = C(n, p, p_1, p_2, s)$. Now the desired estimate (1.2) is a consequence of (1.11), (1.12) and (1.14). This proves Lemma 1.1. ∎

2 Application to the stationary Navier-Stokes equations

Let us consider the stationary Navier-Stokes equation in $\mathbb{R}^n$ for $n \geq 3$;

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \pi = f, \\ \text{div } u = 0, \end{cases} \tag{NS}$$

where $u = u(x) = (u^1(x), \cdots, u^n(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, respectively, while $f = f(x) = (f^1(x), \cdots, f^n(x))$ denotes the given external force. we rewrite (NS) to the generalized form by means of the abstract setting of the functional analysis. Let P be the projection operator from L^p onto the solenoidal space $L^p_\sigma \equiv \{u \in L^p; \text{div } u = 0\}$. It is known that P has the expression $P = \{P_{jk}\}_{1\leq j,k\leq n}$ with $P_{jk} = \delta_{jk} + R_jR_k,\ j,k = 1, \cdots, n$, where δ_{jk} denotes the Kronecker symbol and $R_k = \frac{\partial}{\partial x_k}(-\Delta)^{-\frac{1}{2}}$ denotes the Riesz transform. Since R_k, $k = 1, 2, \cdots, n$ is a bounded operator in L^p for $1 < p < \infty$, P is also bounded from L^p onto L^p_σ for $1 < p < \infty$. However, P is *unbounded* in L^p for $p = 1$ and for $p = \infty$. On the other hand, we have

Proposition 2.1 *P is bounded in the homogeneous Besov space $\dot{B}^s_{p.q}$ for all $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$.*

Proof. For the proof, it suffices to show that the Riesz transforms $R_k (k = 1, 2, \cdots, n)$ are bounded in $\dot{B}^s_{p.q}$ for all $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. It should be noted by the Hausdorff-Young inequality that

$$\begin{aligned} \|\varphi_j * R_k f\|_{L^p} &= \left\| \sum_{l=j-1}^{j+1} \varphi_l * R_k(\varphi_j * f) \right\|_{L^p} \\ &\leq \sum_{l=j-1}^{j+1} \left\| \mathcal{F}^{-1}\left(\hat{\varphi}_l(\xi)\frac{i\xi_k}{|\xi|}\right) * \varphi_j * f \right\|_{L^p} \\ &\leq 3\|\Phi_k\|_{L^1}\|\varphi_j * f\|_{L^p}, \qquad k = 1, \cdots, n \end{aligned}$$

for all $1 \leq p \leq \infty$ and for all $j \in \mathbb{Z}$ with $\Phi_k = \mathcal{F}^{-1}(\phi(\xi)\frac{i\xi_k}{|\xi|})$ in L^1, from which we see that R_k, $k = 1, \cdots, n$ is bounded in $\dot{B}^s_{p,q}$ even for $p = 1$ and $p = \infty$. This proves Proposition 2.1. ∎

Since we need to find the solution u of (NS) with div $u = 0$, let us introduce the space $\dot{\mathcal{B}}^s_{p,q} \equiv PB^s_{p,q}$. Since $Pu = u$, $P(\nabla\pi) = 0$ and since P commutes with $-\Delta$, application of P to both sides of (NS) yields that $-\Delta u + P(u \cdot \nabla u) = Pf$. Since div $u = 0$, it holds that $u \cdot \nabla u = \nabla \cdot u \otimes u$, and hence we see that u can be expressed by

$$\begin{aligned} u &= (-\Delta)^{-1}P(u \cdot \nabla u) + (-\Delta)^{-1}Pf \\ &= P(-\Delta)^{-1}\nabla \cdot (u \otimes u) + P(-\Delta)^{-1}f \\ &= K(u \otimes u) + P(-\Delta)^{-1}f, \end{aligned} \tag{E}$$

where $K \equiv P(-\Delta)^{-1}\nabla\cdot$ may be regarded as the Fourier multiplier with the differential order -1. More precisely, $Kg = (Kg_1, \cdots, Kg_n)$ has an expression

$$Kg_j(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sum_{k,l=1}^{n}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\frac{1}{|\xi|^2}i\xi_l\mathcal{F}g_{kl}(\xi)d\xi, \quad j = 1, \cdots, n$$

for $n \times n$ tensors $g = (g_{kl})_{1 \le k,l \le n}$. Then we have the following proposition.

Proposition 2.2 *([8, Proposition 1.1]) Let $1 \le p \le p_0$ and $-\infty < s_0 \le s+1 < \infty$ satisfy $s_0 - n/p_0 - 1 = s - n/p$. Let $1 \le q \le \infty$. K is a bounded operator from $\dot{B}^s_{p,q}$ to $\dot{\mathcal{B}}^{s_0}_{p_0,q}$ with the estimate*

$$\|Kg\|_{\dot{B}^{s_0}_{p_0,q}} \le C\|g\|_{\dot{B}^s_{p,q}}, \tag{2.1}$$

for all $g \in \dot{B}^s_{p,q}$, where $C = C(n, p, p_0, q, s, s_0)$.

Proof. Since the projection P is bounded from $\dot{B}^{s_0}_{p_0,q}$ onto $\dot{\mathcal{B}}^{s_0}_{p_0,q}$, it suffices to show that $K' \equiv (-\Delta)^{-1}\nabla$ with the expression

$$K'g_k(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{|\xi|^2} \sum_{l=1}^n i\xi_l \mathcal{F}g_{kl}(\xi)d\xi, \quad k = 1, \cdots, n$$

is a bounded operator from $\dot{B}^s_{p,q}$ to $\dot{B}^{s_0}_{p_0,q}$ with such an estimate as (2.1).

Let us first consider the case $1 \le q < \infty$. We define $1 \le r \le \infty$ by $1/r = 1 - (1/p - 1/p_0)$. By the Hausdorff-Young inequality, we have that

$$\begin{aligned}
\|K'g\|_{\dot{B}^{s_0}_{p_0,q}} &= \left\{ \sum_{j\in\mathbb{Z}} \left(2^{s_0 j}\|\varphi_j * K'g\|_{L^{p_0}}\right)^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j\in\mathbb{Z}} \left(2^{s_0 j}\|\tilde{\varphi}_j * \varphi_j * K'g\|_{L^{p_0}}\right)^q \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{j\in\mathbb{Z}} \left(2^{s_0 j}\|K'\tilde{\varphi}_j * \varphi_j * g\|_{L^{p_0}}\right)^q \right\}^{\frac{1}{q}} \\
&\le \left\{ \sum_{j\in\mathbb{Z}} \left(2^{s_0 j}\|K'\tilde{\varphi}_j\|_{L^r}\|\varphi_j * g\|_{L^p}\right)^q \right\}^{\frac{1}{q}},
\end{aligned} \tag{2.2}$$

where $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. It is easy to see that

$$K'\tilde{\varphi}_j(x) = 2^{-j}2^{jn}\Psi(2^j x) \quad \text{with} \quad \Psi \equiv \sum_{l=1}^n \mathcal{F}^{-1}\left(\frac{i\xi_l}{|\xi|^2}\sum_{k=-1}^{1}\phi(2^{-k}\xi)\right),$$

which yields that

$$\|K'\tilde{\varphi}_j\|_{L^r} \le 2^{-j}2^{n(1-\frac{1}{r})j}\|\Psi\|_{L^r} \le C2^{-j+n(\frac{1}{p}-\frac{1}{p_0})j} = C2^{(s-s_0)j},$$

where $C = C(n, p, p_0)$ is independent of $j \in \mathbb{Z}$. Notice that $\Psi \in \mathcal{S}$ because supp $\sum_{k=-1}^{1}\phi(2^{-k}\xi) \subset \{\xi \in \mathbb{R}^n; 2^{-2} \le |\xi| \le 2^2\}$. Hence it follows from (2.2) that

$$\|K'g\|_{\dot{B}^{s_0}_{p_0,q}} \le C\left\{ \sum_{j\in\mathbb{Z}} \left(2^{sj}\|\varphi_j * g\|_{L^p}\right)^q \right\}^{\frac{1}{q}} = C\|g\|_{\dot{B}^s_{p,q}},$$

where $C = C(n, p, p_0, q, s, s_0)$. In case $q = \infty$, the proof is quite similar to the above, so we may omit it. This proves Proposition 2.2. ∎

Our main result in this section now reads as follows.

Theorem 2.1 *([8, Theorem 1.2]) Let $n \geq 3$. For every $1 \leq p < n$ and $1 \leq q \leq \infty$ there is a constant $\delta = \delta(n,p,q) > 0$ such that if $f \in \dot{B}_{p,q}^{-3+\frac{n}{p}}$ satisfies $\|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} < \delta$, then there exists a solution $u \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ of (E). Moreover, there exists a constant $\eta = \eta(n,p,q) > 0$ such that if u and v are two solutions of (E) in the class $\dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ satisfying $\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq \eta$, $\|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq \eta$, then it holds that $u \equiv v$.*

In the case $n/2 < p < n$, a similar result to Theorem 2.1 has been obtained by Cunanan-Okabe-Tsutsui [5]. An immediate consequence of the above theorem is the existence of self-similar solutions.

Corollary 2.1 *([8, Corollary 1.3]) Let $n \geq 3$. Let $1 \leq p < n$ and $q = \infty$. If $f \in \dot{B}_{p,\infty}^{-3+\frac{n}{p}}$ is a homogeneous function with degree -3, i.e., $f(\lambda x) = \lambda^{-3} f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ and if f satisfies $\|f\|_{\dot{B}_{p,\infty}^{-3+\frac{n}{p}}} < \delta$, then the solution u given by Theorem 2.1 is a homogeneous function with degree -1, i.e., $u(\lambda x) = \lambda^{-1} u(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$, which means that u may be regarded as a self-similar solution of (NS).*

The following lemma of the bilinear estimate plays an important role for the proof of our main theorem.

Lemma 2.1 *([8, Lemma 2.3]) Let $n \geq 3$ and let $1 \leq p < n$, $1 \leq q \leq \infty$. For $u, v \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ we have $K(u \otimes v) \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ with the estimate*

$$\|K(u \otimes v)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C \|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}, \tag{2.3}$$

where $C = C(n,p,q)$.

Proof. Taking $p = p_0$, $s = -2 + n/p$ in Proposition 2.2, we have that $s_0 = -1 + n/p$, and so it holds that

$$\|K(u \otimes v)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C \|u \otimes v\|_{\dot{B}_{p,q}^{-2+\frac{n}{p}}}, \tag{2.4}$$

where $C = C(n,p,q)$.

Let us first consider the case for $n \geq 3$ and $1 \leq p < n/2$. Take p_1 and p_2 in such a way

$$p_1 = p, \quad n < p_2, \quad p' \equiv \frac{p}{p-1} \leq p_2.$$

We define p_0 and s_0 by

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_0 = \frac{n}{p_0} - 2. \tag{2.5}$$

Since $1 \le p < n/2$, we have that

$$1 \le p_0 \le p, \quad 0 < s_0, \quad \left(-2+\frac{n}{p}\right) - \frac{n}{p} = s_0 - \frac{n}{p_0}. \tag{2.6}$$

It should be noted that the above (2.6) yields $1 \le p_0 \le p < n/2$, which necessarily implies that $n \ge 3$. Hence it follows from Proposition 1.2 that

$$\|u \otimes v\|_{\dot{B}^{-2+\frac{n}{p}}_{p,q}} \le C\|u \otimes v\|_{\dot{B}^{s_0}_{p_0,q}}. \tag{2.7}$$

Since $n < p_2$, we have $\alpha \equiv 1 - n/p_2 > 0$, and we have by Lemma 1.1 (i) that

$$\|u \otimes v\|_{\dot{B}^{s_0}_{p_0,q}} \le C(\|u\|_{\dot{B}^{s_0+\alpha}_{p_1,q}}\|v\|_{\dot{B}^{-\alpha}_{p_2,q}} + \|u\|_{\dot{B}^{-\alpha}_{p_2,q}}\|v\|_{\dot{B}^{s_0+\alpha}_{p_1,q}}). \tag{2.8}$$

Since $p = p_1$, $p < p_2$ and since

$$s_0 + \alpha - \frac{n}{p_1} = -1 = \left(-1+\frac{n}{p}\right) - \frac{n}{p}, \quad -\alpha - \frac{n}{p_2} = -1 = \left(-1+\frac{n}{p}\right) - \frac{n}{p},$$

it follows from Proposition 1.2 that

$$\dot{B}^{-1+\frac{n}{p}}_{p,q} \hookrightarrow \dot{B}^{s_0+\alpha}_{p_1,q}, \quad \dot{B}^{-1+\frac{n}{p}}_{p,q} \hookrightarrow \dot{B}^{-\alpha}_{p_2,q}.$$

Hence we obtain from (2.8) that

$$\|u \otimes v\|_{\dot{B}^{s_0}_{p_0,q}} \le C\|u\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}}\|v\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}}. \tag{2.9}$$

Now, the desired estimate (2.3) is a consequence of (2.4), (2.7) and (2.9).

We next consider the case for $n \ge 3$ and $n/2 \le p < n$. In such a case, we take p_1 and p_2 so that

$$p_1 = p, \quad n < p_2 < \frac{np}{2p-n}.$$

Define p_0 and s_0 by (2.5). Since

$$\frac{1}{p} < \frac{1}{p_0} = \frac{1}{p} + \frac{1}{p_2} < \frac{2}{n} + \frac{1}{n} \le 1,$$
$$s_0 = \frac{n}{p_0} - 2 = n\left(\frac{1}{p_2} - \left(\frac{2}{n} - \frac{1}{p}\right)\right) > 0,$$

we have (2.6), so it holds (2.7). Since $\alpha \equiv 1 - n/p_2 > 0$, implied by $n < p_2$, in the same way as in the above case, we obtain (2.9), which yields the desired estimate (2.3). This proves Lemma 2.1. ∎

Proof of Theorem 2.1. We first prove the existence of the solution to (E). We solve (E) by the successive approximation. For that purpose, let us define the approximating solutions $\{u_j\}$ of (E) by

$$\begin{cases} u_0 = P(-\Delta)^{-1}f, \\ u_{j+1} = K(u_j \otimes u_j) + u_0, \ j = 0, 1, \cdots. \end{cases} \tag{2.10}$$

Since $f \in \dot{B}_{p,q}^{-3+\frac{n}{p}}$, we see that $u_0 \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$. Assume that $u_j \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$. By Lemma 2.1, we have that $u_{j+1} \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ with the estimate

$$\|u_{j+1}\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C\|u_j\|^2_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} + \|u_0\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}, \tag{2.11}$$

where $C = C(n,p,q)$ is independent of j. By induction, it holds that $u_j \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ for all $j = 0, 1, \cdots$. Taking $M_j = \|u_j\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}$, we have by (2.11) that

$$M_{j+1} \leq CM_j^2 + M_0, \quad j = 0, 1, \cdots . \tag{2.12}$$

By the standard argument we see from (2.12) that under the condition

$$M_0 < \frac{1}{4C}, \tag{2.13}$$

the sequence $\{M_j\}_{j=0}^{\infty}$ is subject to the estimate

$$M_j \leq \alpha \equiv \frac{1 - \sqrt{1 - 4CM_0}}{2C}, \ j = 0, 1, \cdots . \tag{2.14}$$

Take $w_j = u_{j+1} - u_j$, and we have

$$\begin{aligned} w_j &= K(u_j \otimes u_j) - K(u_{j-1} \otimes u_{j-1}) \\ &= K(u_j \otimes w_{j-1}) + K(w_{j-1} \otimes u_{j-1}). \end{aligned}$$

Letting $L_j = \|w_j\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}$, we have similarly to (2.12) that

$$\begin{aligned} L_j &\leq C(M_j + M_{j-1})L_{j-1} \\ &\leq 2C\alpha L_{j-1}. \end{aligned}$$

Therefore, it holds that

$$L_j \leq (2C\alpha)^j L_0, \quad j = 1, 2, \cdots .$$

By the definition of α in (2.14), we see that

$$2C\alpha = 1 - \sqrt{1 - 4cM_0} < 1,$$

and hence it holds that

$$\sum_{j=0}^{\infty} L_j < \infty, \tag{2.15}$$

which implies that u_j converges to some u in $\dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$. Since

$$M_0 = \|A^{-1}Pf\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C\|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} < C\delta, \tag{2.16}$$

with $C = C(n,p)$, by taking $\delta = \delta(n,p,q)$ sufficiently small, we see from the above estimate that the condition (2.13) is fulfilled provided $\|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} \leq \delta$. Now, letting $j \to \infty$ in (2.10), we see from Lemma 2.1 that the limit $u \in \dot{\mathcal{B}}_{p,q}^{-1+\frac{n}{p}}$ is a solutions of (E).

We next consider the uniqueness. Let $u \in \dot{\mathcal{B}}^{-1+\frac{n}{p}}_{p,q}$ and $v \in \dot{\mathcal{B}}^{-1+\frac{n}{p}}_{p,q}$ be the solutions of (E) such that $\|u\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} \le \eta$, $\|v\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} \le \eta$. It follows from Lemma 2.1 that

$$\begin{aligned}\|u-v\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} &= \|K(u\otimes(u-v)) + K((u-v)\otimes v)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} \\ &\le C(\|u\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} + \|v\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}})\|u-v\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}} \\ &\le 2C\eta\|u-v\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}}.\end{aligned}$$

By taking $\eta > 0$ sufficiently small to satisfy $2C\eta < 1$, we obtain $u - v = 0$. This completes the proof of Theorem 2.1. ∎

3 $L^p - L^q$ estimates of the Stokes semigroup in Besov spaces

We first investigate the behavior of the heat semigroup in the homogeneous Besov spaces.

Proposition 3.1 (i) *Let $1 \leqq p \leqq q \leqq \infty$, $1 \leqq r \leqq \infty$ and $s_0 \leqq s_1$. It holds that*

$$\|e^{t\Delta}a\|_{\dot{B}^{s_1}_{q,r}} \leqq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}(s_1-s_0)}\|a\|_{\dot{B}^{s_0}_{p,r}}$$

for all $a \in \dot{B}^{s_0}_{p,r}$ and all $0 < t < \infty$ with a constant $C = C(n,p,q,r,s_0,s_1)$.
(ii) *Let $s_0 < s_1$, $1 \leqq p \leqq \infty$. It holds that*

$$\|e^{t\Delta}a\|_{\dot{B}^{s_1}_{p,1}} \leqq Ct^{-\frac{1}{2}(s_1-s_0)}\|a\|_{\dot{B}^{s_0}_{p,\infty}}$$

for all $a \in \dot{B}^{s_0}_{p,\infty}$ and for all $0 < t < \infty$ with a constant $C = C(n,p,s_0,s_1)$.
(iii) *Let $s_0 < s_1$ and $1 \leqq p \leqq q \leqq \infty$. It holds that*

$$\|e^{t\Delta}a\|_{\dot{B}^{s_1}_{q,1}} \leqq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}(s_1-s_0)}\|a\|_{\dot{B}^{s_0}_{p,\infty}}$$

for all $a \in \dot{B}^{s_0}_{p,\infty}$ and for all $0 < t < \infty$ with a constant $C = C(n,p,q,s_0,s_1)$.

For the proof, see [10, Lemma 2.2] and [9, Lemma 2.2].

The following theorem characterizes the class of the initial data a in the homogeneous Besov space in the case that $e^{t\Delta}a$ belongs to *the Serrin class* in the generalized Lorentz space in time.

Theorem 3.1 *([12, Lemma 2.1])* (i) *Let $n < p < \infty$ and $1 \leqq q \leqq \infty$. For $a \in \dot{B}^{-1+\frac{n}{p}}_{p,q}$ it holds that $e^{t\Delta}a \in L^{\alpha,q}(0,\infty;\dot{B}^0_{r,1})$ for all $p \leqq r \leqq \infty$ and $2 \leqq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$ with the estimate*

$$\Big\| \|e^{t\Delta}a\|_{\dot{B}^0_{r,1}} \Big\|_{L^{\alpha,q}(0,\infty)} \leqq C\|a\|_{\dot{B}^{-1+\frac{n}{p}}_{p,q}}, \tag{3.1}$$

where $C = C(n.p,q,r)$. In particular, if $a \in \dot{B}^{-1+\frac{n}{p}}_{p,s}$ for $\frac{2}{s} + \frac{n}{p} = 1$ with $n < p < \infty$, then it holds that $e^{t\Delta}a \in L^s(0,\infty;\dot{B}^0_{p,1})$.

(ii) *Assume that* $a \in \mathcal{S}'$ *satisfies*

$$e^{t\Delta}a \in L^{\alpha,q}(0,\infty;L^r).$$

for $n < r \leqq \infty$ *and* $2 \leqq \alpha < \infty$ *with* $\frac{2}{\alpha} + \frac{n}{r} = 1$ *and for* $1 < q \leqq \infty$*. Then it holds that* $a \in \dot{B}^{-1+\frac{n}{r}}_{r,q}$ *with the estimate*

$$\|a\|_{\dot{B}^{-1+\frac{n}{r}}_{r,q}} \leqq C\|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)}, \tag{3.2}$$

where $C = C(n,r,q)$*.*

Proof. The special case when $q = \alpha$ was proved by [1, Theorem 2.34]. Here we give another proof based on the real interpolation.

(i) We take p_0, p_1 and $0 < \theta < 1$ in such a way that

$$n < p_0 < p < p_1 \leqq \infty, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

For every $a \in \dot{B}^{-1+\frac{n}{p_i}}_{p,\infty}$, $i = 0,1$, it follows from Proposition 3.1(iii) that

$$\|e^{t\Delta}a\|_{\dot{B}^0_{r,1}} \leqq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}(0-(-1+\frac{n}{p_i}))}\|a\|_{\dot{B}^{-1+\frac{n}{p_i}}_{p,\infty}} = Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}+\frac{1}{n}-\frac{1}{p_i}\right)}\|a\|_{\dot{B}^{-1+\frac{n}{p_i}}_{p,\infty}}, \quad i = 0,1. \tag{3.3}$$

Let us define α_0 and α_1 in such a way that

$$\frac{1}{\alpha_i} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r} + \frac{1}{n} - \frac{1}{p_i}\right), \quad i = 0,1. \tag{3.4}$$

Since $n < p_0 < p < p_1$ and since $p \leqq r \leqq \infty$, we easily verify that $1 < \alpha_i < \infty$ for $i = 0,1$, and obtain from (3.3) that the mappings

$$\dot{B}^{-1+\frac{n}{p_i}}_{p,\infty} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}^0_{r,1}} \in L^{\alpha_i,\infty}(0,\infty), \quad i = 0,1.$$

are bounded sub-additive operators. Here $L^{\alpha_i,q}(0,\infty)$ denotes the Lorentz space on $(0,\infty)$ (see, e.g., Bergh-Löfström [2, Chapter 5]). Then it follows from the real interpolation theorem that

$$(\dot{B}^{-1+\frac{n}{p_0}}_{p,\infty}, \dot{B}^{-1+\frac{n}{p_1}}_{p,\infty})_{\theta,q} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}^0_{r,1}} \in (L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,q} \tag{3.5}$$

is also bounded sub-additive. Since

$$(\dot{B}^{-1+\frac{n}{p_0}}_{p,\infty}, \dot{B}^{-1+\frac{n}{p_1}}_{p,\infty})_{\theta,q} = \dot{B}^{-1+\frac{n}{p}}_{p,q}, \quad (L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,q} = L^{\alpha,q}(0,\infty)$$

with α defined by

$$\frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1} = -\frac{n}{2r} + \frac{1}{2},$$

we conclude from (3.5) that

$$\dot{B}^{-1+\frac{n}{p}}_{p,q} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}^0_{r,1}} \in L^{\alpha,q}(0,\infty)$$

is a bounded sub-additive operator for $\frac{2}{\alpha}+\frac{n}{r}=1$ with $p \leqq r \leqq \infty$, which implies (3.1). This proves (i).

(ii) Let us first consider the case $1<q<\infty$. We make use of the following characterization of the equivalent norm of the homogeneous Besov space $\dot{B}^{1-\frac{n}{r}}_{r',q'}$ due to Triebel [13]:

$$\|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',q'}} \backsimeq \left\{\int_0^\infty (t^{1-\frac{1}{2}(1-\frac{n}{r})}\|(-\Delta)e^{t\Delta}\varphi\|_{L^{r'}})^{q'}\frac{dt}{t}\right\}^{\frac{1}{q'}}, \tag{3.6}$$

where we have used the relation $\frac{2}{\alpha}+\frac{n}{r}=1$ with $1-\frac{n}{r}=\frac{2}{\alpha}>0$.

For $a\in\mathcal{S}'$, we take a dual coupling with $\varphi\in\mathcal{S}$. Since

$$e^{t\Delta}\varphi-\varphi=\int_0^t \frac{\partial}{\partial\tau}e^{\tau\Delta}\varphi\,d\tau=-\int_0^t(-\Delta)e^{\tau\Delta}\varphi\,d\tau,$$

φ is expressed by $\varphi=e^{t\Delta}\varphi+\int_0^t(-\Delta)e^{\tau\Delta}\varphi\,d\tau$. We consider the coupling

$$|\langle a,\varphi\rangle| \leqq |\langle a,e^{t\Delta}\varphi\rangle|+\int_0^t|\langle a,(-\Delta)e^{\tau\Delta}\varphi\rangle|d\tau =: I_1(t)+I_2(t). \tag{3.7}$$

By (3.6) and the Hölder inequality, it holds that

$$\begin{aligned}
I_2(t) &\leqq \int_0^t |\langle e^{\frac{\tau}{2}\Delta}a,(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\rangle|d\tau\\
&\leqq \int_0^t \|e^{\frac{\tau}{2}\Delta}a\|_{L^r}\|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}}d\tau\\
&\leqq \int_0^t \tau^{-1+\frac{1}{q'}+\frac{1}{2}(1-\frac{n}{r})}\|e^{\frac{\tau}{2}\Delta}a\|_{L^r}\tau^{1-\frac{1}{q'}-\frac{1}{2}(1-\frac{n}{r})}\|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}}d\tau\\
&\leqq \left[\int_0^t (\tau^{-1+\frac{1}{q'}+\frac{1}{2}(1-\frac{n}{r})}\|e^{\frac{\tau}{2}\Delta}a\|_{L^r})^q\,d\tau\right]^{\frac{1}{q}}\left[\int_0^t(\tau^{1-\frac{1}{q'}-\frac{1}{2}(1-\frac{n}{r})}\|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}})^{q'}d\tau\right]^{\frac{1}{q'}}\\
&\leqq \left[\int_0^t (\tau^{\frac{1}{2}(1-\frac{n}{r})}\|e^{\frac{\tau}{2}\Delta}a\|_{L^r})^q\,\frac{d\tau}{\tau}\right]^{\frac{1}{q}}\left[\int_0^t(\tau^{1-\frac{1}{2}(1-\frac{n}{r})}\|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}})^{q'}\frac{d\tau}{\tau}\right]^{\frac{1}{q'}}\\
&\leqq \left[\int_0^t (\tau^{\frac{1}{\alpha}}\|e^{\tau\Delta}a\|_{L^r})^q\,\frac{d\tau}{\tau}\right]^{\frac{1}{q}}\|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',q'}}.
\end{aligned}$$

Since $\{e^{\tau\Delta}\}_{\tau\geq 0}$ is a contraction semigroup in L^r, we see that $t\in(0,\infty)\to\|e^{t\Delta}a\|_{L^r}$ is a non-negative and non-increasing function. Hence it is easy to see that

$$\left[\frac{\alpha}{q}\int_0^t(\tau^{\frac{1}{\alpha}}\|e^{\tau\Delta}a\|_{L^r})^q\,\frac{d\tau}{\tau}\right]^{\frac{1}{q}}=\|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)},$$

which yields that

$$I_2(t)\leqq C\|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)}\|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',q'}},$$

for all $0 < t < \infty$ and all $\varphi \in \mathcal{S}$ with $C = C(n, r, q)$. Since $a \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, it is easy to see that $I_1(t) \to 0$ as $t \to \infty$. Hence, letting $t \to \infty$ in both sides of (3.7), we obtain that

$$|\langle a, \varphi\rangle| \leqq C\|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)}\|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',q'}}$$

for all $\varphi \in \mathcal{S}$. Since $\dot{B}^{-1+\frac{n}{r}}_{r,q} = (\dot{B}^{1-\frac{n}{r}}_{r',q'})^*$ and since $\mathcal{S}$ is dense in $\dot{B}^{1-\frac{n}{r}}_{r',q'}$, it follows from the above estimate that

$$\|a\|_{\dot{B}^{-1+\frac{n}{r}}_{r,q}} = \sup_{\varphi\in\mathcal{S},\ \|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',q'}}=1} |\langle a, \varphi\rangle| \leqq C\|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)},$$

which implies (3.2).

Next, we consider the case $q = \infty$. Notice that $\dot{B}^{-1+\frac{n}{r}}_{r,\infty} = (\dot{B}^{1-\frac{n}{r}}_{r',1})^*$. Again by the characterization of the norm $\dot{B}^{1-\frac{n}{r}}_{r,1}$, we see that

$$\|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',1}} \backsimeq \int_0^\infty t^{1-\frac{1}{2}(1-\frac{n}{r})}\|(-\Delta)e^{t\Delta}\varphi\|_{L^{r'}}\frac{dt}{t}. \tag{3.8}$$

Since

$$\|e^{t\Delta}a\|_{L^{\alpha,\infty}(0,\infty;L^r)} = \sup_{0<t<\infty} t^{\frac{1}{\alpha}}\|e^{t\Delta}a\|_{L^r},$$

in the same manner as in the above case $1 < q < \infty$, we see easily that

$$I_2(t) \leqq C\|e^{t\Delta}a\|_{L^{\alpha,\infty}(0,\infty;L^r)}\|\varphi\|_{\dot{B}^{1-\frac{n}{r}}_{r',1}},$$

for all $0 < t < \infty$ and all $\varphi \in \mathcal{S}$ with $C = C(n, r)$, from which, as in the same way as the above case, we obtain the desired estimate. ∎

We next consider the maximal regularity theorem for the heat equation in the homogeneous Besov space. To this end, let us first consider the homogeneous heat equation.

Proposition 3.2 *([11, Lemma 2.1]) Let $1 < p < \infty$, $1 < \alpha < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$. Assume that $1 \le r \le p$ satisfies*

$$\frac{n}{p} \le \frac{n}{r} < \frac{2}{\alpha} + \frac{n}{p}.$$

For $a \in \dot{B}^k_{r,q}$ with $k = 2 + n/r - (2/\alpha + n/p - s)$, it holds that

$$\Delta e^{t\Delta}a \in L^{\alpha,q}(0, \infty; \dot{B}^s_{p,1})$$

with the estimate

$$\|\,\|\Delta e^{t\Delta}a\|_{\dot{B}^s_{p,1}}\|_{L^{\alpha,q}(0,\infty)} \le C\|a\|_{\dot{B}^k_{r,q}},$$

where $C = C(n, p, \alpha, s, q)$.

Proof. Since $n/r < 2/\alpha + n/p$, we have that $k < s+2$. Hence taking $\theta \in (0,1)$ and $k_0 < k < k_1 < s+2$ so that $k = (1-\theta)k_0 + \theta k_1$. By Proposition 3.1(iii) it holds that

$$\|\Delta e^{t\Delta} a\|_{\dot{B}^s_{p,1}} = \|e^{t\Delta} a\|_{\dot{B}^{s+2}_{p,1}} \le C t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}(s+2-k_i)} \|a\|_{\dot{B}^{k_i}_{r,\infty}}$$

for $i = 0, 1$, and hence we see that the mapping

$$a \in \dot{\mathcal{B}}^{k_i}_{r,\infty} \mapsto \|\Delta e^{t\Delta} a\|_{\dot{B}^s_{p,1}} \in L^{\alpha_i,\infty}(0,\infty)$$

is a bounded sub-additive operator for

$$\frac{1}{\alpha_i} = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}) + \frac{1}{2}(s+2-k_i), \quad i = 0, 1.$$

Then it follows from the real interpolation theorem that

$$a \in (\dot{\mathcal{B}}^{k_0}_{r,\infty}, \dot{\mathcal{B}}^{k_1}_{r,\infty})_{\theta,q} \to \|\Delta e^{t\Delta} a\|_{\dot{B}^s_{p,1}} \in (L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,q}.$$

Since $(\dot{\mathcal{B}}^{k_0}_{r,\infty}, \dot{\mathcal{B}}^{k_1}_{r,\infty})_{\theta,q} = \dot{\mathcal{B}}^k_{r,q}$ and since $(L^{\alpha_0,\infty}(0,\infty), L^{\alpha_1,\infty}(0,\infty))_{\theta,q} = L^{\alpha,q}(0,\infty)$, implied by

$$\begin{aligned}
\frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1} &= (1-\theta)\left(\frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) + \frac{1}{2}(s+2-k_0)\right) + \theta\left(\frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) + \frac{1}{2}(s+2-k_1)\right) \\
&= \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) + \frac{1}{2}\left(s+2-(1-\theta)k_0 - \theta k_1\right) \\
&= \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) + \frac{1}{2}\left(s+2-k\right),
\end{aligned}$$

we conclude that the mapping

$$a \in \dot{\mathcal{B}}^k_{r,q} \to \|\Delta e^{t\Delta} a\|_{\dot{B}^s_{p,1}} \in L^{\alpha,q}(0,\infty)$$

is a bounded sub-additive operator, which yields the desired result. This proves Proposition 3.2. ∎

Now, we are in a position to state the maximal regularity theorem for the Stokes equations.

Theorem 3.2 *([11, Theorem 1]) Let $1 < p < \infty$, $1 < \alpha < \infty$, $1 \le \beta \le \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$. Assume that $1 \le r \le \infty$ satisfies*

$$\frac{n}{p} \le \frac{n}{r} < \frac{2}{\alpha} + \frac{n}{p}. \tag{3.9}$$

For every $a \in \dot{B}^k_{r,q}$ with $k = 2 + n/r - (2/\alpha + n/p - s)$ and every $f \in L^{\alpha,q}(0,T;\dot{B}^s_{p,\beta})$ with $0 < T \le \infty$, there exists a unique solution u of

$$\text{(S)} \qquad \begin{cases} \dfrac{du}{dt} - \Delta u = Pf \quad a.e.\ t \in (0,T)\ in\ \dot{B}^s_{p,\beta}, \\ u(0) = a \quad in\ \dot{B}^k_{r,q} \end{cases}$$

in the class

$$u_t,\ \Delta u \in L^{\alpha,q}(0,T;\dot{\mathcal{B}}^s_{p,\beta}).$$

Moreover, such a solution u is subject to the estimate

$$\|u_t\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\beta})} + \|\Delta u\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\beta})} \le C(\|a\|_{\dot{B}^k_{r,q}} + \|f\|_{L^{\alpha,q}(0,T;\dot{B}^s_{p,\beta})}), \tag{3.10}$$

where $C = C(n,p,\alpha,q,\beta,s,r)$ *is a constant independent of* $0 < T \le \infty$.

Proof. Step 1. Let us first prove in case $a = 0$. By the usual maximal regularity theorem in $\dot{H}^s_p$ for $s_0 < s < s_1 \le k+2$, for every $f \in L^\alpha(0,T;\dot{H}^{s_i}_p)$ $(i = 0,1)$ with $0 < T \le \infty$, there exists a unique solution u of (S) in the class

$$u_t,\ -\Delta u \in L^\alpha(0,T;\dot{H}^{s_i}_p)$$

with the estimate

$$\|u_t\|_{L^\alpha(0,T;\dot{H}^{s_i}_p)} + \|\Delta u\|_{L^\alpha(0,T;\dot{H}^{s_i}_p)} \le C\|f\|_{L^\alpha(0,T;\dot{H}^{s_i}_p)}, \quad i = 0,1,$$

where $C = C(n,p,\alpha,s_0,s_1)$ is independent of T. For the detail, see, e.g., Giga-Sohr [6, Theorem 2.1]. This implies that the mapping

$$S : f \in L^\alpha(0,T;\dot{H}^{s_i}_p) \to (u_t,-\Delta u) \in L^\alpha(0,T;\dot{H}^{s_i}_p)^2, \quad i = 0,1$$

is a bounded linear operator with its operator norm independent of T. Hence by the real interpolation, S extends a bounded operator from $L^\alpha(0,T;(\dot{H}^{s_0}_p,\dot{H}^{s_1}_p)_{\theta,\beta})$ to $L^\alpha(0,T;(\dot{H}^{s_0}_p,\dot{H}^{s_1}_p)_{\theta,\beta})^2$ for all $1 \le \beta \le \infty$.

Since $(\dot{H}^{s_0}_p,\dot{H}^{s_1}_p)_{\theta,\beta} = \dot{B}^s_{p,\beta}$ with $s = (1-\theta)s_0 + \theta s_1$, we see that

$$S : f \in L^\alpha(0,T;\dot{B}^s_{p,\beta}) \to (u_t,-\Delta u) \in L^\alpha(0,T;\dot{B}^s_{p,\beta})^2$$

is a bounded operator with its operator norm independent of T. Taking $\alpha_0 < \alpha < \alpha_1$ and $0 < \theta < 1$ so that $1/\alpha = (1-\theta)/\alpha_0 + \theta/\alpha_1$, we see that

$$\begin{aligned} S : f &\in (L^{\alpha_0}(0,T;\dot{B}^s_{p,\beta}), L^{\alpha_1}(0,T;\dot{B}^s_{p,\beta}))_{\theta,q} \\ &\to (u_t,-\Delta u) \in (L^{\alpha_0}(0,T;\dot{B}^s_{p,\beta}), L^{\alpha_1}(0,T;\dot{B}^s_{p,\beta}))^2_{\theta,q} \end{aligned}$$

is a bounded operator with its operator norm independent of T. Since

$$(L^{\alpha_0}(0,T;\dot{B}^s_{p,\beta}), L^{\alpha_1}(0,T;\dot{B}^s_{p,\beta}))_{\theta,q} = L^{\alpha,q}(0,T;\dot{B}^s_{p,\beta}),$$

we obtain the desired result with the estimate (3.10) for $a = 0$.

Step 2. For $a \in \dot{\mathcal{B}}^k_{r,q}$ and $f \in L^{\alpha,q}(0,T;\dot{B}^s_{p,\beta})$, we see that

$$u(t) = e^{t\Delta}a + Sf(t), \quad 0 < t < T$$

solves (S). Since $\dot{\mathcal{B}}^s_{p,1} \subset \dot{\mathcal{B}}^s_{p,\beta}$, the desired result with the estimate (3.10) is a consequence of Proposition 3.2 and the argument of the above Step 1. This completes the proof of Theorem 3.2. ∎

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