Construction and stability analysis of one-peak symmetric stationary solutions to the Schnakenberg model with heterogeneity

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1 Introduction and Main Results

In this paper, based on a recent work [5], we present our study on the existence and the linear stability of stationary solutions for the following Schnakenberg model:

$$u_t = \varepsilon^2 u_{xx} + d\varepsilon - u + g(x)u^2 v, \quad x \in (-1,1),\ t > 0, \tag{1}$$

$$\varepsilon v_t = Dv_{xx} + \frac{1}{2} - \frac{c}{\varepsilon} g(x)u^2 v, \quad x \in (-1,1),\ t > 0, \tag{2}$$

$$u_x(\pm 1) = v_x(\pm 1) = 0, \tag{3}$$

where d and c are positive constants, ε^2 and D are positive diffusion coefficients. $u(x,t)$ and $v(x,t)$ represent the density of two chemical substances. Here, $g(x)$ is a positive function, which represents the reaction speed of the chemical reaction at $x \in (-1,1)$ and may vary on the location x, for example by the effect of temperature.

Our system (1)-(3) is obtained from the original Schnakenberg model:

$$U_t = D_1 U_{xx} + a - U + g(x)U^2 V, \quad x \in (-1,1),\ t > 0,$$

$$V_t = D_2 V_{xx} + b - g(x)U^2 V, \quad x \in (-1,1),\ t > 0,$$

$$U_x(\pm 1) = V_x(\pm 1) = 0$$

by using the spacial scaling: $c = \frac{1}{4b^2}, d = ac^{-\frac{1}{2}} = 2ab$, and

$$U = \frac{1}{2b\varepsilon} u, \quad V = 2b\varepsilon v, \quad D_1 = \varepsilon^2, \quad D_2 = \frac{D}{\varepsilon}.$$

Especially, we treat sufficiently small ε and a fixed D, i.e. the ratio of diffusion coefficients $\frac{D}{\varepsilon^2}$ is large (cf. Turing's diffusion-driven instability). Moreover, (2) means that v reacts very rapidly than u in our model.

Inspired by the work of Iron, Wei and Winter [3] which studied in the case $d = 0$ and $g(x) = 1$, the purpose of our study is to investigate the effect of symmetric heterogeneity $g(x)$, namely $g(x) = g(-x)$, on the linear stability of stationary solutions for (1)-(3) rigorously. To state our main results, we prepare some notations. Let w_0 be the unique solution of

$$w_0'' - w_0 + w_0^2 = 0, \quad x \in \mathbb{R},$$
$$w_0 > 0,\ w_0(0) = \max_{\mathbb{R}} w_0,\ \lim_{|y|\to\infty} w_0(y) = 0.$$

It is known that w_0 is unique and can be written explicitly $w_0(y) = \frac{3}{2}(\cosh\frac{y}{2})^{-2}$. Let w be the unique solution of the following problem:

$$w'' - w + g(0)w^2 = 0, \quad x \in \mathbb{R},$$
$$w > 0,\ w(0) = \max_{\mathbb{R}} w,\ \lim_{|y|\to\infty} w(y) = 0.$$

Then it is easy to see $w(y) = g(0)^{-1}w_0(y)$. Let χ be a cut-off function:

$$\chi \in C_0^\infty(\mathbb{R})\ ,\ 0 \le \chi \le 1\ ,\ \chi(x) = \begin{cases} 1, & |x| < \frac{1}{4}, \\ 0, & |x| > \frac{1}{2}. \end{cases}$$

Define symmetric function spaces: for each $a \in (0, \infty)$,

$$L_s^2(-a,a) := \big\{u \in L^2(-a,a) \bigm| u(x) = u(-x)\big\},$$
$$H_s^2(-a,a) := \big\{u \in H^2(-a,a) \bigm| u(x) = u(-x), u'(\pm a) = 0\big\}.$$

Let $I := (-1,1)$ and $I_\varepsilon := \big(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\big)$ for $\varepsilon > 0$. We also use the following notation for the rescaling: for a function $u : I \to \mathbb{R}$, define $\overline{u}(y) := u(\varepsilon y)$ $(y \in I_\varepsilon)$.

The steady-state problem for (1)$\sim$(3) is the following:

$$0 \ = \varepsilon^2 u'' + d\varepsilon - u + g(x)u^2 v, \quad x \in (-1,1), \tag{4}$$
$$0 \ = Dv'' + \tfrac{1}{2} - \tfrac{c}{\varepsilon} g(x)u^2 v, \quad x \in (-1,1), \tag{5}$$
$$u'(\pm 1) = v'(\pm 1) = 0. \tag{6}$$

First, we state the existence of a one-peak solution.

Theorem 1 *Fix $D < +\infty$ arbitrarily. Assume that $g(x)$ is positive, Lipschitz continuous and satisfies $g(x) = g(-x)$. Then, there exists a sufficiently small $\varepsilon_1 > 0$ such that, for $0 < \varepsilon < \varepsilon_1$, (4)$\sim$ (6) admits a symmetric one-peak solution $(u_\varepsilon(x), v_\varepsilon(x)) \in H_s^2(I_\varepsilon) \times H_s^2(I_\varepsilon)$, where $u_\varepsilon(x)$ concentrates at $x = 0$. Moreover, $u_\varepsilon(x)$ takes the following asymptotic form:*

$$u_\varepsilon(x) = w_\varepsilon(x) + \phi_\varepsilon(x), \tag{7}$$

where

$$w_\varepsilon(x) := \frac{1}{\xi_0} w\Big(\frac{x}{\varepsilon}\Big)\chi(x), \quad \xi_0 := cg(0)\int_{\mathbb{R}} w^2(y)dy$$

and $\overline{\phi_\varepsilon} \in H^2_s(I_\varepsilon)$ *such that*

$$||\overline{\phi_\varepsilon}||_{H^2(I_\varepsilon)} \le C\sqrt{\varepsilon} \tag{8}$$

holds for some constant $C > 0$ *independent of* ε*. Also,* $v_\varepsilon(x)$ *satisfies*

$$v_\varepsilon(0) = \xi_0 + O(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0. \tag{9}$$

Moreover, there exists $v_0 \in H^1(I)$ *such that* $v_\varepsilon \rightharpoonup v_0$ *weakly in* $H^1(I)$*, where* v_0 *satisfies*

$$\begin{gathered} -Dv_0''(x) = \tfrac{1}{2} - \delta_0(x), \quad x \in (-1,1), \\ v_0(0) = \xi_0, \quad v_0'(\pm 1) = 0 \end{gathered}$$

and $\delta_0(x)$ *is the Dirac's delta function.*

Next, we study the linear stability of the solutions $(u_\varepsilon, v_\varepsilon)$ constructed in Theorem 1. We linearize the system (1)$\sim$ (3) at $(u_\varepsilon, v_\varepsilon)$ and obtain the following eigenvalue problem:

$$\varepsilon^2\varphi_\varepsilon'' - \varphi_\varepsilon + 2gu_\varepsilon v_\varepsilon \varphi_\varepsilon + gu_\varepsilon^2\psi_\varepsilon = \lambda_\varepsilon\varphi_\varepsilon, \quad x \in (-1,1), \tag{10}$$

$$D\psi_\varepsilon'' - \frac{2c}{\varepsilon}gu_\varepsilon v_\varepsilon\varphi_\varepsilon - \frac{c}{\varepsilon}gu_\varepsilon^2\psi_\varepsilon = \varepsilon\lambda_\varepsilon\psi_\varepsilon, \quad x \in (-1,1), \tag{11}$$

$$\varphi_\varepsilon'(\pm 1) = \psi_\varepsilon'(\pm 1) = 0,$$

where, λ_ε is an eigenvalue, and $(\varphi_\varepsilon, \psi_\varepsilon) \neq (0,0)$ is an eigenfunction. We say that the solution $(u_\varepsilon, v_\varepsilon)$ is stable if $\mathrm{Re}\lambda_\varepsilon < 0$ holds for all eigenvalues and unstable if there exists an eigenvalue satisfying $\mathrm{Re}\lambda_\varepsilon > 0$. We have the following result on the stability.

Theorem 2 *Fix* $D < +\infty$*. Let* $\varepsilon > 0$ *be sufficiently small. Let* $(u_\varepsilon, v_\varepsilon)$ *be the solution given in Theorem 1 . Then, we have the following* **for large eigenvalues***, namely* $\lambda_\varepsilon \to \lambda_0 \neq 0$*:*

(1) $(u_\varepsilon, v_\varepsilon)$ *is stable for any* $D < +\infty$*, namely* $\mathrm{Re}(\lambda_\varepsilon) < 0$ *holds.*

Furthermore, let $g \in C^3(-1,1)$*. Then, we have the following* **for small eigenvalues***, namely* $\lambda_\varepsilon \to 0$*:*

(2) If $g''(0) \le 0$*, then* $(u_\varepsilon, v_\varepsilon)$ *is stable for any* $D < +\infty$*.*

(3) If $g''(0) > 0$*,* $(u_\varepsilon, v_\varepsilon)$ *is stable for* $D < D_1$*,* $(u_\varepsilon, v_\varepsilon)$ *is unstable for* $D > D_1$*, where,* $D_1 > 0$ *is*

$$D_1 := \frac{1}{2c\int_{\mathbb{R}} w_0^2} \cdot \frac{g^2(0)}{g''(0)} = \frac{1}{12c} \cdot \frac{g^2(0)}{g''(0)}.$$

In fact, we have the following asymptotic behavior of λ_ε *as* $\varepsilon \to 0$*:*

$$\lambda_\varepsilon = \varepsilon^2 \frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}} (w')^2} \Big(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \Big) + O(\varepsilon^{\frac{5}{2}}). \tag{12}$$

Remark 1 *Note* $\int_{\mathbb{R}} w_0^2 = 6$ *and*

$$\xi_0 = cg(0) \int_{\mathbb{R}} w^2(y)dy = cg(0)^{-1} \int_{\mathbb{R}} w_0^2(y)dy = 6cg(0)^{-1}.$$

Hence, for the case $g''(0) > 0$ *the condition* $D < D_1$ *is equivalent to*

$$\Big(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \Big) < 0.$$

Remark 2 *Since we are concerned with the existence of unstable eigenvalues, we can assume that* $\mathrm{Re}\lambda_\varepsilon \geq -\frac{1}{4}$ *for example. We can show that eigenvalues* λ_ε *are uniformly bounded under the assumption* $\mathrm{Re}\lambda_\varepsilon \geq -\frac{1}{4}$*. Therefore, we can assume that there exists a* λ_0 *such that* $\lambda_\varepsilon \to \lambda_0$ *as* $\varepsilon \to 0$*, taking a subsequence if necessary.*

Remark 3 *In the case* $g(x) = 1$ *and* $d = 0$*, by the result of Iron, Wei, and Winter[3], the one-peak symmetric solution is stable for any* $D > 0$*. Compared with the case* $g(x) = 1$*, Theorem 2 reveals the strong influence of the heterogeneity* $g(x)$ *on the stability of the one-peak symmetric solution. We should mention that a similar destabilization effect of the heterogeneity has been studied for the Gierer-Meinhardt system(see [8]). We note that our results also cover the case* $d = 0$*. We emphasize that, even in the case* $g(x) = 1$*, the remainder estimate* $O(\varepsilon^{\frac{5}{2}})$ *for small eigenvalues is more precise than the result of Iron, Wei and Winter. We also emphasize that for the case* $d > 0$ *we need to take care of the remainder terms more carefully, compared with the case* $d = 0$*.*

Even for non-symmetric heterogeneity $g(x)$, we can expect similar results. However, we need more computations and left to future works. For the related works with some heterogeneity in other Turing systems, see for example [2], [6], [7] , [8] and the references therein. Recently, for a given $N \in 2, N \in \mathbb{N}$ and a given symmetric $\frac{2}{N}$-periodic function $g(x)$ in the interval $I = (-1, 1)$, one of the authors studied the existence of multi-peak symmetric solutions and its stability in details (see [4]). We also mention that Ao and Liu[1] studied recently another heterogeneity effect on the existence and its stability for the Schnakenberg model with precursors.

2 Outline of the Proof of Theorem 1

2.1 Heuristic explanation of the choice of ξ_0

Before giving the outline of the proof of Theorem 1, we explain briefly why we choose ξ_0 as follows in Theorem 1:

$$\xi_0 := cg(0) \int_{\mathbb{R}} w^2(y)dy.$$

Suppose $\overline{u_\varepsilon}$ and $\overline{v_\varepsilon}$ are uniformly bounded. Then by the equation for v:

$$-D\overline{v_\varepsilon}'' = \frac{\varepsilon^2}{2} - \varepsilon c\overline{g}\overline{u_\varepsilon}^2\overline{v_\varepsilon}, \tag{13}$$

we have $|D\overline{v_\varepsilon}''(y)| \leq C\varepsilon$. Since $\overline{v_\varepsilon}$ is symmetric, we have $\overline{v_\varepsilon}'(0) = 0$. Therefore, for fixed $R > 0$ we have $|\overline{v_\varepsilon}'(y)| \leq CR\varepsilon$ $(|y| \leq R)$. This implies $\overline{v_\varepsilon}(y) \sim C_0$ $(|y| \leq R)$ for some positive constant C_0. On the other hand $\overline{u_\varepsilon}$ satisfies

$$-\overline{u_\varepsilon}'' = d\varepsilon - \overline{u_\varepsilon} + \overline{g}\overline{u_\varepsilon}^2\overline{v_\varepsilon}, \ y \in I_\varepsilon = (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}).$$

Now we expect $\overline{u_\varepsilon}(y) \sim A_0 w(y) := u_\infty(y)$. Then we have

$$-u_\infty''(y) + u_\infty(y) = g(0)u_\infty(y)^2 C_0, \ y \in \mathbb{R}.$$

So if we take $w(y)$ to be a solution to $-w'' + w = g(0)w^2$, we must have $A_0C_0 = 1$. Now integrating (13), we have

$$0 = 1 - c\int_{I_\varepsilon} \overline{g}\overline{u_\varepsilon}^2\overline{v_\varepsilon}\, dy.$$

So taking the limit, we would have

$$1 = c\int_{\mathbb{R}} g(0)A_0^2 w(y)^2 C_0\, dy.$$

Therefore, since $A_0C_0 = 1$ we should have

$$A_0 = \frac{1}{cg(0)\int_{\mathbb{R}} w^2\, dy}.$$

Thus if we define $\xi_0 := cg(0)\int_{\mathbb{R}} w^2(y)dy$, then we have

$$u_\varepsilon(x) \sim \frac{1}{\xi_0}w(\frac{x}{\varepsilon}) \quad \text{and} \quad v_\varepsilon(0) \sim \xi_0.$$

2.2 Outline of the construction by using the contraction mapping principle.

Let $u = w_\varepsilon + \phi$ with $\overline{\phi}(y) \in B(C_0)$, where

$$w_\varepsilon(x) := \frac{1}{\xi_0}w\Big(\frac{x}{\varepsilon}\Big)\chi(x), \quad \xi_0 := cg(0)\int_{\mathbb{R}} w^2(y)dy$$

and

$$B(C_0) := \left\{\overline{\phi} \in H_s^2(I_\varepsilon) \ \middle|\ ||\overline{\phi}||_{H^2(I_\varepsilon)} \leq C_0\sqrt{\varepsilon}\ , \ \overline{\phi}'(\pm\frac{1}{\varepsilon}) = 0\right\}, \tag{14}$$

where the constant C_0 is independent of ε, which will be chosen suitably later. Then, we can find a unique solution $v := T[u] = T[w_\varepsilon + \phi]$ of the second equation (5):

$$-Dv'' + \frac{c}{\varepsilon}g(x)u^2 v = \frac{1}{2}, \ x \in (-1, 1), \quad v'(\pm 1) = 0.$$

We seek a unique $\phi(x) \in H_s^2(I)$ such that $(u(x), v(x)) = (w_\varepsilon + \phi, T[w_\varepsilon + \phi])$ satisfies the first equation (4). Substituting $u = w_\varepsilon + \phi, v = T[w_\varepsilon + \phi]$ into the first equation (4):

$$-\varepsilon^2 u'' = d\varepsilon - u + g(x)u^2 T[u],\ x \in (-1,1), \quad u'(\pm 1) = 0,$$

we have

$$\varepsilon^2\phi'' - \phi + 2gw_\varepsilon\phi T[w_\varepsilon + \phi] + gw_\varepsilon^2 T[w_\varepsilon + \phi] + \varepsilon^2 w_\varepsilon'' - w_\varepsilon + d\varepsilon + g\phi^2 T[w_\varepsilon + \phi] = 0. \quad (15)$$

Using the Fréchet derivative $R_\varepsilon[\phi] = \langle T'[w_\varepsilon], \phi\rangle$, we have

$$S_\varepsilon[\phi] + gw_\varepsilon^2 T[w_\varepsilon] + \varepsilon^2 w_\varepsilon'' - w_\varepsilon + d\varepsilon + N_1[\phi] = 0,$$

where

$$S_\varepsilon[\phi] := \varepsilon^2\phi'' - \phi + 2gT[w_\varepsilon]w_\varepsilon\phi + R_\varepsilon[\phi]gw_\varepsilon^2 \quad (16)$$

and $N_1[\phi]$ is the higher order term. Here, in the y-variable, using $\overline{w_\varepsilon}'' - \overline{w_\varepsilon} = -\xi_0 g(0)\overline{w_\varepsilon}^2 + O(e^{-\frac{1}{4\varepsilon}})$ we rewrite as follows:

$$\overline{S_\varepsilon}[\overline{\phi}] + \overline{g}\ \overline{w_\varepsilon}^2\overline{T[w_\varepsilon]} - g(0)\xi_0\overline{w_\varepsilon}^2 + d\varepsilon + O(e^{-\frac{1}{4\varepsilon}}) + \overline{N_1[\phi]} = 0, \quad (17)$$

where

$$\overline{S_\varepsilon}[\overline{\phi}] := \overline{S_\varepsilon[\phi]} = \overline{\phi}'' - \overline{\phi} + 2\overline{g}\ \overline{T[w_\varepsilon]}\ \overline{w_\varepsilon}\ \overline{\phi} + \overline{R_\varepsilon[\phi]}\ \overline{g}\ \overline{w_\varepsilon}^2.$$

Now we have the following invertibility of the operator $\overline{S_\varepsilon}:\ H_s^2(I_\varepsilon) \to L_s^2(I_\varepsilon)$.

Lemma 1 ([5, Lemma 3.2]) *There exist $\varepsilon_0 > 0$ and $\lambda > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the following inequality holds:*

$$\left|\left|\overline{S_\varepsilon}[\overline{\phi}]\right|\right|_{L^2(I_\varepsilon)} \geq \lambda\left|\left|\overline{\phi}\right|\right|_{H^2(I_\varepsilon)},\ \overline{\phi} \in H_s^2(I_\varepsilon)\ . \quad (18)$$

Furthermore, $\mathrm{Ran}(\overline{S_\varepsilon}) = L_s^2(I_\varepsilon)$ *holds.*

Thus we have

$$\overline{\phi} = -\overline{S_\varepsilon}^{-1}\left[\overline{I[w_\varepsilon]}\right] - \overline{S_\varepsilon}^{-1}\left[\overline{N_1[\phi]}\right] =: M_\varepsilon[\overline{\phi}], \quad (19)$$

where

$$\overline{I[w_\varepsilon]} := \overline{g}\ \overline{w_\varepsilon}^2\overline{T[w_\varepsilon]} - g(0)\xi_0\overline{w_\varepsilon}^2 + d\varepsilon + O(e^{-\frac{1}{4\varepsilon}}).$$

If we choose $C_0 > 0$ large enough such that

$$||\overline{S_\varepsilon}^{-1}\left[\overline{I[w_\varepsilon]}\right]||_{H^2(I_\varepsilon)} \leq \frac{C_0}{2}\sqrt{\varepsilon} \quad (20)$$

we can show that M_ε is a contraction mapping on $B(C_0)$ for small $\varepsilon > 0$. Actually, we can choose C_0 so that $C_0 > \frac{4(C_1+d)}{\lambda}$, where C_1 is the constant will appear in Corollary 1 later. (Note that the constants $\lambda > 0$ and C_1 depend only on $w(x), g(x)$ and the fixed parameters $c > 0, D > 0$.) Thus, there exists a unique $\overline{\phi} \in B(C_0)$ which satisfies the desired equation.

2.3 Basic estimates, including the estimates for $T[w_\varepsilon + \phi]$.

We note the following estimates, which play key roles throughout this work.

Lemma 2 ([5, Lemma 2.8, 2.9]) *Fix $C_0 > 0$. For each $\overline{\phi} \in B(C_0)$, let $\eta(x) \in H^2(-1,1)$ satisfy*

$$-D\eta'' + \frac{cg(w_\varepsilon + \phi)^2}{\varepsilon}\eta = \frac{h}{\varepsilon},\ x \in (-1,1), \qquad \eta'(\pm 1) = 0, \tag{21}$$

where $h(x)$ is a given function on $L^2(-1,1)$. Then, the following estimates hold:

(1) $||\overline{\eta}||_{L^\infty(I_\varepsilon)} \leq C||\overline{h}||_{L^1(I_\varepsilon)}$.

(2) $||\overline{\eta}'||_{L^2(I_\varepsilon)} \leq C\sqrt{\varepsilon}||\overline{h}||_{L^1(I_\varepsilon)}$.

(3) $|\overline{\eta}(y)\overline{g}(y) - \overline{\eta}(0)\overline{g}(0)| \leq C\sqrt{\varepsilon|y|}||\overline{h}||_{L^1(I_\varepsilon)},\ y \in I_\varepsilon$.

Here, the constant C is independent of ε. Furthermore, if we have a uniform bound $\|\overline{h}\|_{L^1(I_\varepsilon)} \leq M$, then we have

$$\eta(0) = \xi_0 \int_{I_\varepsilon} \overline{h}\, dy + O(\sqrt{\varepsilon}) \quad \text{as} \quad \varepsilon \to 0. \tag{22}$$

By using Lemma 2, we can obtain the following estimate which allows the estimate (20).

Corollary 1 *There exists a constant C_1 such that the following estimates hold*

$$\|\overline{T[w_\varepsilon]}\|_{L^\infty(I_\varepsilon)} \leq C_1, \qquad |T[w_\varepsilon](0) - \xi_0| \leq C_1\sqrt{\varepsilon},$$

$$\|\overline{g}\ \overline{w_\varepsilon}^2\overline{T[w_\varepsilon]} - g(0)\xi_0\overline{w_\varepsilon}^2\|_{L^2(I_\varepsilon)} \leq C_1\sqrt{\varepsilon}.$$

(Proof.) By (1) , (3) of Lemma 2 and (22) as $\phi = 0$ and $h(x) = \frac{\varepsilon}{2}$, there exists a constant C_1, independent of ε and C_0, such that the following estimates hold:

$$\|\overline{T[w_\varepsilon]}\|_{L^\infty(I_\varepsilon)} \leq C_1, \quad |T[w_\varepsilon](0) - \xi_0| \leq C_1\sqrt{\varepsilon},$$

$$|\overline{T[w_\varepsilon]}(y)\overline{g}(y) - \overline{T[w_\varepsilon]}(0)\overline{g}(0)| \leq C_1\sqrt{\varepsilon|y|},\ y \in I_\varepsilon.$$

Thus we have

$$\begin{aligned}
&\overline{w_\varepsilon}^2(y)|\overline{T[w_\varepsilon]}(y)\overline{g}(y) - \xi_0\overline{g}(0)| \\
&\leq \overline{w_\varepsilon}^2(y)\Big(|\overline{T[w_\varepsilon]}(y)\overline{g}(y) - \overline{T[w_\varepsilon]}(0)\overline{g}(0)| + \overline{g}(0)|T[w_\varepsilon](0) - \xi_0|\Big) \\
&\leq g(0)C_1\overline{w_\varepsilon}^2(y)(\sqrt{\varepsilon|y|} + \sqrt{\varepsilon}) \leq g(0)\frac{C_1}{\xi_0^2}w^2(y)(\sqrt{\varepsilon|y|} + \sqrt{\varepsilon}),\ y \in I_\varepsilon.
\end{aligned}$$

This implies the desired estimate.

3 Global pointwise estimates for solutions

Since $\overline{u_\varepsilon}(y) = \overline{w_\varepsilon}(y) + \overline{\phi_\varepsilon}(y)$ with $\|\overline{\phi_\varepsilon}\|_{H^2(I_\varepsilon)} \le C\sqrt{\varepsilon}$, we easily have

$$|\overline{u_\varepsilon}(y)| \le C\sqrt{\varepsilon} + Ce^{-\frac{|y|}{\sqrt{2}}},\ y \in I_\varepsilon$$

by using the Sobolev's embedding theorem. However, this estimate is not enough to treat several error terms in the stability analysis. We need the following pointwise estimates for the solution $(u_\varepsilon, v_\varepsilon)$ in our stability analysis.

Lemma 3 ([5, Proposition 4.2]) *There exists a constant C, which is independent of ε, such that the following estimates hold:*
(i)

$$\|\overline{v_\varepsilon}'\|_{L^\infty(I_\varepsilon)} \le C\varepsilon,$$

(ii)

$$|\overline{u_\varepsilon}(y)| \le C(d\varepsilon + e^{-\frac{1}{\sqrt{2}\varepsilon}} + e^{-\frac{|y|}{\sqrt{2}}}),\ y \in I_\varepsilon,$$

(iii)

$$|\overline{u_\varepsilon}'(y)| \le C(d^2\varepsilon^2 + e^{-\frac{\sqrt{2}}{\varepsilon}} + e^{-\frac{|y|}{2}}),\ y \in I_\varepsilon.$$

These estimates can be obtained by using comparison arguments. In particular, $\overline{u_\varepsilon}(y)$ and $\overline{u_\varepsilon}'(y)$ are exponentially small near the boundary of I_ε if $d = 0$. We also have the following uniform bounds:

$$\|\overline{u_\varepsilon}\|_{L^\infty(I_\varepsilon)} \le C, \quad \|\overline{v_\varepsilon}\|_{L^\infty(I_\varepsilon)} \le C, \quad \|\overline{u_\varepsilon}\|_{L^2(I_\varepsilon)} \le C, \quad \|\overline{u_\varepsilon}'\|_{L^2(I_\varepsilon)} \le C.$$

4 Outline of the Proof of Theorem 2

We may assume that $\|\overline{\varphi_\varepsilon}\|_{H^2(I_\varepsilon)} = 1$. By the extension theorem we have $\|\overline{\varphi_\varepsilon}\|_{H^2(\mathbf{R})} \le C$. So, there exists a subsequence and $\overline{\varphi} \in H^2(\mathbf{R})$ such that $\overline{\varphi_\varepsilon}$ converges to $\overline{\varphi}$ weakly in $H^2(\mathbf{R})$ and strongly in $C^1_{loc}(\mathbf{R})$.

Lemma 4 (Boundedness of unstable eigenvalues, [5, Proposition 4.2]) *Assume* $\mathrm{Re}(\lambda_\varepsilon) \ge -\frac{1}{4}$. *Then, we have the following:*
(1) $\overline{\varphi} \ne 0$.
(2) There exists a constant C, independent of ε, such that $|\lambda_\varepsilon| \le C$.

By this Lemma, we may assume $\lambda_\varepsilon \to \lambda_0$ for some constant λ_0. We consider two cases:
(a) **large eigenvalue**: i.e. $\lambda_\varepsilon \to \lambda_0 \ne 0$.
(b) **small eigenvalue**: i.e. $\lambda_\varepsilon \to 0$.

4.1 Stability analysis for large eigenvalues

Lemma 5 *Assume* $\lambda_\varepsilon \to \lambda_0 \neq 0$. *Then, we have*

$$\overline{\varphi}''(y) - \overline{\varphi}(y) + 2w_0(y)\overline{\varphi}(y) - 2\frac{\int w_0\overline{\varphi}\,dz}{\int w_0^2\,dz}w_0(y)^2 = \lambda_0\overline{\varphi}(y).$$

Then, by the well-known lemma of Wei and Winter (see Lemma 2.2 in [3], or [8]) for nonlocal eigenvalue problem above, we can conclude $\mathrm{Re}\lambda_0 < 0$. So for sufficiently small $\varepsilon > 0$ we have $\mathrm{Re}\lambda_\varepsilon < 0$, namely λ_ε is a stable eigenvalue.

(Sketch of the proof of Lemma 5.) From the equation (11) for ψ_ε, we can show $\|\psi_\varepsilon\|_{H^1(I)} \leq C$ and apply Lemma 2 to obtain

$$\overline{\psi_\varepsilon}(0) = \psi_\epsilon(0) = \xi_0\int_{I_\epsilon}(-2c\overline{gu_\varepsilon v_\varepsilon\varphi_\varepsilon} - \varepsilon^2\lambda_\varepsilon\overline{\psi_\varepsilon})\,dy + O(\sqrt{\varepsilon})$$

$$\to \psi(0) = -2cg(0)\xi_0\int_{\mathbb{R}} w\overline{\varphi}\,dy = -2\xi_0^2\frac{\int_{\mathbb{R}} w\overline{\varphi}\,dy}{\int_{\mathbb{R}} w^2\,dy} \text{ as } \varepsilon \to 0. \tag{23}$$

Here we used $\xi_0 = cg(0)\int_{\mathbb{R}} w^2\,dy$. On the other hand, from the equation (10), we have

$$\overline{\varphi_\varepsilon}'' - \overline{\varphi_\varepsilon} + 2\overline{gu_\varepsilon v_\varepsilon\varphi_\varepsilon} + \overline{gu_\varepsilon}^2\overline{\psi_\varepsilon} = \lambda_\varepsilon\overline{\varphi_\varepsilon} \text{ on } I_\varepsilon.$$

Then , for any $\zeta \in C_0^\infty(\mathbb{R})$, we have

$$\int_{I_\varepsilon}\left(\overline{\varphi_\varepsilon}'' - \overline{\varphi_\varepsilon} + 2\overline{gu_\varepsilon v_\varepsilon\varphi_\varepsilon} + \overline{gu_\varepsilon}^2\overline{\psi_\varepsilon}\right)\zeta(y)\,dy = \int_{\mathbb{R}}\lambda_\varepsilon\overline{\varphi_\varepsilon}\zeta(y)\,dy.$$

Taking $\varepsilon \to 0$ and using Lemma 2, we obtain

$$\int_{\mathbb{R}}\left(\overline{\varphi}''(y) - \overline{\varphi}(y) - +2g(0)w(y)\overline{\varphi}(y) + \frac{g(0)}{\xi_0^2}w^2(y)\psi(0)\right)\zeta(y)\,dy = \lambda_0\int_{\mathbb{R}}\overline{\varphi}(y)\zeta(y)\,dy. \tag{24}$$

Since $w(y) = g(0)^{-1}w_0(y)$, (23) and (24) yield Lemma 5.

4.2 Stability analysis for small eigenvalues

We have the following key precise asymptotic for small eigenvalues λ_ε which yields the proof of Theorem 2.

Proposition 1 *Assume .* $\lambda_\varepsilon \to 0$. *Then, as* $\varepsilon \to 0$, *the asymptotic form of* λ_ε *is given as follows:*

$$\lambda_\varepsilon = \varepsilon^2\frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}}(w')^2}\Big(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3}\Big) + O(\varepsilon^{\frac{5}{2}}). \tag{25}$$

(Sketch of the proof of Proposition 1.) We denote $u_{\epsilon,1}(x) = u_\varepsilon(x)\chi(x)$. To show Proposition 1, let us decompose

$$\varphi_\varepsilon(x) = \varepsilon a_\varepsilon u'_{\varepsilon,1}(x) + \varphi_\varepsilon^\perp(x), \tag{26}$$

where a_ε is some complex number and $\varphi_\varepsilon^\perp$ satisfies

$$\overline{\varphi_\varepsilon^\perp} \perp \mathcal{K}_\varepsilon \text{ in } L^2(I_\varepsilon), \quad \mathcal{K}_\varepsilon := \text{span}\left\{\overline{u_{\varepsilon,1}}'\right\} \subset H^2(I_\varepsilon).$$

In y-variable, we have

$$\overline{\varphi_\varepsilon}(y) = a_\varepsilon \overline{u_{\varepsilon,1}}'(y) + \overline{\varphi_\varepsilon^\perp}(y).$$

Similarly, we decompose

$$\psi_\varepsilon(x) = \varepsilon a_\varepsilon \psi_{\varepsilon,1}(x) + \psi_\varepsilon^\perp(x). \tag{27}$$

Here, $\psi_{\varepsilon,1}$ is a unique solution of

$$D\psi''_{\varepsilon,1} - \frac{c}{\varepsilon}g(x)u_\varepsilon^2\psi_{\varepsilon,1} - \frac{2c}{\varepsilon}g(x)v_\varepsilon u'_{\varepsilon,1} = \varepsilon\lambda_\varepsilon\psi_{\varepsilon,1},\ x \in (-1,1), \quad \psi'_{\varepsilon,1}(\pm 1) = 0, \tag{28}$$

and $\psi_\varepsilon^\perp$ is defined by $\psi_\varepsilon^\perp := \psi_\varepsilon - \varepsilon a_\varepsilon \psi_{\varepsilon,1}$. We have the following formula for λ_ε:

Lemma 6 ([5, Lemma 6.1])

$$J_1 + J_2 + J_3 + J_4 + O(|a_\varepsilon|\varepsilon^3) = \lambda_\varepsilon a_\varepsilon \xi_0^{-2}\int_{\mathbb{R}}(w')^2 dy + O(\sqrt{\varepsilon}\lambda_\varepsilon|a_\varepsilon|). \tag{29}$$

where J_i $(i = 1,2,3,4)$ are defined as follows:

$$J_1 := a_\varepsilon \int_{I_\varepsilon}\left(\varepsilon\overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon}'\right)\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}' dy - a_\varepsilon \int_{I_\varepsilon} \overline{g}'\overline{v_\varepsilon}\ \overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}' dy,$$

$$J_2 := -\int_{I_\varepsilon}\left(\overline{g}'\overline{v_\varepsilon} + \overline{g}\ \overline{v_\varepsilon}'\right)\overline{u_{\varepsilon,1}}^2\ \overline{\varphi_\varepsilon^\perp} dy,$$

$$J_3 := \int_{I_\varepsilon}\overline{g}\ \overline{u_\varepsilon}^2\overline{\psi_\varepsilon^\perp}\ \overline{u_{\varepsilon,1}}' dy, \quad J_4 := \int_{I_\varepsilon}\overline{r_\varepsilon}\ \overline{\varphi_\varepsilon^\perp} dy.$$

Here, $\overline{r_\varepsilon}$ is a function satisfying $\overline{r_\varepsilon}(y) = 0$ on $|y| \leq \frac{1}{4\varepsilon}$ and $\overline{r_\varepsilon}(y) = O(\varepsilon^2)$ on $\frac{1}{4\varepsilon} \leq |y| \leq \frac{1}{\varepsilon}$.

This is obtained by multiplying $\overline{u_{\varepsilon,1}}'$ and integration by parts. Among them, J_1 is the leading term to determine the precise asymptotic for λ_ε. The following Proposition 2 is important and decide the asymptotic behavior of J_1.

Proposition 2 ([5, Proposition 6.2]) *The following estimates hold:*

(1) $\left|\left|\left(\varepsilon\overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon}'\right)\overline{g}\ \overline{u_\varepsilon}^2\right|\right|_{L^1(I_\varepsilon)} \leq C\varepsilon^2$.

(2) $|\varepsilon\overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon}'(y)| \leq C\varepsilon^2|y|$.

(3) $\varepsilon\overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon}'(y) = \varepsilon^2 y \cdot \dfrac{cg(0)\xi_0^{-1}}{2D}\displaystyle\int_{\mathbb{R}} w^2 dt + O(\varepsilon^{\frac{5}{2}}|y|).$

Here, the constant C is independent of $\varepsilon > 0$.

These are obtained by the representation of $\overline{v_\varepsilon}'(y)$ and $\overline{\psi_{\varepsilon,1}}(y)$ by using Dirichlet and Neumann Green functions, respectively. By Proposition 2, we have the following.

Proposition 3 *It holds that*

$$J_1 = a_\varepsilon \varepsilon^2\Big(-\frac{g(0)\int_{\mathbb{R}} w^3}{6D\xi_0^3} + \frac{g''(0)\int_{\mathbb{R}} w^3}{3\xi_0^2}\Big) + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}),$$

where the constant C is independent of $\varepsilon > 0$.

(Sketch of the proof of Proposition 3.) By (3) of Proposition 2, we have

$$a_\varepsilon \int_{I_\varepsilon} (\varepsilon\overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon}'(y))\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'\,dy$$

$$= a_\varepsilon\varepsilon^2 \frac{cg(0)\xi_0^{-1}}{2D}\Big(\int_{\mathbb{R}} w^2\,dx\Big)\int_{\mathbb{R}} y\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'\,dy + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$$

$$= -a_\varepsilon\varepsilon^2 \frac{cg(0)^2\int_{\mathbb{R}} w^2\,dx}{6D\xi_0^4}\int_{\mathbb{R}} w^3\,dx + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}) = -a_\varepsilon\varepsilon^2\frac{g(0)}{6D\xi_0^3}\int_{\mathbb{R}} w^3\,dx + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}).$$

Here, we used

$$cg(0)\int_{\mathbb{R}} w(y)^2\,dy = \xi_0, \qquad \int_{\mathbb{R}} yw(y)^2w'(y)\,dy = -\frac{1}{3}\int_{\mathbb{R}} w(y)^3\,dy.$$

Since by using $\overline{g}'(y) = \varepsilon^2 y g''(0) + O(\varepsilon^3|y|^2)$ we also have

$$\int_{I_\varepsilon} \overline{g}'\overline{v_\varepsilon}\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'\,dy = -\varepsilon^2\frac{g''(0)\xi_0^{-2}}{3}\int_{\mathbb{R}} w^3\,dx + O(\varepsilon^{\frac{5}{2}}), \tag{30}$$

we can conclude that

$$J_1 = a_\varepsilon\int_{I_\varepsilon}\big(\varepsilon\overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon}'\big)\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'dy - a_\varepsilon\int_{I_\varepsilon}\overline{g}'\overline{v_\varepsilon}\,\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'dy$$

$$= a_\varepsilon\varepsilon^2\Big(-\frac{g(0)\int_{\mathbb{R}} w^3}{6D\xi_0^3} + \frac{g''(0)\int_{\mathbb{R}} w^3}{3\xi_0^2}\Big) + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}).$$

To estimate the remainder terms J_2, J_3, and J_4, we need the following several estimates.

Lemma 7 ([5, Lemma 6.6]) *For $\psi_\varepsilon^\perp$, it holds that:*

(1) $||\overline{\psi_\varepsilon^\perp}||_{L^\infty(I_\varepsilon)} \le C||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}.$

(2) $||\overline{\psi_\varepsilon^\perp}'||_{L^2(I_\varepsilon)} \le C\sqrt{\varepsilon}||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}.$

Basically, these estimates can be obtained by applying Lemma 2 for $\psi_\varepsilon^\perp$. Lemma 7 implies the following estimates.

Lemma 8 ([5, Lemma 6.8]) *For J_i $(i = 2, 3, 4)$, it holds that:*

(1) $|J_2| \leq C\varepsilon ||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

(2) $|J_3| \leq C\sqrt{\varepsilon} ||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

(3) $|J_4| \leq C\varepsilon^{\frac{3}{2}} ||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

Here, the constant C is independent of $\varepsilon > 0$.

By using Proposition 2 again and the inveritibility of some operator $\tilde{L}_\varepsilon$, which is close to the operator S_ε, we can show the following estimates.

Lemma 9 ([5, Lemma 6.9]) *The following hold:*

(1) $|a_\varepsilon| \neq 0$.

(2) $||\overline{\varphi_\varepsilon^\perp}||_{L^2} \leq C|a_\varepsilon|\varepsilon^{\frac{3}{2}}$.

Combing these estimates, we arrive at the remainder estimates for J_2, J_3 and J_4.

Lemma 10 $J_2 = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$, $J_3 = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$, $J_4 = O(|a_\varepsilon|\varepsilon^3)$.

Estimates for J_2 and J_4 follow directly from Lemma 8 and 9. However, for J_3, Lemma 8 and 9 yield just $J_3 = (|a_\varepsilon|\varepsilon^2)$, which is not enough. Actually, we need the following refined estimate to get the correct estimate for J_3.

$$\overline{\psi_\varepsilon^\perp}(y) - \overline{\psi_\varepsilon^\perp}(0) = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}|y|). \tag{31}$$

This is obtained by the representation of $\overline{\psi_\varepsilon^\perp}(y)$ by using the Neumann Green function. Now, by using Proposition 3, Lemma 10 and Lemma 6, we can complete the proof of Proposition 1.

5 Further Remarks

We give two remarks.

Remark 4 *Assume $g(x)$ is Lipschitz continuous and $g \in C^3((-1, 0])$ and $g \in C^3([0, 1))$, respectively. Let $g'(+0) := \lim_{x>0, x\to 0} g'(x)$ and $g'(-0) := \lim_{x<0, x\to 0} g'(x) = -g'(+0)$ by the symmetry. When $g'(+0) \neq g'(-0)$, the stability of the solution is determined by* *the sign of $g'(+0)$. First, note that using $\overline{g}'(y) = \varepsilon g'(+0) + \varepsilon^2 y g''(+0) + O(\varepsilon^3|y|^2)$ for $y > 0$ and $\overline{g}'\overline{v_\varepsilon u_\varepsilon}^2\overline{u_\varepsilon}'$ is an even function, we can compute*

$$\int_{I_\varepsilon} \overline{g}'\overline{v_\varepsilon u_\varepsilon}^2\overline{u_\varepsilon}'\,dy = 2\int_0^{\frac{1}{\varepsilon}} \overline{g}'\overline{v_\varepsilon u_\varepsilon}^2\overline{u_\varepsilon}'\,dy$$

$$= \varepsilon g'(+0)\xi_0^{-2}\int_0^{+\infty} w(y)^2 w'(y)\,dy + O(\varepsilon^{\frac{3}{2}}) = -\varepsilon\frac{g'(+0)w(0)^3}{3\xi_0^2} + O(\varepsilon^{\frac{3}{2}}).$$

Thus, in the computation of the small eigenvalue λ_ε, the leading term of J_1 become as follows:

$$J_1 = a_\varepsilon\frac{\varepsilon}{3\xi_0^2}g'(+0)w(0)^3 + O(|a_\varepsilon|\varepsilon^{\frac{3}{2}}).$$

Compare with (30) and Proposition 3 for the case $g \in C^3(-1,1)$. This implies

$$\lambda_\varepsilon = \varepsilon\frac{g'(+0)}{3\int_{\mathbb{R}} w'(x)^2\,dx}(w(0))^3 + O(\varepsilon^{\frac{3}{2}}).$$

Therefore, the solution is unstable if $g'(+0) > 0$ and stable if $g'(+0) < 0$.

Remark 5 (Boundary peal solution and its stability) *For a given Lipschitz continuous positive function $g(x)$, we can construct a boundary peak solution $(u_\varepsilon, v_\varepsilon)$ on the interval $I := (-1,1)$. Because, consider an extension of $g(x)$ on the interval $\tilde{I} := (-1,3)$, which is symmetric with respect to $x = 1$. We denote it by $\tilde{g}(x)$. For this function $\tilde{g}$, we can construct solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ to the corresponding Schnakenberg system on the interval $\tilde{I}$, which is symmetric with respect to $x = 1$. Restricting this solution on the original interval I, we obtain a boundary peak solution $(u_\varepsilon, v_\varepsilon)$. For the stability of this boundary peak solution, let us consider the linearized eigenvalue problem on I. We denote by λ_ε and $(\varphi_\varepsilon, \psi_\varepsilon)$ the eigenvalue and the associated eigenfunctions, respectively. Now, extending the eigenfunction $(\varphi_\varepsilon, \psi_\varepsilon)$ on the interval $\tilde{I} = (-1,3)$ to be symmetric with respect to $x = 1$. Then by the Neumann boundary condition at $x = 1$, this extended function $(\tilde{\varphi_\varepsilon}, \tilde{\psi_\varepsilon})$ is an eigenfunction associated with the eigenvalue λ_ε on the interval $\tilde{I}$. Then, we can apply our theorem to study the stability of the boundary peak solution $(u_\varepsilon, v_\varepsilon)$. Namely, assuming $\tilde{g}(x)$ is C^3 function, $\tilde{g}''(1)$ determine the stability of the boundary peak solution constructed in this way.*

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