

# Large exponent asymptotics for one dimensional fully nonlinear diffusions of power type

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## Abstract

In this note, we discuss limit behavior for fully nonlinear diffusion of power type in one space dimension. It turns out that, as the exponent tends to infinity, the solution converges locally uniformly to a unique limit function that is independent of the time variable. We rescale the time variable to characterize the limit as a unique viscosity solution of a fully nonlinear singular parabolic equation with jump discontinuity. Such asymptotic behavior is closely related to applications in math models of image denoising and collapsing sandpiles.

## 1 Introduction

This note is a simplified presentation of our work on large exponent behavior for power curvature flow [12, 10]. Although some parts of our results apply to the Euclidean space of general dimensions, in this paper we restrict ourselves in one space dimension so as to give a clearer view of the topic. Moreover, we choose to consider a heat-type problem rather than a general parabolic operator so that we can better clarify our basic idea.

The equation we are concerned with is as follows:

$$\text{(PH)} \quad \begin{cases} u_t - |u_{xx}|^{\alpha-1} u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}, \end{cases} \quad (1.1)$$

where  $\alpha > 0$  is a given exponent and  $u_0$  is a given Lipschitz initial value. (Below we denote by  $C^{0,1}(\mathbb{R})$  the class of  $k$  times differentiable functions with  $k$ -th derivatives Lipschitz continuous.) We are interested in the limit behavior of the unique solution of  $u_\alpha$  of (PH) as  $\alpha \rightarrow \infty$ .

In addition to the Lipschitz continuity, we need to further assume that  $u_0$  has small second derivatives near space infinity, that is,  $u_0 \in C_b^{0,1}(\mathbb{R})$ , where

$$C_b^{0,1}(\mathbb{R}) = \left\{ f \in C^{0,1}(\mathbb{R}) : \text{there exist } \lambda \in (0, 1) \text{ and a bounded open } \mathcal{K} \subset \mathbb{R} \text{ such that } -\lambda \leq f_{xx} \leq \lambda \text{ in the viscosity sense in } \mathbb{R} \setminus \overline{\mathcal{K}} \right\}. \quad (1.3)$$

Our main result is as follows.

**Theorem 1.1** (Large exponent behavior). *Assume that  $u_0 \in C_b^{0,1}(\mathbb{R})$ . Let  $u_\alpha$  be the unique solution of (PH). Then there exists  $U_\infty \in C_b^{0,1}(\mathbb{R})$  such that  $u_\alpha(x, t) \rightarrow U_\infty(x)$  locally uniformly for all  $(x, t) \in \mathbb{R} \times (0, \infty)$  as  $\alpha \rightarrow \infty$ .*

This result amounts to saying that the large exponent limit does not depend on  $t$  and an initial layer forms in this limit process. When  $u_0 \in C^{1,1}(\mathbb{R}) \cap C_b^{0,1}(\mathbb{R})$ , the large exponent limit  $U_\infty$  can be characterized via a layer analysis as follows. Let

$$L_0 = \max \{ \|(u_0)_{xx}\|_{L^\infty(\mathbb{R})}, 1 \}. \quad (1.4)$$

We can rescale the solution  $u_\alpha$  of (PH) by choosing arbitrarily

$$0 < \tau < \frac{1}{L_0} \quad (1.5)$$

and setting

$$U_\alpha(x, t) = u_\alpha \left( x, \frac{t^\alpha}{\alpha} \right) \quad \text{for } (x, t) \in \mathbb{R} \times [\tau, \infty). \quad (1.6)$$

Then it is not difficult to verify that  $U_\alpha$  is the unique solution of

$$U_t - |tU_{xx}|^{\alpha-1} U_{xx} = 0 \quad \text{in } \mathbb{R} \times (\tau, \infty) \quad (1.7)$$

with the initial condition

$$U_\alpha(x, \tau) = u_\alpha \left( x, \frac{\tau^\alpha}{\alpha} \right). \quad (1.8)$$

We can show that the large exponent limit  $U_\infty$  exists and can be determined by

$$U_\infty(x) = \lim_{\alpha \rightarrow \infty} U_\alpha(x, 1) \quad \text{for } x \in \mathbb{R}.$$

Letting  $\alpha \rightarrow 0$  in (1.7)–(1.8) formally gives rise to the following fully nonlinear parabolic equation

$$U_t = F(t, U_{xx}) \quad \text{in } \mathbb{R} \times (\tau, \infty) \quad (1.9)$$

with

$$U(\cdot, \tau) = u_0 \quad \text{in } \mathbb{R}, \quad (1.10)$$

where  $F : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is defined to be

$$F(t, z) = \begin{cases} 0 & \text{if } |z| < 1/t, \\ \text{sgn}(z)/t & \text{if } |z| = 1/t, \\ \infty & \text{if } z > 1/t, \\ -\infty & \text{if } z < -1/t. \end{cases} \quad (1.11)$$

When  $u_0$  is of class  $C^{1,1}$ , it turns out that, as  $\alpha \rightarrow \infty$ ,  $U_\alpha$  does converge to the unique continuous solution  $U$  of (1.9)–(1.10) locally uniformly in  $\mathbb{R} \times [\tau, 1]$  and  $u_\alpha(x, t)$  converges to  $U_\infty(x) = U(x, 1)$  for all  $(x, t) \in \mathbb{R} \times (0, \infty)$

The main difficulty lies at the uniqueness issue for the limit problem (1.9)–(1.10), since the operator  $F$  is highly singular and does not fit into the classical viscosity solution theory [7]. We use the notion of envelope viscosity solutions and establish a comparison principle for this problem. However, we need to assume that the second derivative of  $u_0$

is small near space infinity so that  $u_\alpha$  hardly evolves outside a compact set. It essentially gives us an extra boundary condition for the limit of  $U_\alpha$  and facilitates our analysis. Once the comparison principle for (1.9) is established, the convergence of  $u_\alpha$  then follows the standard Barles-Perthame procedure [4].

We remark that our results can be used as an alternative approach to study a model describing collapsing sandpiles [3, 8]. Also, large exponent asymptotics for a more general class of equations than (1.1), especially the power curvature flow, has applications in image processing [2, 1, 13, 5]. We refer the reader to [12, 10] for more details on these applications. It is also an important topic to investigate the limit behavior as  $\alpha \rightarrow 0$ , which was discussed in [6, 11].

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## 2 The Power-type Diffusion in One Dimension

Let us quickly review the wellposedness of (PH). Using the theory of viscosity solutions [7], one can show that for every fixed  $\alpha > 0$  there exists a unique viscosity solution  $u_\alpha \in C(\mathbb{R} \times [0, \infty))$  of (PH) satisfying the following growth condition: for any  $T > 0$  there exists  $C_T > 0$  such that

$$|u_\alpha(x, t)| \leq C_T(1 + |x|) \quad \text{for all } (x, t) \in \mathbb{R} \times [0, T].$$

In particular, the following comparison result holds.

**Theorem 2.1** (Comparison principle for power-type heat equation). *Fix  $\alpha \geq 1$ . Suppose that  $u \in USC(\mathbb{R} \times [0, \infty))$  and  $v \in LSC(\mathbb{R} \times [0, \infty))$  are respectively a locally bounded subsolution and a locally bounded supersolution of (1.1). Assume that either  $u(\cdot, 0)$  or  $v(\cdot, 0)$  is Lipschitz in  $\mathbb{R}$ . Assume in addition that for any  $T > 0$  there exists  $C_T > 0$  such that*

$$|u(x, t)| + |v(x, t)| \leq C_T(1 + |x|) \quad \text{for all } (x, t) \in \mathbb{R} \times [0, T].$$

*If  $u(x, 0) \leq v(x, 0)$  in  $\mathbb{R}$ , then*

$$u(x, t) \leq v(x, t) \quad \text{for all } (x, t) \in \mathbb{R} \times [0, \infty).$$

We refer the reader to [10, Appendix A] for a complete proof of this result and detailed discussion on the existence of solutions of (PH) by Perron's method.

Moreover, we have the following result on the Lipschitz preserving property.

**Lemma 2.2** (Lipschitz regularity preserving property). *Assume that  $u_0 \in C^{0,1}(\mathbb{R})$ . Let  $u_\alpha$  be the unique viscosity solution of (PH) for every  $\alpha > 0$ . Then  $u_\alpha(\cdot, t)$  is Lipschitz in  $\mathbb{R}$  for any  $t \geq 0$ . Moreover,*

$$\|(u_\alpha)_x(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|(u_0)_x\|_{L^\infty(\mathbb{R})}$$

*holds for any  $t \geq 0$ .*

The proof is based on comparison between  $u_\alpha$  and a translation of  $u_\alpha$  in space.

### 3 Large exponent behavior

We next discuss the large exponent behavior for (PH) with  $u_0 \in C_b^{0,1}(\mathbb{R})$  defined in (1.3). Our method is based on a rescaling technique described in Section 1.

#### 3.1 Rescaled evolution

We assume that  $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$  and recall  $L_0, \tau$  as given in (1.4) and (1.5). We also recall the function  $U_\alpha$  given by (1.6), which solves (1.7)–(1.8).

Let us rigorously study the behavior of  $U_\alpha$ , which is supposed to be related to (1.9)–(1.10), where  $F$  is given in (1.11). We start with a definition of viscosity solutions to the limit equation. We recall in [7] the notion of semijets  $P^{2,\pm}u(x_0, t_0)$  and  $\overline{P}^{2,\pm}u(x_0, t_0)$  at a given point  $(x_0, t_0)$ .

**Definition 3.1** (Definition of viscosity solutions). Let  $\mathcal{O}$  be an open subset of  $\mathbb{R} \times (0, \infty)$ . A locally bounded upper semicontinuous (resp., lower semicontinuous) function  $U : \mathcal{O} \rightarrow \mathbb{R}$  is called a subsolution (resp., supersolution) of

$$U_t = F(t, U_{xx}) \quad \text{in } \mathcal{O} \quad (3.1)$$

with  $F$  given by (1.11) if

$$z \geq -1/t_0, \quad \text{and } \eta \leq 0 \text{ if } -1/t_0 \leq z < 1/t_0.$$

$$\text{(resp., } z \leq 1/t_0, \quad \text{and } \eta \geq 0 \text{ if } -1/t_0 < z \leq 1/t_0.)$$

holds for any  $(\eta, p, z) \in \overline{P}^{2,+}u(x_0, t_0)$  (resp.,  $(\tau, p, z) \in \overline{P}^{2,-}u(x_0, t_0)$ ). A locally bounded function  $u$  is a solution if it is both a subsolution and a supersolution.

It is equivalent to use the test functions instead of the semijets to define solutions as above. We next give a comparison principle for the Cauchy problem (1.9)–(1.10). A comparison principle for more general equations of the same type can be found in [10, Theorem 4.1].

**Theorem 3.2** (Comparison principle). *Fix  $\tau > 0$  and  $T > \tau$ . Suppose that  $u \in USC(\mathbb{R} \times [\tau, T])$  and  $v \in LSC(\mathbb{R} \times [\tau, T])$  are bounded and respectively a subsolution and a supersolution of (3.1) with  $F$  given by (1.11) and  $\mathcal{O} = \mathbb{R} \times (\tau, T)$ . Assume that  $u(\cdot, t)$  and  $v(\cdot, t)$  are Lipschitz continuous in  $\mathbb{R}$  uniformly for all  $\tau \geq 0$ . Assume in addition that there exists an open bounded subset  $\mathcal{I} \subset \mathbb{R}$  such that  $u \leq v$  holds in  $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, T)$ . If  $u \leq v$  in  $\mathbb{R} \times \{\tau\}$ , then  $u \leq v$  in  $\mathbb{R} \times [\tau, T)$ .*

*Proof.* Assume by contradiction that  $u - v > 0$  somewhere in  $\mathbb{R} \times [\tau, \infty)$ . Then there exist  $\sigma > 0$  small such that

$$(x, t) \mapsto u(x, t) - v(x, t) - \frac{\sigma}{T - t}$$

attains a positive value in  $\mathbb{R} \times [\tau, T)$ , which implies that

$$\sup_{(x,y,t) \in \mathbb{R}^2 \times [0,T)} u(x, t) - v(x, t) - \frac{\sigma}{T - t} \geq \mu$$

for some  $\mu > 0$ , since  $u \leq v$  in  $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, T]$ . For any  $\varepsilon > 0$ , set

$$\Phi(x, y, t) = u(x, t) - v(y, t) - \frac{(x - y)^2}{2\varepsilon} - \frac{\sigma}{T - t}.$$

The Lipschitz continuity of  $u(\cdot, t)$  and  $v(\cdot, t)$  implies that there exists  $L > 0$  such that

$$|u(x, t) - u(y, t)| \leq L|x - y|, \quad |v(x, t) - v(y, t)| \leq L|x - y|$$

for all  $x, y \in \mathbb{R}$ . Using again the fact that  $u \leq v$  in  $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, T]$ , we have

$$u(x, t) - v(y, t) \leq L|x - y| + \sup_{\mathcal{I} \times [\tau, T]} (|u| + |v|) \quad (3.2)$$

for all  $x, y \in \mathbb{R}$  and  $t \in [\tau, T]$ . It follows that  $\Phi$  is bounded from above; suppose that

$$\sup_{\mathbb{R}^2 \times [\tau, T]} \Phi = \mu_\varepsilon.$$

It is clear that  $\mu_\varepsilon > \mu$ . The supremum can actually be attained. Indeed, for any fixed  $\varepsilon > 0$ , suppose that there exists a maximizing sequence  $(x_m, y_m, t_m) \in \mathbb{R}^2 \times [\tau, T]$  with  $m > 0$  large such that

$$\Phi(x_m, y_m, t_m) \geq \mu_\varepsilon - \frac{1}{m}. \quad (3.3)$$

In light of the boundedness of  $u - v$ , we may use the relation

$$u(x_m, t_m) - v(y_m, t_m) - \frac{2\sigma}{T - t_m} - \frac{1}{m} \geq \frac{(x_m - y_m)^2}{2\varepsilon}$$

together with (3.2) to deduce that

$$|x_m - y_m| \leq C\varepsilon^{\frac{1}{2}}$$

for some  $C > 0$  independent of  $\varepsilon$  and  $m$ . Suppose that  $x_m, y_m \in \mathbb{R} \setminus \mathcal{I}$ . Then we have

$$\Phi(x_m, y_m, t_m) \leq u(x_m, t_m) - v(y_m, t_m) \leq u(x_m, t_m) - u(x_m, t_m) \leq CL\varepsilon^{\frac{1}{2}}.$$

This contradicts (3.3) when  $\varepsilon > 0$  is taken small enough to satisfy  $CL\varepsilon^{\frac{1}{2}} < \mu$ . The above argument also implies that  $x_m, y_m$  are bounded in  $\mathbb{R}$  uniformly for all  $\varepsilon, m$ . Hence, by the upper semicontinuity of  $\Phi$ , the maximum of  $\Phi$  can be attained at the limit of a convergent subsequence of  $(x_m, y_m, t_m)$ .

Let  $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^2 \times [\tau, T]$  be a maximizer of  $\Phi$ . Taking a subsequence such that  $(\hat{x}, \hat{y}, \hat{t}) \rightarrow (x^*, x^*, t^*)$  for some  $(x^*, t^*) \in \mathbb{R} \times [\tau, \infty)$  as  $\varepsilon \rightarrow 0$ , we have

$$u(x^*, t^*) - v(x^*, t^*) \geq \mu.$$

Since  $u(\cdot, \tau) \leq v(\cdot, \tau)$ , we deduce that  $t^* \neq \tau$ , which implies that  $\hat{t} > \tau$  when  $\varepsilon > 0$  is sufficiently small. Take

$$\Phi_b(x, y, t) = u(x, t) - v(y, t) - \frac{(x - y)^2}{2\varepsilon} - b(t - \hat{t})^2 - \frac{\sigma}{T - t},$$

where we choose  $b \in \mathbb{R}$  satisfying

$$0 < b < \frac{\sigma}{2T^3} \quad (3.4)$$

for our later use. Then it is clear that  $\Phi_b$  attains the same maximum value  $\mu_\varepsilon$  at  $(\hat{x}, \hat{y}, \hat{t})$  as well. Moreover, any maximizer  $(x, y, t)$  of  $\Phi_b$  clearly fulfills that  $t = \hat{t}$ ; in other words, any maximizer of  $\Phi_b$  is a maximizer of  $\Phi$ .

Hence, setting

$$K = \{(x, y) \in \mathbb{R}^2 : \Phi_b(x, y, \hat{t}) = \mu_\varepsilon\},$$

we can express the set of maximizers of  $\Phi_b$  as  $K \times \{\hat{t}\}$ .

Let  $0 < a < \min\{1, \mu_\varepsilon\}$ . We consider

$$\Psi_a(x, y, t) := u(x, t) - v(y, t) - \frac{(x - y)^2}{2\varepsilon} + ax^2 - b(t - \hat{t})^2 - \frac{\sigma}{T - t}.$$

Since  $\Phi_b$  has a maximum attained strictly at a compact subset  $K \times \{\hat{t}\}$  of  $\mathbb{R}^2 \times (\tau, T)$  and  $\Psi_a(x, y, t) = \Phi_b(x, y, t) + aQ(x)$  with  $Q$  bounded, a standard argument [9, Lemma 2.2.5] yields that  $\Psi_a(x, y, t)$  attains a local maximum  $(x_a, y_a, t_a) \in \mathbb{R}^2 \times (\tau, T)$  close to  $K \times \{\hat{t}\}$  for all  $a > 0$  small. We therefore can take a convergent subsequence, still indexed by  $a$  for simplicity, such that  $(x_a, y_a, t_a) \rightarrow (x_0, y_0, \hat{t})$  as  $a \rightarrow 0$  for some  $(x_0, y_0) \in K$ . Below we turn back to use  $(\hat{x}, \hat{y})$  instead of  $(x_0, y_0)$  to denote the limit of  $(x_a, y_a)$  for our convenience.

We next apply the Crandall-Ishii lemma (cf. [7, Theorem 8.3]) for  $\Psi_a$  at  $(x_a, y_a, t_a, s_a)$  to obtain

$$(\eta_1, p_1, z_1) \in \overline{P}^{2,+} u(x_a, t_a), \quad (\eta_2, p_2, z_2) \in \overline{P}^{2,-} v(y_a, t_a)$$

satisfying

$$\eta_1 - \eta_2 = 2b(t_a - \hat{t}) + \frac{\sigma}{(T - t_a)^2}, \quad (3.5)$$

$$p_1 = -2ax_a + \frac{x_a - y_a}{\varepsilon}, \quad p_2 = \frac{x_a - y_a}{\varepsilon}$$

and

$$-\frac{3}{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} z_1 + 2a & 0 \\ 0 & -z_2 \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad (3.6)$$

Note that (3.5) implies that

$$\eta_1 - \eta_2 > 0 \quad (3.7)$$

due to the requirement on  $b$  as given in (3.4).

By (3.6), we deduce that

$$z_1 - z_2 \leq -2a. \quad (3.8)$$

Using the maximality of  $\Psi_a$  at  $(x_a, y_a, t_a)$ , we now adopt a part of the definition of subsolutions and supersolutions to obtain

$$z_1 \geq -\frac{1}{t_a}, \quad z_2 \leq \frac{1}{t_a}. \quad (3.9)$$

By (3.8), we therefore have

$$\begin{aligned} z_1 &\leq \frac{1}{t_a} + z_1 - z_2 \leq \frac{1}{t_a} - 2a < \frac{1}{t_a}, \\ z_2 &\geq -\frac{1}{t_a} + z_2 - z_1 \geq -\frac{1}{t_a} + 2a > \frac{1}{t_a}. \end{aligned}$$

It thus follows from the remaining part of definition of sub- and supersolutions that

$$\eta_1 \leq 0 \leq \eta_2$$

which is obviously a contradiction to (3.7).  $\square$

### 3.2 Large exponent limit

We first consider the behavior of  $U_\alpha$  near space infinity. Since  $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ , we may regularize  $u_0$  to obtain existence of  $0 < \lambda_0 < 1$ , an open bounded subset  $\mathcal{I} \subset \mathbb{R}$  and two family of functions  $\psi_\alpha^\pm \in C^\infty(\mathbb{R})$  ( $\alpha \geq 1$ ) such that

$$\psi_\alpha^- \leq u_0 \leq \psi_\alpha^+ \quad \text{in } \mathbb{R},$$

$$\sup_{x \in \mathbb{R} \setminus \mathcal{I}} \{ |\psi_\alpha^+(x) - u_0(x)| + |\psi_\alpha^-(x) - u_0(x)| \} \leq \frac{1}{\alpha}$$

and

$$\max \left\{ \sup_{\mathbb{R}} |(\psi_\alpha^+)_{xx}|, \sup_{\mathbb{R}} |(\psi_\alpha^-)_{xx}| \right\} \leq \lambda_0. \quad (3.10)$$

It follows that

$$w_\alpha^+(x, t) = \psi_\alpha^+(x) + \lambda_0^\alpha t, \quad w_\alpha^-(x, t) = \psi_\alpha^-(x) - \lambda_0^\alpha t$$

are respectively a super- and subsolution of (1.1), which by Theorem 2.1 yields that  $w_\alpha^- \leq u_\alpha \leq w_\alpha^+$  in  $\mathbb{R} \times [0, \infty)$ . Therefore, we have

$$u_0(x) - \frac{1}{\alpha} - \frac{(\lambda_0 t)^\alpha}{\alpha} \leq U_\alpha(x, t) \leq u_0(x) + \frac{1}{\alpha} + \frac{(\lambda_0 t)^\alpha}{\alpha}$$

for any  $(x, t) \in (\mathbb{R} \setminus \mathcal{I}) \times [\tau, \infty)$  and thus we obtain the following result.

**Lemma 3.3** (Convergence outside a compact set). *Assume that  $u_0 \in C_c^{1,1}(\mathcal{T})$ . Let  $u_\alpha$  be the solution of (GP) and  $U_\alpha$  be as in (1.6). Let  $\tau$  be given as in (1.5). Then there exist an open bounded subset  $I \subset \mathbb{R}$  and  $\lambda_0 \in (0, 1)$  such that  $U_\alpha \rightarrow 0$  locally uniformly in  $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, 1/\lambda_0)$  as  $\alpha \rightarrow \infty$ .*

Our first main result is as follows.

**Theorem 3.4** (Large exponent convergence for the rescaled equation). *Assume that  $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ . Let  $\tau > 0$  be given in (1.5). Let  $u_\alpha$  be the unique solution of (PH) and  $U_\alpha$  be as in (1.6). Let  $\lambda_0 \in (0, 1)$  be given by (3.10). Then  $U_\alpha \rightarrow U$  locally uniformly in  $\mathbb{R} \times [\tau, 1/\lambda_0)$  as  $\alpha \rightarrow \infty$ , where  $U$  is the unique continuous solution of (3.1) and (1.10) with  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$  and  $F$  given by (1.11).*

We use the so-called Barles-Perthame procedure [4] to prove the convergence. Let us take the half relaxed limits of  $U_\alpha$  in  $\mathbb{R} \times [\tau, 1/\lambda_0)$ :

$$\bar{U} = \limsup_{\alpha \rightarrow \infty}^* U_\alpha, \quad \underline{U} = \liminf_{\alpha \rightarrow \infty}^* U_\alpha. \quad (3.11)$$

**Proposition 3.5** (Sub- and supersolution properties of relaxed limits). *Let  $\tau > 0$ . Assume that  $U_\alpha$  is a solution of (1.7). Then  $\bar{U}$  and  $\underline{U}$  as in (3.11) are respectively a subsolution and a supersolution of (1.9).*

*Proof.* We only prove that  $\bar{U}$  is a subsolution. The supersolution part can be proved using a symmetric argument. Let  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$ .

Suppose that there exist  $(x_0, t_0) \in \mathcal{O}$  and  $\phi \in C^2(\mathcal{O})$  such that  $\bar{U} - \phi$  attains a strict maximum at  $(x_0, t_0)$ . By [9, Lemma 2.2.5],  $U_\alpha - \phi$  has a local maximum at  $(x_\alpha, t_\alpha)$  with  $(x_\alpha, t_\alpha) \rightarrow (x_0, t_0)$  as  $\alpha \rightarrow \infty$ . Since  $U_\alpha$  is a subsolution of (1.7), we have

$$\phi_t(x_\alpha, t_\alpha) \leq t_\alpha^{\alpha-1} |\phi_{xx}(x_\alpha, t_\alpha)|^{\alpha-1} \phi_{xx}(x_\alpha, t_\alpha). \quad (3.12)$$

It follows that

$$\liminf_{\alpha \rightarrow \infty} (t_\alpha \phi_{xx}(x_\alpha, t_\alpha)) \geq -1,$$

which implies that  $\phi_{xx}(x_0, t_0) \geq -1/t_0$ . Moreover, if

$$-1/t_0 \leq \phi_{xx}(x_0, t_0) < 1/t_0,$$

then sending  $\alpha \rightarrow \infty$  in (3.12) yields  $\phi_t(x_0, t_0) \leq 0$ .  $\square$

**Proposition 3.6** (Verification of initial value). *Assume that  $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ . Let  $\tau > 0$ . Let  $u_\alpha$  be the solution of (PH). Let  $U_\alpha$  be as in (1.6). Then  $\bar{U}$  and  $\underline{U}$  in (3.11) satisfy*

$$\bar{U}(x, \tau) \leq u_0 \leq \underline{U}(x, \tau) \quad \text{for all } x \in \mathbb{R}.$$

*Proof.* In order to show  $\bar{U}(\cdot, \tau) \leq u_0$  in  $\mathbb{R}$ , we construct supersolutions of (1.1) and (1.7). Set

$$w(x, t) = u_0(x) + L_0^\alpha t \quad \text{for } (x, t) \in \mathbb{R} \times [0, \infty)$$

and

$$W(x, t) = U_\alpha(x, \tau) + \frac{L_0^\alpha}{\alpha} (t^\alpha - \tau^\alpha) \quad \text{for } (x, t) \in \mathbb{R} \times [\tau, \infty),$$

where  $L_0$  is as in (1.4).

We claim that  $w$  and  $W$  are respectively supersolution of (1.1) and (1.7) for all  $\alpha > 0$ . To prove this, we may use a standard mollification to approximate  $u_0$  with  $u_0^\varepsilon \in C^\infty(\mathbb{R})$  so that for any  $\delta > 0$ , we get

$$|(u_0^\varepsilon)_{xx}| \leq L_0 + \delta \quad \text{in } \mathbb{R}$$

when  $\varepsilon > 0$  is sufficiently small. It is then easily verified that

$$w^\varepsilon(x, t) = u_0^\varepsilon(x) + \frac{(L_0 + \delta)^\alpha}{\alpha} t$$

is a supersolution of (1.1). Adopting the standard stability of viscosity solutions, we deduce that  $w$  is also a supersolution of (1.1) by letting  $\varepsilon \rightarrow 0$  and then  $\delta \rightarrow 0$ . The claim for  $W$  can be analogously proved.

Moreover, since  $w(\cdot, 0) = u_0$  in  $\mathbb{R}$ , we have

$$u_\alpha \leq w \quad \text{in } \mathbb{R} \times [0, \infty)$$

for all  $\alpha > 0$  by Theorem 2.1. A similar argument, combined with a comparison result for (1.7), yields that

$$U_\alpha \leq W \quad \text{in } \mathbb{R} \times [\tau, \infty).$$



Hence, we can use the condition (1.5) on  $\tau$  to deduce that

$$U_\alpha(x, \tau) = u_\alpha\left(x, \frac{\tau_\alpha}{\alpha}\right) \leq w\left(x, \frac{\tau_\alpha}{\alpha}\right) \leq u_0(x) + \frac{(L_0\tau)^\alpha}{\alpha} \leq u_0(x) + \frac{1}{\alpha}$$

for all  $x \in \mathbb{R}$  and all  $\alpha \geq 1$ , which in turn implies that

$$U_\alpha(x, t) \leq W(x, t) \leq u_0(x) + \frac{1}{\alpha} + \frac{L_0^\alpha}{\alpha}(t^\alpha - \tau^\alpha).$$

In view of (1.5) again, letting  $\alpha \rightarrow \infty$ , we end up with

$$\overline{U}(x, \tau) \leq u_0(x) \quad \text{for all } x \in \mathbb{R}.$$

We can similarly show that

$$\underline{U}(x, \tau) \geq u_0(x) \quad \text{for all } x \in \mathbb{R}.$$

□

We are now in a position to prove Theorem 3.4.

*Proof.* As shown in Proposition 3.5, the half relaxed limits  $\overline{U}$  and  $\underline{U}$  are respectively a subsolution and a supersolution of (3.1) with  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$ . In view of Proposition 3.6, we have  $\overline{U}(\cdot, 0) = \underline{U}(\cdot, 0) = u_0$  in  $\mathbb{R}$ .

Since  $u_\alpha(x, t)$  preserves the Lipschitz continuity of  $u_0$  in  $x$  for all  $t \in [0, \infty)$ , we easily see that  $U_\alpha$ ,  $\overline{U}$  and  $\underline{U}$  are all Lipschitz in space with the same Lipschitz constant as  $u_0$ . In addition, by Lemma 3.3, we have  $\overline{U} = \underline{U}$  in  $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, 1/\lambda_0)$  for some bounded open set  $\mathcal{I} \subset \mathbb{R}$ .

Using Theorem 3.2 with  $T = 1/\lambda_0$ , we end up with  $\overline{U} \leq \underline{U}$  in  $\mathbb{R} \times [\tau, 1/\lambda_0)$ . Since by definition it is clear that  $\overline{U} \geq \underline{U}$ , this yields that  $\overline{U} = \underline{U}$  in  $\mathbb{R} \times [\tau, 1/\lambda_0)$ . We thus obtain the locally uniform convergence of  $U_\alpha$  as  $\alpha \rightarrow \infty$ . □

We have shown that there exists a unique solution of (3.1) and (1.10) with  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$ . The following is then an immediate consequence of Theorem 3.4 and (1.6).

**Theorem 3.7** (Large exponent convergence for  $C^{1,1}$  initial data). *Assume that  $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ . Let  $\tau$  be given by (1.5) and  $\lambda_0$  be as in Lemma 3.3. Let  $u_\alpha$  be a unique solution of (PH). Let  $U$  be the unique solution of (3.1) with (1.10) and  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$ . Then  $u_\alpha(x, t) \rightarrow U_\infty(x) := U(x, 1)$  locally uniformly for all  $(x, t) \in \mathbb{R} \times (0, \infty)$  as  $\alpha \rightarrow \infty$ .*

We next show that the limit of  $u_\alpha$  above does not depend on the choice of  $\tau$ .

**Proposition 3.8** (Independence of choices of initial time). *Assume that  $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ . Let  $L_0$  be given by (1.4). Assume that  $0 < \tau_1 < \tau_2 < 1/L_0$ . Let  $\mathcal{O}_i = \mathbb{R} \times (\tau_i, 1/\lambda_0)$  and let  $U_i$  be the unique solution of (3.1) and (1.10) with  $\mathcal{O} = \mathcal{O}_i$  for  $i = 1, 2$ . Then  $U_1(x, \tau_2) = U_2(x, \tau_2)$  for all  $x \in \mathbb{R}$ . In particular,  $U_1 = U_2$  in  $\mathbb{R} \times [\tau_2, 1/\lambda_0)$ .*

*Proof.* Since  $\tau_1 < \tau_2 < 1/L_0$ , it is easily seen that

$$t|(u_0)_{xx}| < 1 \quad \text{a.e. in } \mathbb{R} \tag{3.13}$$

for all  $t \in (\tau_1, \tau_2)$ , which is equivalent to saying that

$$-1/\tau_2 < (u_0)_{xx} < 1/\tau_2$$

holds in  $\mathbb{R}$  in the viscosity sense. It follows that

$$U(x, t) = u_0(x) \tag{3.14}$$

is a solution of (3.1) with  $\mathcal{O} = \mathbb{R} \times (\tau_1, \tau_2)$ . As a consequence of the uniqueness result (Theorem 3.2) and the continuity of  $U_1$ , we have  $U_1(x, t) = u_0(x)$  for all  $(x, t) \in \mathbb{R} \times [\tau_1, \tau_2]$ . Since  $U_1$  and  $U_2$  share the same value at  $t = \tau_2$ , it is obvious that  $U_1 = U_2$  in  $\mathbb{R} \times [\tau_2, 1/\lambda_0)$ .  $\square$

We finally drop the  $C^{1,1}$  regularity of  $u_0$  and prove the more general convergence result in Theorem 1.1.

*Proof of Theorem 1.1.* For a general initial value  $u_0 \in C_b^{0,1}(\mathbb{R})$ , we can approximate it by regularization with the standard mollifier. We may assume that there exist a bounded open set  $\mathcal{I} \subset \mathbb{R}$  and a sequence  $u_{0,m} \in C^\infty(\mathbb{R}) \cap C_b^{0,1}(\mathbb{R})$  such that  $u_{0,m}$  converges uniformly to  $u_0$  in  $\mathbb{R}$  and

$$|(u_{0,m})_{xx}(x)| \leq \lambda \quad \text{for any } x \in \mathbb{R} \setminus \mathcal{I}$$

for some  $\lambda \in (0, 1)$ . Note that for every  $m > 0$ , we can obtain a large exponent limit  $U_{\infty,m}$  corresponding to the solution  $u_{\alpha,m}$  of (PH) with the initial condition  $u_{0,m}$ . Moreover, by Theorem 2.1, we have

$$\sup_{\mathbb{R} \times [0, \infty)} |u_{\alpha,m} - u_{\alpha,m'}| \leq \sup_{\mathbb{R}} |u_{0,m} - u_{0,m'}|$$

for all  $m, m' > 0$  and  $\alpha \geq 1$ , which implies, by letting  $\alpha \rightarrow \infty$ , that

$$\sup_{\mathbb{R}} |U_{\infty,m} - U_{\infty,m'}| \leq \sup_{\mathbb{R}} |u_{0,m} - u_{0,m'}|.$$

In other words,  $U_{\infty,m}$  is a Cauchy consequence in  $C_b^{0,1}(\mathbb{R})$ . Since  $U_{\infty,m} = u_0$  in  $\mathbb{R} \setminus \mathcal{I}$  for all  $m > 0$ , we can find a unique limit  $U_\infty$  of  $U_{\infty,m}$  in  $C_b^{0,1}(\mathbb{R})$ . It is then easy to see that  $u_\alpha(x, t) \rightarrow U_\infty(x)$  locally uniformly in  $\mathbb{R} \times [\tau, \infty)$  as  $\alpha \rightarrow \infty$ .  $\square$

Let us finally discuss the large exponent behavior of solution to (PH) when  $u_0$  is Lipschitz and convex in  $\mathbb{R}$ .

Suppose that  $u_0$  is smooth, Lipschitz and convex in  $\mathbb{R}$ . Recall that  $U_\infty = U(\cdot, 1)$ , where  $U$  is the solution of (3.1) with  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$  and the initial value  $u_0$ . It turns out that  $U_\infty$  satisfies

$$-W_{xx} + 1 \geq 0 \quad \text{in } \mathbb{R} \tag{3.15}$$

in the viscosity sense. To see this, suppose that there exist  $x_0 \in \mathbb{R}$  and  $\phi \in C^2(\mathbb{R})$  such that  $U_\infty - \phi$  attains a strict minimum over  $\mathbb{R}$  at  $x_0$ . Then for any  $\beta > 0$  large,

$$(x, t) \mapsto U(x, t) - \phi(x) + \beta|t - 1|^2$$

attains a local minimum at  $(x_\beta, t_\beta) \in \mathbb{R} \times (0, \infty)$  with  $(x_\beta, t_\beta) \rightarrow (x_0, 1)$  as  $\beta \rightarrow \infty$ . It then follows from Definition 3.1 that

$$\phi_{xx}(x_\beta) \leq 1/t_\beta,$$

which yields that  $\phi_{xx}(x_0) \leq 1$  by passing to the limit as  $\beta \rightarrow \infty$ .

In addition, thanks to the convexity of  $u_0$ , we can show that  $u_0$  is a stationary subsolution of (PH), which by Theorem 2.1 implies that  $u_\alpha(x, t) \geq u_0(x)$  for all  $(x, t) \in \mathbb{R} \times [0, \infty)$ . In view of Theorem 3.7, passing to the limit as  $\alpha \rightarrow \infty$ , we deduce that

$$U_\infty \geq u_0 \quad \text{in } \mathbb{R}. \quad (3.16)$$

Moreover, we can show that  $U_\infty \leq W$  for any supersolution  $W$  of (3.15) that satisfies  $W \geq u_0$ . To prove this, it suffices to show the same result for  $W$  that has at most a linear growth at space infinity, since one can reduce the value of  $W$  to meet this condition. It is not difficult to verify that  $W$  itself is a supersolution of (3.1) with  $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$  with  $\tau, \lambda \in (0, 1)$  depending on the regularity of  $u_0$ . Then by Theorem 3.2, we get

$$U(x, t) \leq W(x) \quad \text{for any } (x, t) \leq \mathbb{R} \times (\tau, 1/\lambda),$$

which implies that

$$U_\infty(x) = U(x, 1) \leq W(x) \quad \text{for any } x \in \mathbb{R}. \quad (3.17)$$

Our argument above amounts to saying that  $U_\infty$  is the minimal supersolution of the obstacle problem

$$\min\{-U_{xx} + 1, U - u_0\} = 0 \quad \text{in } \mathbb{R}; \quad (3.18)$$

that is,

$$U_\infty(x) = \inf\{U(x) : U \text{ is a viscosity supersolution of (3.18)}\}. \quad (3.19)$$

For  $u_0$  that is only Lipschitz and convex without smoothness, we can obtain the same result by approximation. This result is consistent with [12, Theorem 8.3].

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