On test function method for semilinear wave equations with scale-invariant damping

東京理科大学理工学部数学科　側島 基宏 (Motohiro Sobajima)
Department of Mathematics, Faculty of Science and Technology,
Tokyo University of Science

1. Introduction

In this paper we consider the following semilinear wave equation with a space-dependent damping term

$$
(1.1)\qquad \begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \dfrac{a}{|x|}\partial_t u(x,t) = |u(x,t)|^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = \varepsilon f(x),\ \partial_t u(x,0) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases}
$$

where $N \geq 3$ ($N \in \mathbb{N}$), $a \geq 0$ and $1 < p < \frac{N-2}{N-4}$ ($1 < p < \infty$ for $N = 3, 4$). The initial data (f, g) is assumed to be smooth enough and compactly supported, that is, $f, g \in C_0^\infty(\mathbb{R}^N)$ with

$$
\operatorname{supp}(f,g) = \operatorname{supp} f \cup \operatorname{supp} g \subset \overline{B}(0, R_0) = \{x \in \mathbb{R}^N \ ;\ |x| \leq R_0\}.
$$

The parameter $\varepsilon > 0$ describes the smallness of initial data.

The semilinear wave equation ($a = 0$) has been studied from the pioneering work by John [**5**]. In [**5**], the problem (1.1) with $N = 3$ and $a = 0$ is discussed and the following assertion is shown

(i) If $1 < p < 1 + \sqrt{2}$, then there exists a pair (f, g) such that the problem does not have global-in-time solutions of (1.1) for all ε.

(ii) If $p > 1 + \sqrt{2}$, then there exists a global-in-time solution of (1.1) with small ε.

After that, there are many subsequent papers dealing with the N-dimensional semilinar wave equation ($a = 0$) (see e.g., Kato [**6**], Yordanov–Zhang [**9**], and Zhou [**10**]). For the N-dimensional case, the following is proved in the literature.

(i) If $1 < p \leq p_S(N)$, then there exists a pair (f, g) such that the problem does not have global-in-time solutions of (1.1) for all ε.

(ii) If $p > p_S(N)$, then there exists a global-in-time solution of (1.1) with small ε.

Here the exponent $p_S(n)$ is called the Strauss exponent defined as

$$
\begin{aligned}
\gamma(n,p) &:= 2 + (n+1)p - (n-1)p^2, \\
p_S(n) &:= \sup\{p > 1 \ ;\ \gamma(n,p) > 0\} \\
&= \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)} \qquad (n > 1).
\end{aligned}
$$

The study of maximal existence time (lifespan)

$$T_\varepsilon = T(\varepsilon f, \varepsilon g) = \sup\{T > 0\,; \text{there exists a solution of (1.1) in } (0,T)\}.$$

of blowup solutions to (1.1) has been also studied (see Lindblad [**7**], Takamura–Wakasa [**8**] and there references therein) as

$$T_\varepsilon \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \text{if } N=1,\ 1<p<\infty, \\ C\varepsilon^{-\frac{p-1}{3-p}} & \text{if } N=2,\ 1<p<2, \\ Ca(\varepsilon^{-1}) & \text{if } N=2,\ p=2, \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } N=2,\ 2<p<p_S(2), \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } N\ge 3,\ 1<p<p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } N\ge 2,\ p=p_S(N), \end{cases} \tag{1.2}$$

where $a(s)$ denotes the inverse of the function $s(a) = a\sqrt{1+\log(1+a)}$. Therefore the blowup phenomena for solutions to (1.1) with small initial data and their lifespan estimate is already established.

If $a > 0$, then the there are few works dealing with global existence and blowup of solutions to (1.1). If the damping term is milder, that is, we consider the problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + (1+|x|^2)^{-\frac{\alpha}{2}}\partial_t u(x,t) = |u(x,t)|^p, & (x,t)\in\mathbb{R}^N\times(0,T), \\ u(x,0)=\varepsilon f(x),\ \partial_t u(x,0) = \varepsilon g(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.3}$$

with $\alpha\in[0,1)$, then Ikehata–Todorova–Yordanov [**3**] consider the global existence and blowup of solutions to (1.3). In this case, they proved

(i) If $1 < p \le 1 + \frac{2}{N-\alpha}$, then there exists a pair (f,g) such that the problem does not have global-in-time solutions of (1.1) for all ε.

(ii) If $p > 1 + \frac{2}{N-\alpha}$, then there exists a global-in-time solution of (1.1) with small ε.

This means the situation is close to the parabolic problem

$$\begin{cases} \partial_t v(x,t) - (1+|x|^2)^{\frac{\alpha}{2}}\Delta v(x,t) = 0, & (x,t)\in\mathbb{R}^N\times(0,T), \\ u(x,0)=\varepsilon f(x), & x\in\mathbb{R}^N \end{cases} \tag{1.4}$$

which has an unbounded diffusion. The case $\alpha = 1$ is more delicate. The linear problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + a(1+|x|^2)^{-\frac{1}{2}}\partial_t u(x,t) = 0, & (x,t)\in\mathbb{R}^N\times(0,\infty), \\ u(x,0)=u_0(x),\ \partial_t u(x,0) = u_1(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.5}$$

for $a > 0$. Ikehata–Todorova–Yordanov [**4**] discussed the decay property of energy function

$$\int_{\mathbb{R}^N} \Big(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\Big)\,dx \le \begin{cases} C(1+t)^{-a} & \text{if } 0<a<N, \\ C_\delta(1+t)^{-N+\delta} & \text{if } a\ge N. \end{cases}$$

Therefore the situation strongly depends on the size of the constant a in front of the damping term $(1+|x|^2)^{-\frac{1}{2}}\partial_t u$.

Here we would like to consider the nonlinear problem (1.1) with $a > 0$. It is remarkable that the equation in (1.1) has the scale-invariance, that is, if u satisfies the equation on (1.1), then the scaled function $u_\lambda(x,t) = \lambda^{-\frac{2}{p-1}}u(\lambda x, \lambda t)$ also satisfies (1.1). This kind of structure helps us to analyse the dynamics of solutions.

Actually, in Ikeda–Sobajima [**1**] the finite time blowup of solutions is proved. More precisely, they showed

Proposition 1.1 ([**1**]). *Let $N \geq 3$ and let f, g be nonnegative, smooth and compactly supported with $g \not\equiv 0$. If $1 < p < \infty$ for $N = 3, 4$, $1 < p < \frac{N-2}{N-4}$ for $N \geq 5$, then there exists a unique solution*

$$u \in W^{2,\infty}([0,T_\varepsilon); L^2(\mathbb{R}^N)) \cap W^{1,\infty}([0,T_\varepsilon); H^1(\mathbb{R}^N)) \cap L^\infty([0,T_\varepsilon); H^2(\mathbb{R}^N)).$$

Here T_ε stands for the maximal existence time of solutions. Moreover, if $0 < a < \frac{(N-1)^2}{N+1}$ and $\frac{N}{N-1} < p \leq p_S(N+a)$, then the maximal existence time T_ε of solution u is finite. In particular, the following estimates hold: there exists a positive constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$,

$$T_\varepsilon \leq \begin{cases} C_\delta \varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}-\delta} & \text{if } p_S(N+a+2) < p < p_S(N+a), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N+a), \end{cases}$$

where C and C_δ are positive constants independent of ε and $C_\delta \to \infty$ as $\delta \to 0$.

We conjecture that $p_S(N+a)$ is the critical exponent for the problem (1.1) at least for small a, that is, it is expected that $p > p_S(N+a)$ implies the global existence for suitable initial data. From this viewpoint, it is natural that Proposition 1.1 gives the blowup result for the "critical" case $p = p_S(N+a)$ with an estimate for T_ε of exponential type. However, in the subcritical case $\frac{N}{N-1} < p < p_S(N+a)$, the expected estimates should be $T_\varepsilon \leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}}$ (without δ-loss) which could not prove in [**1**].

The purpose of this paper is to deal with the estimate for T_ε of solutions to (1.1) in the subcritical case $\frac{N}{N-1} < p < p_S(N+a)$. The result is the following.

Theorem 1.1. *Let f, g be nonnegative, smooth and compactly supported with $g \not\equiv 0$ and let u be the solution of (1.1) in Proposition 1.1. Then there exists a positive constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$,*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(\frac{2}{p-1}-N+1)^{-1}} & \text{if } \frac{N}{N-1} < p < \frac{N+1}{N-1}, \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}} & \text{if } \frac{N+1}{N-1} < p < p_S(N+a), \end{cases}$$

where C is a positive constant independent of ε.

Remark 1.1. We can directly check the following identity:

$$p_S(N+a_*) = \frac{N+1}{N-1}, \quad a_* = \frac{(N-1)^2}{N+1}.$$

(see also Ikeda–Sobajima [**1**]). Therefore we have $0 \le a < a_*$ implies $\frac{N+1}{N-1} < p_S(N+a)$. At this moment, we may regard Theorem 1.1 as an extension of the result for (upper) lifespan estimates for the usual semilinear wave equations ($a = 0$) with small initial data.

The proof is based on a test function method for wave equations developed in Ikeda–Sobajima–Wakasa [**2**]. In particular, for the problem (1.1) we use positive solutions to the corresponding linear conjugate equation

$$\partial_t^2\Phi - \Delta\Phi - \frac{a}{|x|}\partial_t\Phi = 0.$$

In Section 2, we prove Theorem 1.1 by using positive solutions to the corresponding conjugate equation $\partial_t^2\Phi - \Delta\Phi - \frac{a}{|x|}\partial_t\Phi = 0$.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we use the following structure. We will only give an idea for the proof.

Lemma 1. *Let u be a solution of (1.1). Assume that for every $t \ge 0$, $u(t)$ is compactly supported. Then for every $T \in (0, T_\varepsilon)$ and $\Phi \in C^\infty(\mathbb{R}^N \times [0, T_\varepsilon))$ satisfying $\partial_t\Phi(\cdot, T) = \Phi(\cdot, T) = 0$,*

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^N}\Big(g + \frac{a}{|x|}f\Big)\Phi(x,0) - f(x)\partial_t\Phi(x,0)\,dx + \int_0^T\int_{\mathbb{R}^N}|u|^p\Phi\,dx\,dt\\ &= \int_0^T\int_{\mathbb{R}^N}u\Big(\partial_t^2\Phi - \Delta\Phi - \frac{a}{|x|}\partial_t\Phi\Big)\,dx\,dt.\end{aligned}$$

Sketch of the proof. Multiplying the equation in (1.1) and Φ and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p\Phi\,dx &= \int_{\mathbb{R}^N}\Big(\partial_t^2u - \Delta u + \frac{a}{|x|}\partial_t u\Big)\Phi\,dx.\\ &= \frac{d}{dt}\int_{\mathbb{R}^N}\Big(\partial_t u + \frac{a}{|x|}u\Big)\Phi - u\partial_t\Phi\,dx + \int_{\mathbb{R}^N}u\Big(\partial_t^2\Phi - \frac{a}{|x|}\partial_t\Phi\Big) - \Delta u\Phi\,dx.\end{aligned}$$

Employing integration by parts and integrating it over $[0, T]$, we obtain the desired equality. □

Next, we fix $\eta \in C^\infty([0,\infty); [0,1])$ as follows:

$$\eta(s) = \begin{cases}1 & \text{if } s \le 1/2,\\ \text{decreasing} & \text{if } 1/2 < s < 1,\\ 0 & \text{if } s \ge 1,\end{cases} \qquad \eta_T(t) = \eta(t/T).$$

Since $\varphi(x,t) = 1$ satisfies $\partial_t^2\varphi - \Delta\varphi - \frac{a}{|x|}\partial_t\varphi = 0$, we first choose $\Phi = \varphi\eta_T^{2p'} = \eta_T^{2p'}$, where $p' = p/(p-1)$ is the Hölder conjugate of p. Then we have the following.

Lemma 2. *Let f, g be nonnegative and smooth with* $\operatorname{supp}(f,g) \subset \overline{B}(0,R_0)$ *and* $g \not\equiv 0$. *If $T_\varepsilon > 2R_0$, then for every $T \in (2R_0, T_\varepsilon)$,*

$$C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\,dx\,dt \le CT^{(N-1-\frac{2}{p-1})\frac{1}{p'}} \left(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\,dx\,dt\right)^{\frac{1}{p}},$$

where $C_{f,g} = \int_{\mathbb{R}^N} g + a|x|^{-1} f\,dx > 0$. In particular, we have

$$C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\,dx\,dt \le C^p T^{N-1-\frac{2}{p-1}}.$$

Sketch of the proof. Applying Lemma 1 with $\Phi = \eta^{2p'}$, we have

$$\begin{aligned}
&C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\,dx\,dt \\
&= \int_{T/2}^T \int_{\mathbb{R}^N} u\Big(\partial_t^2 \eta_T^{2p'} - \Delta \eta_T^{2p'} - \frac{a}{|x|}\partial_t \eta_T^{2p'}\Big)\,dx\,dt. \\
&\le C_1 \int_{T/2}^T \int_{\operatorname{supp} u(t)} |u| \eta_T^{2p'-2}\Big(\frac{1}{T^2} + \frac{1}{T|x|}\Big)\,dx\,dt. \\
&\le C_1 \left(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\,dx\,dt\right)^{\frac{1}{p}} \left(\int_{T/2}^T \int_{B(0,R_0+t)} \Big(\frac{1}{T^2} + \frac{1}{T|x|}\Big)^{p'}\,dx\,dt\right)^{\frac{1}{p'}}.
\end{aligned}$$

It should be mentioned that the restriction $p > \frac{N}{N-1}$ comes from the integrability of $|x|^{-p'}$ in $B(R_0+t)$. The remaining part is just a straight forward computation. □

Next, to find a good test function, we introduce

$$\widetilde{\varphi}(x,t) = (2R_0 + t + |x|)^{-\frac{N-1-a}{2}} (2R_0 + t - |x|)^{-\frac{N-1+a}{2}}, \quad x \in B(0, 2R_0 + t),$$

which is a self-similar solution of the equation $\partial_t^2 \Phi - \Delta\Phi - \frac{a}{|x|}\partial_t \Phi = 0$ given by

$$\Phi_\beta(x,t) = (2R_0 + t + |x|)^{-\beta} F\left(\beta, \frac{N-1+a}{2}, N-1; \frac{2|x|}{2R_0 + t - |x|^2}\right)$$

with a particular choice $\beta = N-1$. The function $F(\cdot,\cdot,\cdot,z)$ stands for the Gauss hypergeometric function (Φ_β for general β is introduced in [**1**]). But because of the simple structure of $\widetilde{\varphi}$, by direct computation we can verify that $\widetilde{\varphi}$ satisfies the linear conjugate equation $\partial_t^2 \widetilde{\varphi} - \Delta\widetilde{\varphi} - \frac{a}{|x|}\partial_t \widetilde{\varphi} = 0$ on $\operatorname{supp} u$. The following lemma is a consequence of the choice of $\Phi = \widetilde{\varphi}\eta_T^{2p'}$. This lemma can be understood as the concentration phenomena to the wave front $\{|x| \sim t\}$ for the wave equation (with scale-invariant damping term).

Lemma 3. *Let f, g be nonnegative and smooth with* $\operatorname{supp}(f,g) \subset \overline{B}(0,R_0)$ *and* $g \not\equiv 0$. *If $T_\varepsilon > 2R_0$, then for every $T \in (2R_0, T_\varepsilon)$,*

$$\delta \varepsilon^p T^{N - \frac{N-1+a}{2}p} \le \int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\,dx\,dt.$$

where δ is a positive constant independent of ε.

Sketch of the proof. Applying Lemma 1 with $\Phi = \widetilde{\varphi}\eta^{2p'}$, we have

$$\begin{aligned}
\widetilde{C}_{f,g}\varepsilon &\le \varepsilon \int_{\mathbb{R}^N} \Big(g + \frac{a}{|x|}f\Big)\widetilde{\varphi}(x,0) - f(x)\partial_t\widetilde{\varphi}(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} |u|^p \widetilde{\varphi}\eta_T^{2p'}\,dx\,dt \\
&= \int_{T/2}^T \int_{\mathbb{R}^N} u\Big(\partial_t^2\eta_T^{2p'}\widetilde{\varphi} + 2\partial_t\eta_T^{2p'}\partial_t\widetilde{\varphi} - \frac{a}{|x|}\partial_t\eta_T^{2p'}\widetilde{\varphi}\Big)\,dx\,dt. \\
&\le C_2 \int_{T/2}^T \int_{\operatorname{supp} u(t)} |u|\eta_T^{2p'-2}\Big(\frac{\widetilde{\varphi}}{T^2} + \frac{\widetilde{\varphi}}{T|x|} + \frac{\partial_t\widetilde{\varphi}}{T}\Big)\,dx\,dt. \\
&\le C_1 \left(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p\eta_T^{2p'}\,dx\,dt\right)^{\frac{1}{p}} \left(\int_{T/2}^T \int_{B(0,R_0+t)} \Big(\frac{\widetilde{\varphi}}{T^2} + \frac{\widetilde{\varphi}}{T|x|} + \frac{\partial_t\widetilde{\varphi}}{T}\Big)^{p'}\,dx\,dt\right)^{\frac{1}{p'}},
\end{aligned}$$

where we have used $\partial_t\widetilde{\varphi} \le 0$ and the conjugate equation for $\widetilde{\varphi}$. The remaining part is just a straight forward computation. ☐

Finally, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that $T_\varepsilon > 2R_0$. Then combining Lemmas 2 and 3, we already have the following inequality:for every $T \in (2R_0, T_\varepsilon)$,

$$C_{f,g}\varepsilon + \delta\varepsilon^p T^{N-\frac{N-1+a}{2}p} \le CT^{N-1-\frac{2}{p-1}}.$$

Then we see that if $p < \frac{N+1}{N-1}$, then $\kappa = -(N-1-\frac{2}{p-1}) > 0$ and therefore

$$T \le \left(\frac{C}{C_{f,g}\varepsilon}\right)^{\frac{1}{\kappa}}.$$

On the other hand, if $p < p_S(N+a)$, then $\frac{N-1+a}{2} - 1 - \frac{2}{p-1} = -\frac{\gamma(N+a,p)}{2(p-1)} < 0$ and therefore

$$T \le \left(\frac{C}{\delta\varepsilon^p}\right)^{\frac{2(p-1)}{\gamma(N+a,p)}}.$$

Since T_ε is the maximal existence time, we can choose T arbitrary close to T_ε. This means that T_ε satisfies the same estimate as T as above. The proof is complete. ☐

References

[1] M. Ikeda, M. Sobajima, *Life-span of blowup solutions to semilinear wave equation with space-dependent critical damping*, to appear in Funkcialaj Ekvacioj, `arXiv:1709.04401`.

[2] M. Ikeda, M. Sobajima, K. Wakasa, *Blow-up phenomena of semilinear wave equations and their weakly coupled systems*, J. Differential Equations **267** (2019), 5165–5201.

[3] R. Ikehata, G. Todorova, B. Yordanov, *Critical exponent for semilinear wave equations with space-dependent potential*, Funkcial. Ekvac. **52** (2009), 411–435.

[4] R. Ikehata, G. Todorova, B. Yordanov, *Optimal decay rate of the energy for wave equations with critical potential*, J. Math. Soc. Japan **65** (2013), 183–236.

[5] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–268.

[6] T.Kato, *Blow up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl. Math. **33** (1980), 501–505.

[7] H. Lindblad, *Blow-up for solutions of* $\Box u = |u|^p$ *with small initial data*, Comm. Partial Differential Equations **15** (1990), 757–821.

[8] H. Takamura, K. Wakasa, *The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high dimensions*, J. Differential Equations **251** (2011), 1157–1171.

[9] B. Yordanov, Q.S. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. **231** (2006), 361–374.

[10] Y. Zhou, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions*, Chin. Ann. Math. Ser. B **28** (2007), 205–212.