The Tanaka Instability of Traveling Waves in Hamiltonian Systems

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Abstract

This paper reviews the linear instability of nonlinear traveling waves in Hamiltonian systems subject to superharmonic perturbations. Tanaka's instability, characterized by a zero eigenvalue with geometric multiplicity of one and algebraic multiplicity equal to or greater than four, occurs in the presence of translationally symmetric traveling wave solutions at the extrema of energy with respect to wave speed. The theory finds application in the study of the superharmonic instability of wave equations with a Hamiltonian structure, such as water waves with constant vorticity.

1 Introduction

Traveling waves are ubiquitous phenomena that arise as special subsets of solutions to differential equations describing the time evolution of diverse physical systems, such as fluids or plasmas. Object of the present paper is a type of traveling wave instability, called the Tanaka instability, which occurs in Hamiltonian systems. This instability was first observed by Tanaka [1], who numerically showed that a steep gravity wave subject to superharmonic disturbances (perturbations with a wavenumber that is an integer multiple of the fundamental mode of oscillation) destabilizes in correspondence of the point where energy is stationary with respect to wave speed. This fact was unexpected because earlier analysis carried out by Longuet-Higgins [2] suggested that instability would arise in correspondence of the wave steepness with maximum wave speed. The Tanaka instability was later explained by Saffman [3], who performed a linear stability analysis and showed that any traveling wave solution of a canonical Hamiltonian system with both translational and reflectional symmetry and subject to superharmonic perturbations possesses a zero eigenvalue with algebraic multiplicity equal to or greater than four at the extrema of energy with respect to wave speed. This result was generalized in [4], where it was shown that the reflectional symmetry of the wave profile is not essential for the manifestation of the Tanaka instability. Recently, Murashige and Choi [5] observed the occurrence of the Tanaka instability in a different system (water waves over a linear shear flow), confirming that this phenomenon represents a general feature of Hamiltonian dynamics.

The Tanaka instability can be physically understood as follows. First, due to the postulated translational invariance of traveling waves, such solutions cannot be unique, and the characteristic polynomial associated with the linear stability analysis for the growth of infinitesimal perturbations always has zero as root. Furthermore, the algebraic multiplicity of the zero eigenvalue is always even and at least equal to two because of the Hamiltonian structure of the equations. When energy becomes stationary as a function of wave speed, the algebraic multiplicity of the zero eigenvalue increases to at least four, indicating the coalescence of the zero eigenvalue associated with translational symmetry with a generally non-zero eigenvalue. In other words, the square of the second eigenvalue goes from positive to negative values in correspondence of these energy extrema, implying the onset of instability.

In this paper, we review the Tanaka instability from the standpoint of Hamiltonian mechanics [6, 7] by following the discussion of [4]. While we are concerned with infinite dimensional canonical Hamiltonian systems, we remark that the theory can be generalized to the noncanonical case. In section 2, we derive a Krein-like theorem [8, 9] for the linear stability of traveling waves in canonical Hamiltonian systems subject to superharmonic disturbances. In particular, we show that if σ is an eigenvalue, so is $-\sigma$. In section 3 some examples of infinite dimensional Hamiltonian systems in canonical form are provided, including the nonlinear Schrödinger equation and the water wave system with constant vorticity. The occurrence of the Tanaka instability is discussed in section 4. Finally, some considerations on further relaxation of the conditions for the onset of the Tanaka instability are given in the concluding section.

2 Superharmonic Krein's Theorem for Traveling Waves in Canonical Hamiltonian Systems

In this paper we are concerned with infinite dimensional Hamiltonian systems in canonical form:

$$\frac{\partial \zeta}{\partial t} = -\frac{\delta H}{\delta \eta}, \qquad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \zeta}. \tag{1}$$

Here, $\zeta = \zeta(\boldsymbol{x}, t)$ and $\eta = \eta(\boldsymbol{x}, t)$ are real valued functions of n Cartesian coordinates $\boldsymbol{x} = \left(x^1, ..., x^n\right) \in \mathbb{R}^n$ and time t. Both ζ and η belong to a vector space X equipped with the standard L^2 inner product $\langle f, g\rangle = \int_{\mathbb{R}^n} f g^* \, dV$ where $f, g \in X$ and $*$ stands for the complex conjugate. The Hamiltonian $H = H[\zeta, \eta]$ is a real valued functional of ζ and η. In the following, we shall assume that ζ, η, and H are smooth. The symbol δ denotes functional differentiation. For the purpose of the present paper the first order functional derivative $\delta H/\delta u^i$ of a functional $H = H\left[u^1, u^2, ...\right]$ is defined according to

$$\lim_{\epsilon \to 0} \frac{H\left[u^i + \epsilon r\right] - H\left[u^i\right]}{\epsilon} = \left\langle \frac{\delta H}{\delta u^i}, r \right\rangle, \tag{2}$$

with $r \in X$.

In finite dimensions, the linear stability of a steady solution of a Hamiltonian system in canonical form is described by Krein's theorem [8, 9], which states that the sprectrum of the linearized problem for the growth of infinitesimal perturbations is time-reversible and symmetric, i.e. if $\sigma \in \mathbb{C}$ is an eigenvalue, so must be $-\sigma$, σ^*, and $-\sigma^*$. Krein's theorem can be easily verified in two dimensions: given a steady solution (ζ_0, η_0) of the 2 dimensional Hamilton's equations

$$\dot{\zeta} = -\frac{\partial H}{\partial \eta}, \qquad \dot{\eta} = \frac{\partial H}{\partial \zeta}, \tag{3}$$

the linearized equations for perturbations $(d, f) = (\zeta - \zeta_0, \eta - \eta_0)$ around the steady state are

$$\begin{bmatrix} \dot{d} \\ \dot{f} \end{bmatrix} = A \begin{bmatrix} d \\ f \end{bmatrix}, \qquad A = \begin{bmatrix} -H_{\eta\zeta} & -H_{\eta\eta} \\ H_{\zeta\zeta} & H_{\zeta\eta} \end{bmatrix}. \tag{4}$$

Here, the lower indexes specify partial derivatives, e.g. $H_{\zeta\eta} = \partial^2 H/\partial\zeta\partial\eta$. Furthermore, the partial derivatives of the Hamiltonian $H = H(\zeta, \eta)$ are evaluated at the steady state (ζ_0, η_0). The matrix A satisfies the property:

$$\mathcal{J}_c A^T \mathcal{J}_c = A, \tag{5}$$

where $\mathcal{J}_c$ is the symplectic matrix

$$\mathcal{J}_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6}$$

It follows that the characteristic polynomial $p(\sigma)$ of A is an even function of the eigenvalue σ. Indeed,

$$p(\sigma) = \det(A - \sigma I) = \det\left(\mathcal{J}_c A^T \mathcal{J}_c + \sigma \mathcal{J}_c^2\right) = (\det \mathcal{J}_c)^2 \det\left(A^T + \sigma I\right) = p(-\sigma). \tag{7}$$

In this notation A^T denotes the transpose matrix, while I is the identity matrix. Therefore, if σ is an eigenvalue, so is $-\sigma$. Furthermore, since A is a real matrix, its characteristic polynomial has real coefficients, which implies that σ^* and $-\sigma^*$ are eigenvalues as well (non-real roots of polynomials with real coefficients arise in complex conjugate pairs).

A similar linear stability result applies to traveling wave solutions of system (1) subject to superharmonic perturbations, i.e. perturbations whose wave number is an integer multiple of a fundamental wavenumber k_0. To see this, consider the change of variables with complex coefficients

$$a = \frac{\eta + \mathrm{i}\zeta}{\sqrt{2}}, \qquad b = \frac{\eta - \mathrm{i}\zeta}{\sqrt{2}}. \tag{8}$$

In the new variables, system (1) becomes:

$$\frac{\partial a}{\partial t} = -\mathrm{i}\frac{\delta H}{\delta b}, \qquad \frac{\partial b}{\partial t} = \mathrm{i}\frac{\delta H}{\delta a}. \tag{9}$$

Given a function $a(\boldsymbol{x}, t)$, the Fourier transform $\mathcal{F}[a] = \hat{a}(\boldsymbol{k}, t)$ and the inverse Fourier transform $\mathcal{F}^{-1}[\hat{a}] = a(\boldsymbol{x}, t)$ are defined as

$$\hat{a} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x}, \qquad a = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{a} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}. \tag{10}$$

Observing that

$$\frac{\delta H}{\delta b} = \int_{\mathbb{R}^n} \frac{\delta H}{\delta \hat{b}} \frac{\delta \hat{b}}{\delta b} d\boldsymbol{k} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\delta H}{\delta \hat{b}} e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}, \qquad \frac{\delta H}{\delta a} = \int_{\mathbb{R}^n} \frac{\delta H}{\delta \hat{a}} \frac{\delta \hat{a}}{\delta a} d\boldsymbol{k} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\delta H}{\delta \hat{a}} e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}, \tag{11}$$

and using equations (9) and (10), it follows that

$$\frac{\partial \hat{a}}{\partial t}(\boldsymbol{k}, t) = -\mathrm{i}\frac{\delta H}{\delta \hat{b}}(-\boldsymbol{k}, t), \qquad \frac{\partial \hat{b}}{\partial t}(\boldsymbol{k}, t) = \mathrm{i}\frac{\delta H}{\delta \hat{a}}(-\boldsymbol{k}, t). \tag{12}$$

Furthermore, from the definition of the Fourier transform (10) and the property $a^* = b$, the relationships below hold:

$$\hat{a}(\boldsymbol{k}, t) = \hat{b}^*(-\boldsymbol{k}, t), \qquad \hat{b}(\boldsymbol{k}, t) = \hat{a}^*(-\boldsymbol{k}, t). \tag{13}$$

Then, equation (12) can be written as

$$\frac{\partial \hat{a}}{\partial t} = -\mathrm{i}\left(\frac{\delta H}{\delta \hat{a}}\right)^*, \qquad \frac{\partial \hat{b}}{\partial t} = \mathrm{i}\left(\frac{\delta H}{\delta \hat{b}}\right)^*. \tag{14}$$

Here both $\hat{a}$ and $\hat{b}$ are evaluated at $(\boldsymbol{k}, t)$. Next, suppose that $\boldsymbol{c} \in \mathbb{R}^n$ is the phase speed of a traveling wave and perform a second change of coordinates $(\boldsymbol{x}, t) \to (\boldsymbol{x}', t') = (\boldsymbol{x} - \boldsymbol{c}t, t)$ to a reference frame moving with speed $\boldsymbol{c}$. Setting $s = a(\boldsymbol{x}' + \boldsymbol{c}t, t)$ and $u = b(\boldsymbol{x}' + \boldsymbol{c}t, t)$, we have

$$\hat{a} = \frac{e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}t}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} s e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'} d\boldsymbol{x}' = e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}t}\hat{s}, \qquad \hat{b} = \frac{e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}t}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'} d\boldsymbol{x}' = e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}t}\hat{u}. \tag{15}$$

Using equation (14) and equation (15), we obtain

$$\frac{\partial \hat{s}}{\partial t} = \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}\hat{s} - \mathrm{i}\left(\frac{\delta H}{\delta \hat{s}}\right)^*, \qquad \frac{\partial \hat{u}}{\partial t} = \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}\hat{u} + \mathrm{i}\left(\frac{\delta H}{\delta \hat{u}}\right)^*. \tag{16}$$

Observe that now traveling wave solutions with phase speed $\boldsymbol{c}$ are given by steady solutions of system (16). In the superharmonic setting $\hat{s}$ and $\hat{u}$ are infinite dimensional vectors with components

$$\hat{s} = \left(\hat{s}^1, \hat{s}^2, \ldots\right)^T, \qquad \hat{u} = \left(\hat{u}^1, \hat{u}^2, \ldots\right)^T, \tag{17}$$

where $\hat{s}^j = \hat{s}\,(j\boldsymbol{k}_0, t)$, $\hat{u}^j = \hat{u}\,(j\boldsymbol{k}_0, t)$, $\boldsymbol{k}_0 = k_0\boldsymbol{c}/c$, and $|j| \in \mathbb{N}$. In this notation $c = |\boldsymbol{c}|$. Then, equation (16) takes the form

$$\frac{\partial \hat{s}}{\partial t} = \mathrm{i}c\mathcal{I}\hat{s} - \mathrm{i}\left(\frac{\partial H}{\partial \hat{s}}\right)^*, \qquad \frac{\partial \hat{u}}{\partial t} = \mathrm{i}c\mathcal{I}\hat{u} + \mathrm{i}\left(\frac{\partial H}{\partial \hat{u}}\right)^*. \tag{18}$$

Here, we introduced the diagonal matrix $\mathcal{I}$ with components $\mathcal{I}^i_j = jk_0\delta^i_j$. Let $(\hat{S}, \hat{U})$ denote a steady solution of (18) and consider infinitesimal perturbations $(d, f) = (\hat{s} - \hat{S}, \hat{u} - \hat{U})$ of such equilibrium state. Then, the linearization of system (18) around $(\hat{S}, \hat{U})$ gives the system of equations

$$\begin{aligned} d_t =& \mathrm{i}c\mathcal{I}d - \mathrm{i}\left(H_{\hat{s}\hat{s}}d + H_{\hat{s}\hat{u}}f\right)^*, & \text{(19a)}\\ f_t =& \mathrm{i}c\mathcal{I}f + \mathrm{i}\left(H_{\hat{u}\hat{s}}d + H_{\hat{u}\hat{u}}f\right)^*, & \text{(19b)}\\ d^*_t =& -\mathrm{i}c\mathcal{I}d^* + \mathrm{i}\left(H_{\hat{s}\hat{s}}d + H_{\hat{s}\hat{u}}f\right), & \text{(19c)}\\ f^*_t =& -\mathrm{i}c\mathcal{I}f^* - \mathrm{i}\left(H_{\hat{u}\hat{s}}d + H_{\hat{u}\hat{u}}\right). & \text{(19d)} \end{aligned}$$

Here, the matrices $H_{\hat{s}\hat{s}}$, $H_{\hat{s}\hat{u}}$, $H_{\hat{u}\hat{s}}$, and $H_{\hat{u}\hat{u}}$ are evaluated at the steady state $(\hat{S}, \hat{U})$. Notice that in this notation $\left(H_{\hat{s}\hat{u}}\right)^i_j = \partial^2 H/\partial\hat{s}^i\partial\hat{u}^j$ and so on. By introducing the matrix

$$J = \mathrm{i}\begin{bmatrix} c\mathcal{I} & 0 & -H^*_{\hat{s}\hat{s}} & -H^*_{\hat{s}\hat{u}} \\ 0 & c\mathcal{I} & H^*_{\hat{u}\hat{s}} & \hat{H}^*_{\hat{u}\hat{u}} \\ H_{\hat{s}\hat{s}} & H_{\hat{s}\hat{u}} & -c\mathcal{I} & 0 \\ -H_{\hat{u}\hat{s}} & -H_{\hat{u}\hat{u}} & 0 & -c\mathcal{I} \end{bmatrix}, \tag{20}$$

and defining $v = \left(d, f, d^*, f^*\right)^T$, system (19) can be written as

$$v_t = Jv. \tag{21}$$

Now the linear stability of a traveling wave solution with phase speed $\boldsymbol{c}$ can be examined by solving the eigenvalue problem for J. Next, consider the matrix

$$Q = \begin{bmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix}. \tag{22}$$

Here, I is the identity matrix with the same dimension as $\mathcal{I}$. Q has the following properties

$$Q^2 = -I, \qquad Q^T = -Q, \tag{23}$$

with I the identity matrix in appropriate dimensions. Using these relationships, the following identity can be verified

$$QJ^TQ = J. \tag{24}$$

From (23) and (24), it follows that the characteristic polynomial $p\,(\sigma)$ of J is even:

$$p\,(\sigma) = \det\left(J - \sigma I\right) = \det\left(QJ^TQ + \sigma Q^2\right) = \left(\det Q\right)^2 \det\left(J^T + \sigma I\right) = p\,(-\sigma)\,. \tag{25}$$

Thus, if σ is an eigenvalue, so is $-\sigma$. This behavior, which is a consequence of the Hamiltonian structure of the equations, physically represents time reversal symmetry.

3 Infinite Dimensional Hamiltonian Systems in Canonical Form

In this section we review some examples of infinite dimensional Hamiltonian systems that arise in the canonical form (1) in the context of quantum and fluid mechanics.

3.1 The Schrödinger equation

Consider the 3 dimensional Schrödinger equation for a particle in a potential V

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V\psi. \tag{26}$$

Here, $\hbar$ is the reduced Planck constant, m the particle mass, and ψ the wave function. Equation (26) is a linear partial differential equation for the complex variable ψ. Upon introducing the change of variables (Madelung representation)

$$\psi=\sqrt{\rho}\exp\left\{\frac{i}{\hbar}\theta\right\}, \tag{27}$$

where $\rho=|\psi|^2$ is the probability density in the position basis and θ the phase variable, equation (26) tranlates into a system of two coupled partial differential equations:

$$\frac{\partial\rho}{\partial t}=-\frac{1}{m}\nabla\cdot(\rho\nabla\theta), \tag{28a}$$

$$\frac{\partial\theta}{\partial t}=\frac{\hbar^2}{2m}\left(-\frac{|\nabla\rho|^2}{4\rho^2}-\frac{|\nabla\theta|^2}{\hbar^2}+\frac{\Delta\rho}{2\rho}\right)-V. \tag{28b}$$

System (28b) can be cast in canonical Hamiltonian form

$$\frac{\partial\rho}{\partial t}=\frac{\delta H}{\delta\theta},\qquad \frac{\partial\theta}{\partial t}=-\frac{\delta H}{\partial\rho}, \tag{29}$$

where the Hamiltonian H is given by

$$H=\int_{\mathbb{R}^3}\rho\left(\frac{|\nabla\theta|^2}{2m}+V+\frac{\hbar^2}{8m}|\nabla\log\rho|^2\right)dV. \tag{30}$$

3.2 The nonlinear Schrödinger equation

Consider the 3 dimensional nonlinear Schrödinger equation and its complex conjugate

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+\kappa\psi|\psi|^2,\qquad -i\hbar\frac{\partial\psi^*}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi^*+\kappa\psi^*|\psi|^2 \tag{31}$$

Here, $\kappa\in\mathbb{R}$ is a constant. Define the new variables

$$a=\sqrt{\frac{\hbar}{2}}(\psi+\psi^*),\qquad b=\frac{1}{i}\sqrt{\frac{\hbar}{2}}(\psi-\psi^*). \tag{32}$$

Then, equation (31) can be expressed in canonical Hamiltonian form

$$\frac{\partial a}{\partial t}=\frac{\delta H}{\delta b},\qquad \frac{\partial b}{\partial t}=-\frac{\delta H}{\delta a}, \tag{33}$$

where the Hamiltonian H is given by

$$H=\int_{\mathbb{R}^3}\left(\frac{\hbar^2}{2m}|\nabla\psi|^2+\frac{\kappa}{2}|\psi|^4\right)dV=\int_{\mathbb{R}^3}\left[\frac{\hbar}{4m}\left(|\nabla a|^2+|\nabla b|^2\right)+\frac{\kappa}{8\hbar^2}\left(a^2+b^2\right)^2\right]dV. \tag{34}$$

3.3 Water waves with constant vorticity

Consider a gravity-capillary wave over a linear shear current. The fluid density is assumed to be constant and equal to 1. The domain Ω of the system is 2 dimensional and has boundary $\partial\Omega$. Let (x,y) denote a Cartesian coordinate system with the x axis aligned with the horizontal direction of propagation of the wave and the y axis aligned with the vertical upward direction. We have $\Omega = \left\{(x,y)\in\mathbb{R}^2 \mid 0<x<L,\ d<y<\eta\left(x,t\right)\right\}$, where $L>0$ and $d\leq 0$ are real constants and $\eta\left(x,t\right)$ represents the elevation of the water surface at the point (x,t). The boundary $\partial\Omega = B\cup\Sigma\cup T$ can be divided in three components: the bottom $B = \left\{(x,y)\in\mathbb{R}^2 \mid y=d\right\}$, the water surface $\Sigma = \left\{(x,y)\in\mathbb{R}^2 \mid y=\eta\left(x,t\right)\right\}$, and the vertical boundaries $T = \left\{(x,y)\in\mathbb{R}^2 \mid x=\{0,L\}\right\}$. In Ω, the velocity field has expression

$$\boldsymbol{v} = \nabla\phi - \omega y\,\partial_x. \tag{35}$$

In this notation ϕ is the velocity potential, ∂_x the tangent vector along the x axis, and ω a real constant representing the vorticity of the shear current. The ideal Euler equations for the fluid can be written as

$$\phi_t = f - \omega\psi - \frac{\left|\nabla\phi\right|^2}{2} + \omega y\phi_x - P - gy, \qquad \Delta\phi = 0 \quad \text{in } \Omega, \tag{36}$$

with $f = f\left(t\right)$ an arbitrary function of time, ψ the harmonic conjugate of ϕ, P the pressure, and g the gravitational constant. Equation (36) is supplied with the boundary condition $\phi_y = 0$ at the bottom B, the periodic boundary condition $\boldsymbol{v}\left(0,y,t\right) = \boldsymbol{v}\left(L,y,t\right)$ on the vertical boundaries T, and the dynamic boundary conditions at the water surface Σ

$$\zeta_t = -\frac{\left|\nabla\phi\right|^2}{2} - g\eta + \alpha\frac{\partial}{\partial x}\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right) + \phi_y\eta_t + \omega\eta\phi_x - \frac{\omega}{2}\chi - \frac{\omega^2}{4}\eta^2, \tag{37a}$$

$$\eta_t = \phi_y - \phi_x\eta_x + \omega\eta\eta_x, \tag{37b}$$

where we introduced the real constant α associated with surface tension and the quantities

$$\zeta = \theta - \frac{\omega}{2}\int_0^x \eta\, ds, \qquad \theta = \phi\left(x,\eta,t\right), \qquad \chi = \psi\left(x,\eta,t\right). \tag{38}$$

Notice that here the function χ, which is defined up to an arbitrary function of time, is chosen to be such that $\chi\left(0,t\right) - \left(\omega/2\right)\eta^2\left(0,t\right) = 0$. Furthermore, at the water surface the pressure P does not appear explicitly in the equations since it equals the constant atmospheric pressure and as such it can be canceled by appropriate choice of f. Solving system (37b) is sufficient to determine the potential ϕ in the whole Ω. This is because the continuity equation (the second equation in (36)) implies that ϕ is a harmonic function, and thus it can be computed by solving the relevant boundary value problem for Laplace's equation. System (37b) can be cast in canonical Hamiltonian form (see [10] and [4]). We have

$$\frac{\partial\zeta}{\partial t} = -\frac{\delta H}{\delta\eta}, \qquad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\zeta}, \tag{39}$$

where the Hamiltonian H is given by

$$H\left[\zeta,\eta\right] = \int_0^L\left[\int_d^\eta\left(\frac{\left|\nabla\phi\right|^2}{2} - \omega y\phi_x\right)dy\right]dx + \int_0^L\left(\frac{g}{2}\eta^2 + \frac{\omega^2}{6}\eta^3 + \alpha\sqrt{1+\eta_x^2}\right)dx. \tag{40}$$

4 The Tanaka Instability

The purpose of the present section is to show that if a traveling wave solution of a canonical Hamiltonian system in the form (1) subject to superharmonic perturbations admits a translational symmetry, then Tanaka's instability occurs at the extrema of energy with respect to wave speed. As explained in section 1, this can be shown by determining the conditions under which the zero eigenvalue has algebraic multiplicity greater than two.

Let $\boldsymbol{\xi} \in \mathbb{R}^3$ denote the spatial direction of the translational symmetry of a traveling wave solution (S, U). The existence of this symmetry implies that the functions

$$S_\lambda = S(\boldsymbol{x}' + \lambda \boldsymbol{\xi}), \qquad U_\lambda = U(\boldsymbol{x}' + \lambda \boldsymbol{\xi}), \tag{41}$$

are also traveling wave solutions for any choice of the parameter $\lambda \in \mathbb{R}$. The Fourier transform of the translated solution can be evaluated to be

$$\hat{S}_\lambda = e^{\mathrm{i}\lambda \boldsymbol{k}\cdot\boldsymbol{\xi}} \hat{S}, \qquad \hat{U}_\lambda = e^{\mathrm{i}\lambda \boldsymbol{k}\cdot\boldsymbol{\xi}} \hat{U}. \tag{42}$$

On the other hand, the existence of translational symmetry implies that steady solutions of system (18) are not unique, and the matrix J of equation (20) has a non-trivial null-space corresponding to an eigenvector α with eigenvalue $\sigma = 0$. Using equation (42), the vector α can be computed as below:

$$\alpha = -\frac{d}{d\lambda} \begin{bmatrix} \hat{S}_\lambda \\ \hat{U}_\lambda \\ \hat{S}^*_\lambda \\ \hat{U}^*_\lambda \end{bmatrix}_{\lambda=0} = \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi} \begin{bmatrix} -\hat{S} \\ -\hat{U} \\ \hat{S}^* \\ \hat{U}^* \end{bmatrix}. \tag{43}$$

Restricting to superharmonics, this gives

$$\alpha = \mathrm{i}\frac{\boldsymbol{c}\cdot\boldsymbol{\xi}}{c} \begin{bmatrix} -\mathcal{I}\hat{S} \\ -\mathcal{I}\hat{U} \\ \mathcal{I}\hat{S}^* \\ \mathcal{I}\hat{U}^*. \end{bmatrix} \tag{44}$$

Due to the parity of the characteristic polynomial of the matrix J discussed in section 2, the eigenvalue $\sigma = 0$ must have even algebraic multiplicity. Therefore, there must exist a generalized eigenvector β of rank 2 associated with the eigenvalue $\sigma = 0$. The vector β can be obtained from equation (18) by differentiating the equilibrium system

$$\left(\frac{\partial H}{\partial \hat{s}}\right)^* = c\mathcal{I}\hat{S}, \qquad \left(\frac{\partial H}{\partial \hat{u}}\right)^* = -c\mathcal{I}\hat{U}, \qquad \frac{\partial H}{\partial \hat{s}} = c\mathcal{I}\hat{S}^*, \qquad \frac{\partial H}{\partial \hat{u}} = -c\mathcal{I}\hat{U}^*, \tag{45}$$

with respect to the absolute value c of the wave speed $\boldsymbol{c}$ (in the following, we shall refer to c simply as the wave speed). Notice that in the last equation the derivatives $\partial H/\partial \hat{s}$ and $\partial H/\partial \hat{u}$ are evaluated at the equilibrium $(\hat{S}, \hat{U})$. We have

$$-H^*_{\hat{s}\hat{s}}\frac{d\hat{S}^*}{dc} - H^*_{\hat{s}\hat{u}}\frac{d\hat{U}^*}{dc} + c\mathcal{I}\frac{d\hat{S}}{dc} = -\mathcal{I}\hat{S} \tag{46a}$$

$$H^*_{\hat{u}\hat{s}}\frac{d\hat{S}^*}{dc} + H^*_{\hat{u}\hat{u}}\frac{d\hat{U}^*}{dc} + c\mathcal{I}\frac{d\hat{U}}{dc} = -\mathcal{I}\hat{U}, \tag{46b}$$

$$H_{\hat{s}\hat{s}}\frac{d\hat{S}}{dc} + H_{\hat{s}\hat{u}}\frac{d\hat{U}}{dc} - c\mathcal{I}\frac{d\hat{S}^*}{dc} = \mathcal{I}\hat{S}^* \tag{46c}$$

$$-H_{\hat{u}\hat{s}}\frac{d\hat{S}}{dc} - H_{\hat{u}\hat{u}}\frac{d\hat{U}}{dc} - c\mathcal{I}\frac{d\hat{U}^*}{dc} = \mathcal{I}\hat{U}^*. \tag{46d}$$

This system has the form

$$J\beta = \alpha, \tag{47}$$

with the generalized eigenvector β of rank 2 given by

$$\beta = \frac{\boldsymbol{c}\cdot\boldsymbol{\xi}}{c}\frac{d}{dc}\begin{bmatrix}\hat{S}\\ \hat{U}\\ \hat{S}^*\\ \hat{U}^*.\end{bmatrix}. \tag{48}$$

Assuming that the translational symmetry represents the only degeneracy of the matrix J, the geometric condition for the existence of a generalized eigenvector of rank 3 associated with the eigenvalue $\sigma = 0$ is given by

$$\langle\beta, \tilde{\alpha}\rangle = 0, \tag{49}$$

where $\langle\beta, \tilde{\alpha}\rangle = \beta \cdot \tilde{\alpha}^*$ denotes the standard inner product, and $\tilde{\alpha}$ is the eigenvector of $J^\dagger = (J^*)^T$ associated with the eigenvalue $\sigma = 0$. The condition (49) follows from the fact that any vector β perpendicular to the null-space of the Hermitian conjugate $J^\dagger$ belongs to the image of J, i.e. $\beta = J\gamma$ for some appropriate choice of γ when (49) is satisfied (see for example [4]). If it exists, the vector γ then represents the generalized eigenvector of rank 3. Notice that, due to the parity of the characteristic polynomial, the existence of γ automatically implies that the eigenvalue $\sigma = 0$ has algebraic multiplicity equal to or greater than four. An explicit calculations shows that $\tilde{\alpha}$ has expression

$$\tilde{\alpha} = \mathrm{i}\frac{\boldsymbol{c}\cdot\boldsymbol{\xi}}{c}\begin{bmatrix}\mathcal{I}\hat{S}\\ -\mathcal{I}\hat{U}\\ \mathcal{I}\hat{S}^*\\ -\mathcal{I}\hat{U}^*.\end{bmatrix} \tag{50}$$

Equation (49) thus becomes

$$\langle\beta, \tilde{\alpha}\rangle = -\mathrm{i}\left(\frac{\boldsymbol{c}\cdot\boldsymbol{\xi}}{c}\right)^2\left[\frac{d\hat{S}}{dc}\cdot\mathcal{I}\hat{S}^* - \frac{d\hat{U}}{dc}\cdot\mathcal{I}\hat{U}^* + \frac{d\hat{S}^*}{dc}\cdot\mathcal{I}\hat{S} - \frac{d\hat{U}^*}{dc}\cdot\mathcal{I}\hat{U}\right] = 0. \tag{51}$$

In order to understand the physical meaning of this condition, now consider the derivative of the Hamiltonian H with respect to c at the equilibrium state. Using (45), we have

$$\frac{dH}{dc} = \frac{\partial H}{\partial\hat{s}}\cdot\frac{d\hat{S}}{dc} + \frac{\partial H}{\partial\hat{u}}\cdot\frac{d\hat{U}}{dc} = c\mathcal{I}\hat{S}^*\cdot\frac{d\hat{S}}{dc} - c\mathcal{I}\hat{U}^*\cdot\frac{d\hat{U}}{dc}. \tag{52}$$

We conclude that

$$\langle\beta, \tilde{\alpha}\rangle = -4\mathrm{i}\left(\frac{\boldsymbol{c}\cdot\boldsymbol{\xi}}{c}\right)^2\frac{dH}{dc^2}. \tag{53}$$

Therefore, a sufficient condition for the zero eigenvalue to have algebraic multiplicity of at least four is that the energy H is stationary with respect to wave speed c.

5 Concluding Remarks

In this paper, the Tanaka instability of traveling waves was reviewed from the perspective of Hamiltonian mechanics. This instability occurs in Hamiltonian systems when a traveling wave solution subject to superharmonic disturbances possesses a translational symmetry and its energy is stationary as a function of wave speed. The stationary points are characterized by the zero eigenvalue of the linear stability problem for the growth rate of infinitesimal perturbations. In particular, the algebraic multiplicity of the zero eigenvalue, which is always even due to the Hamiltonian structure of the equations, jumps in correspondence of the energy extrema, indicating the coalescence of the mode corresponding to translational symmetry with a different (generally non-zero) eigenvalue, and causing the onset of instability.

While we have restricted our discussion to canonical Hamiltonian systems in the form (1), it is possible to generalize the theory to the noncanonical Hamiltonian setting. Several infinite dimensional Hamiltonian systems, such as the modon equation [11], the Hasegawa-Mima equation [12], the KdV equation, or magnetohydrodynamics, occur in noncanonical form. Therefore, we expect the Tanaka instability to manifest also for traveling wave solutions of these systems provided that the relevant energy possesses stationary points with respect to wave speed.

A second aspect that deserves comment is the scope of Krein's theorem. In this paper, the restriction of the perturbations to superharmonics enabled the proof of a Krein-like theorem for the linear stability of traveling waves. More precisely, superharmonic perturbations transform the linear stability problem from a continuous one to a discrete one. However, the applicability of a Krein-like theorem with respect to a more general spectrum may allow a further generalization of the conditions for the occurrence of the Tanaka instability.

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