集合値非加法的測度について

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Abstract. Egoroff's theorem and Lusin's theorem are most fundamental theorems in classical measure theory. They established for set-valued measures, which take values in the family of all non-void, closed subsets of a real normed space using Hausdorff metric by several authors. In this talk, we consider these theorems for set valued non-additive measures from the another point of view, using the topological convergence of set sequences.

1. Introduction

Egoroff's theorem and Lusin's theorem are most fundamental theorems in classical measure theory and do not necessary hold in non-additive measure theory without additional conditions. In [1], Wang generalized Egoroff's theorem in case of fuzzy measures. Moreover in [2], Wang and Klir gave another generalization of this result for fuzzy measures, which are null-additive. In [3], Li showed that Egoroff's theorem remain true for fuzzy measures without any other supplementary conditions for them. When a fuzzy measure is not necessarily finite, Li et al. [4] have proved that Egoroff's theorem remains valid on fuzzy measures possessing the order continuity and pseudo-metric generating property. In [5], Murofushi, Uchino and Asahina find the necessary and sufficient condition called the Egoroff condition, which assures that Egoroff's theorem remains valid for real valued non-additive measures, see also Li [6]. In [7, 8], Kawabe extend these results for Riesz space-valued fuzzy measures. In [9], Li and Yasuda proved Lusin's Theorm remains valid for real valued for fuzzy measures, also in [10] Li and Mesir proved Lusin's Theorm remains valid for real valued for monotone measures. For the Lusin's theorem for fuzzy measures on vector (Riesz) space-valued, see [11]. Also these results for an ordered vector space-valued and an ordered topological vector space-valued non-additive measures, see [12, 13]. For informations on real valued non-additive measures, see [2, 14, 15].

Recently, by several authors, Egoroff's theorem and Lusin's theorem are established for non-additive set-valued (multi) measures, which take values in the family of all non-void, closed subsets of real normed spaces. In [16], Precupanu and Gavriluţ investigate Egoroff's theorem in a fuzzy multimeasure in the sense of Hausdorff pseudo metric; see Precupanu and et al. [17]. In [18], Wu and Liu investigate Egoroff's theorem in a set-valued fuzzy measure introduced in Gavriluţ [19].

In this talk, we prove Egoroff's theorem and Lusin's theorem remains valid for non-additive multi measures. In particular, we use a topological convergence with

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respect to set-valued mappings, see [20, 21]. We consider the convergence of point as a weak setting.

2. Preliminaries

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ the set of all natural numbers. We denote by $\mathcal{T}$ the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$. Let X be a non-empty set and $\mathcal{F}$ a σ-field of X. Let Y be a topological vector space (see [22, 23]). Let θ be an origin of Y, and $\mathcal{B}_\theta$ a system of neighborhoods of $\theta \in Y$. Note that for any neighborhood $U \in \mathcal{B}_\theta$, there exists $W \in \mathcal{B}_\theta$ such that W is balanced and satisfy $W \subset V$.

We denote $\mathcal{P}_0(Y)$ be a family of non-empty subsets of Y. Let $\mathcal{P}_{cl}(Y)$ be a family of closed, non-empty subsets of Y. We consider the following two types convergence. Let $\{E_n\} \subset \mathcal{P}_0(Y)$ be a set sequence and $E \in \mathcal{P}_0(Y)$. We say that $\{E_n\}$ is

(A) type (I) convergent to E, if for any $e \in E$ there exists a sequence $\{e_n\}$, which converges to e, that is, for any $U \in \mathcal{B}_0$ there exists a n_0 with $e_n - e \in U$ for any $n \geq n_0$, such that $e_n \in E_n$ for every n;

(B) type (II) convergent to E, if given $j \in \mathbb{N}$, for any sequence $\{e_{n_j}\} \subset Y$, which converges to $e \in Y$, that is, for any $U \in \mathcal{B}_0$ there exists a j_0 with $e_{n_j} - e \in U$ for any $j \geq j_0$, if $e_{n_j} \in E_{n_j}$, then $e \in E$.

If (A) holds, we will write $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty} E_n = E$ and if (B) holds, we will write $\mathrm{Lim}^{(\mathrm{II})}_{n\to\infty} E_n = E$. If both (A) and (B) hold, we will write $\mathrm{Lim}_{n\to\infty} E_n = E$ and said to be Kuratowski convergence [20, 21].

3. The continuity of non-additive multi measures

Definition 1. *Let $(X, \mathcal{F})$ be an arbitrary measurable space, and let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a set-valued mapping. μ is said to be a non-additive multi measure on X if the following conditions (i) and (ii) hold.*

(i) $\mu(\emptyset) = \{\theta\}$,

(ii) *for $A, B \in \mathcal{F}$ with $A \subset B$, $\mu(A) \subset \mu(B)$ (monotonicity).*

Moreover, we consider the following conditions.

Definition 2. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ is said to be*

(i) *continuous from above type (I) if $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \searrow A$;*

(ii) *continuous from below type (I) if $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \nearrow A$;*

(iii) *continuous from above type (II) if $\mathrm{Lim}^{(\mathrm{II})}_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \searrow A$;*

(iv) *continuous from below type (II) if $\mathrm{Lim}^{(\mathrm{II})}_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \nearrow A$.*

(v) *μ has property* (S) *if for any sequence $\{A_n\} \subset \mathcal{F}$ with $\mu(A_n) \to \{\theta\}$, there exists a subsequence $\{A_{n_k}\}$ such that $\mu(\cap_{i=1}^{\infty} \cup_{k=i}^{\infty} A_{n_k}) = \{\theta\}$; see* [25].

(vi) *A non-additive multi measure μ is said to have property weak-(S) if for any $\{E_n\} \subset \mathcal{F}$, with $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty} \mu(E_n) \ni \theta$, there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that $\mu\left(\cap_{j=1}^{\infty} \cup_{i=j}^{\infty} E_{n_i}\right) \ni \theta$. Note that property weak-*(S) *implies property* (S).

Example 3. *Let $(X, \mathcal{F})$ be a measurable space, $m : \mathcal{F} \to R_+$ a non-additive measure on $\mathcal{F}$, $Y = R^2$ and R^2_+ is a positive cone. Consider the order interval with respect to $\mathbb{R}^2_+$ defined by*

$$[a, b]_{\mathbb{R}^2_+} := \{y \in \mathbb{R}^2 \mid y \in (a + \mathbb{R}^2_+) \cap (b - \mathbb{R}^2_+)\},$$

where $a, b \in \mathbb{R}^2$.

Define $\mu(A) := [(0, m(A)), (m(A), m(A))]_{\mathbb{R}^2_+}$ for any $A \in \mathcal{F}$. Then μ is a non-additive multi measure on $\mathcal{F}$.

Definition 4. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ is said to be*

(i) *strongly order continuous type (I), if it is continuous from above at measurable sets of measure zero, that is, for any $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfying $A_n \searrow A$ and $\mu(A) = \{\theta\}$, it holds that $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) = \{\theta\}$;*

(ii) *strongly order semi-continuous type (I), if for any $\{A_n\} \subset \mathcal{F}$ and $A \in \mathcal{F}$ satisfying $A_n \searrow A$ and $\mu(A) \ni \theta$, it holds that $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) \ni \theta$.*

Note that strongly order semi-continuous type (I) implies strongly order continuous type (I).

Definition 5. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ is said to be*

(i) *null-additive, if for any $B \in \mathcal{F}$ with $\mu(B) = \{\theta\}$, then $\mu(A \cup B) = \mu(A)$ for any $A \in \mathcal{F}$;*

(ii) *null-subtractive if for any $B \in \mathcal{F}$ with $\mu(B) = \{\theta\}$, then $\mu(A \setminus B) = \mu(A)$ for any $A \in \mathcal{F}$.*

(iii) *null-null-additive, if for any $A, B \in \mathcal{F}$ with $\mu(A) = \mu(B) = \{\theta\}$, then $\mu(A \cup B) = \{\theta\}$ for any $A \in \mathcal{F}$;*

(iv) *weak null-null-additive, if for any $A, B \in \mathcal{F}$ with $\mu(A) \ni \theta$ and $\mu(B) \ni \theta$, then $\mu(A \cup B) \ni \theta$ for any $A \in \mathcal{F}$;*

(iv) *μ is said to have the weak pseudometric generating property, abbreviated as weak-p.g.p., if for any sequences $\{A_n\}, \{B_n\} \subset \mathcal{F}$, if $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) \ni \theta$ and $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(B_n) \ni \theta$, then $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n \cup B_n) \ni \theta$.*

(iv) *μ is said to have the pseudometric generating property, abbreviated as p.g.p., if for any sequences $\{A_n\}, \{B_n\} \subset \mathcal{F}$, if $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) = \{\theta\}$ and $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(B_n) = \{\theta\}$, then $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n \cup B_n) = \{\theta\}$.*

Lemma 1. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. Then the null-additivity of μ is equivalent to the null-subtractivity of it.*

4. Egoroff's Theorem

Definition 6. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure.*

(1) *A double sequence $\{A_{m,n}\} \subset \mathcal{F}$ is called a weak-μ-regulator if it satisfies the following two conditions.*
  (D1) *$A_{m,n} \supset A_{m,n'}$ whenever $n \le n'$.*
  (D2) *$\mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} A_{m,n}\right) \ni \theta$.*

(2) *A double sequence $\{A_{m,n}\} \subset \mathcal{F}$ is called a μ-regulator if it satisfies the following two conditions.*
  (D1) *$A_{m,n} \supset A_{m,n'}$ whenever $n \le n'$.*

(D2) $\mu\left(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}\right) = \{\theta\}$.

(3) *μ satisfies the weak-Egoroff condition if for any weak-μ-regulator $\{A_{m,n}\}$, there exists a $\tau \in T$ such that $\mu(\cup_{m=1}^{\infty} A_{m,\tau(m)}) \ni \theta$ holds.*

(4) *μ satisfies the Egoroff condition if for any μ-regulator $\{A_{m,n}\}$, there exists a $\tau \in T$ such that $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) = \{\theta\}$ holds.*

Note that Egoroff condition implies weak Egoroff condition.

It is easy to check that the following lemma holds.

Lemma 2. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ satisfies the weak-Egoroff condition (resp. Egoroff condition) if (and only if), for any double sequence $\{A_{m,n}\} \subset \mathcal{F}$ satisfying* (D2) *in* Definition 6 *and the following* (D1$'$)*, it holds that there exists a $\tau \in T$ such that $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) \ni \theta$ (resp. $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) = \{\theta\}$).*

(D1$'$) *$A_{m,n} \supset A_{m',n'}$ whenever $m \geq m'$ and $n \leq n'$.*

Definition 7. *Let $(X, \mathcal{F}, \mu)$ be the non-additive multi measure space, f_n and $f \in \mathcal{F}$ for $n = 1, 2, \ldots$.*

(1) *$\{f_n\}$ is said to converge to f μ-almost everywhere on X, which is denoted by $f_n \overset{a.e.}{\to} f$, if there exists $A \in \mathcal{F}$ such that $\mu(A) = \{\theta\}$ and $\{f_n\}$ converges to f on $X \setminus A$.*

(2) *$\{f_n\}$ is said to converge to f μ-almost uniformly on X, which is denoted by $f_n \overset{a.u.}{\to} f$, if there exists $\{A_\gamma \mid j \in \gamma\} \subset \mathcal{F}$ and there exists $\gamma \in \Gamma$ such that $\mu(A_\gamma) = \{\theta\}$ and $\{f_n\}$ converges to f uniformly on $X \setminus A_\gamma$.*

(3) *We say Egoroff theorem holds if for μ if $\{f_n\}$ converges μ-almost uniformly (μ-a.u.) to f whenever it converges μ-a.e. to the same limit.*

Under the above settings we have the following theorems.

Theorem 8. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. If μ satisfies the Egoroff condition, then it satisfies the weak-Egoroff condition.*

Theorem 9. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. Then the following two conditions are equivalent.*

(1) *μ satisfies the Egoroff condition.*

(2) *The Egoroff theorem holds for μ.*

Theorem 10. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. Assume that there exists $B \in \mathcal{F}$ with $\mu(B) = \{\theta\}$ and for μ-regulator $\{A_{m,n}\}$,*

$$\left(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}\right) \cap B \neq \emptyset$$

holds. If μ satisfies the weak-Egoroff condition, then the Egoroff theorem holds for μ.

5. Sufficient conditions for weak-Egoroff condition

Next we give several sufficient conditions for the establishment of weak-Egoroff condition.

Theorem 11. *We assume that Y is locally convex spaces. Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. If μ satisfies continuous from above type (I), continuous from below type (II), and null-additive, then the weak-Egoroff condition holds for μ.*

Next we consider another sufficient condition.

Definition 12. *The double sequence* $\{r_{m,n}\}$ *of sets in* $\mathcal{P}_{cl}(Y)$ *is called a weak topological regulator if it satisfies the following two conditions.*

(1) $r_{m,n} \supset r_{m,n+1}$ *for any* $m, n \in N$.
(2) *For any* $m \in N$, *it holds that* $\cap_{n=1}^{\infty} r_{m,n} \ni \theta$.

Definition 13. *The double sequence* $\{r_{m,n}\}$ *of sets in* $\mathcal{P}_{cl}(Y)$ *is called a topological regulator if it satisfies the following two conditions.*

(1) $r_{m,n} \supset r_{m,n+1}$ *for any* $m, n \in N$.
(2) *For any* $m \in N$, *it holds that* $\cap_{n=1}^{\infty} r_{m,n} = \{\theta\}$.

Definition 14. *We say that* $\mathcal{P}_{cl}(Y)$ *has property* (EP) *if for any topological regulator* $\{r_{m,n}\}$ *in* $\mathcal{P}_{cl}(Y)$, *there exists a sequence* $\{P_k\}$ *of set in* $\mathcal{P}_{cl}(Y)$ *satisfying the following two conditions.*

(1) $\mathrm{Lim}^{(\mathrm{I})}_{k\to\infty} P_k = \{\theta\}$.
(2) *For any* $k \in N$ *and* $m \in N$, *there exists an* $n_0(m, k) \in N$ *such that* $\{r_{m,n}\} \subset P_k$ *for any* $n \geq n_0(m, k)$.

Definition 15. *We say that* $\mathcal{P}_{cl}(Y)$ *has property weak* (EP) *if for any weak topological regulator* $\{r_{m,n}\}$ *in* $\mathcal{P}_{cl}(Y)$, *there exists a sequence* $\{P_k\}$ *of set in* $\mathcal{P}_{cl}(Y)$ *satisfying the following two conditions.*

(1) $\mathrm{Lim}^{(\mathrm{I})}_{k\to\infty} P_k \ni \theta$.
(2) *For any* $k \in N$ *and* $m \in N$, *there exists an* $n_0(m, k) \in N$ *such that* $\{r_{m,n}\} \subset P_k$ *for any* $n \geq n_0(m, k)$.

Theorem 16. *Let* $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ *be a non-additive multi measure. We assume that* μ *is strongly order semi-continuous type (I) and satisfies property weak-*(S). *We assume that* $\mathcal{P}_{cl}(Y)$ *has property* (EP). *Then* μ *satisfies the weak-Egoroff condition.*

6. Regularity

Let X be a Hausdorff space. Denote by $\mathcal{B}(X)$ the σ-field of all Borel subsets of X, that is, the σ-field generated by the open subsets of X. A non-additive multi measure defined on $\mathcal{B}(X)$ is called a non-additive Borel multi measure on X. First we give a lemma.

Lemma 3. *Let* $\mu : \mathcal{B}(X) \to \mathcal{P}_{cl}(Y)$ *be a non-additive Borel multi measure which is strongly order continuous Type (I) and has property weak-*(S). *We assume that* $\mathcal{P}_{cl}(Y)$ *has property* (EP). *Then the following two conditions are equivalent*:
(i) μ *is null-null-additive.*
(ii) *For any* $U \in \mathcal{B}_0$ *and double sequence* $\{A_{m,n}\} \subset \mathcal{F}$ *satisfying that* $A_{m,n} \downarrow D_m$ *as* $n \to \infty$ *and* $\mu(D_m) = \{\theta\}$ *for each* $m \in N$, *then there exists a sequence* $\{\tau_k\}$ *of elements of* $\mathcal{T}$ *such that* $\mathrm{Lim}^{(\mathrm{I})}_{k\to\infty} \mu\left(\cup_{m=1}^{\infty} A_{m,\tau_k(m)}\right) = \{\theta\}$.

Lemma 4. *Let* $\mu : \mathcal{B}(X) \to \mathcal{P}_{cl}(Y)$ *be a non-additive Borel multi measure which is strongly order semi-continuous Type (I) and has property weak-*(S). *We assume that* $\mathcal{P}_{cl}(Y)$ *has property weak-*(EP). *Then* (i) *implies* (ii):
(i) μ *is weak null-null-additive.*
(ii) *For any* $U \in \mathcal{B}_0$ *and double sequence* $\{A_{m,n}\} \subset \mathcal{F}$ *satisfying that* $A_{m,n} \downarrow D_m$ *as* $n \to \infty$ *and* $\mu(D_m) \ni \theta$ *for each* $m \in N$, *then there exists a sequence* $\{\tau_k\}$ *of*

elements of $\mathcal{T}$ such that $\mathrm{Lim}^{(\mathrm{I})}_{m\to\infty}\mu\left(\cup_{m=1}^{\infty}A_{m,\tau_k(m)}\right)\ni\theta$.

Then we have the following.

Definition 17 ([26]). *Let $\mu:\mathcal{F}\to\mathcal{P}_{cl}(Y)$ be a non-additive Borel multi measure. μ is called weak regular if for any $U\in\mathcal{B}_0$ and $A\in\mathcal{B}(X)$, there exist a sequence of closed set $\{F_U^n\}$ and an open set $\{G_U^n\}$ such that $F_U^n\subset A\subset G_U^n$ and* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(G_U^n\setminus F_U^n)\ni\theta$

Definition 18 ([26]). *Let $\mu:\mathcal{F}\to\mathcal{P}_{cl}(Y)$ be a non-additive Borel multi measure. μ is called regular if for any $U\in\mathcal{B}_0$ and $A\in\mathcal{B}(X)$, there exist sequences of closed sets $\{F_U^n\}$ and open sets $\{G_U^n\}$ such that $F_U^n\subset A\subset G_U^n$ and* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(G_U^n\setminus F_U^n)=\{\theta\}$

Lemma 5. *If μ is regular, then it is weak-regular.*

Theorem 19. *Let X be a metric space and $\mathcal{B}(X)$ a σ-field of all Borel subsets of X. Let $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ be a non-additive Borel multi measure on X which is p.g.p and satisfies weak-Egoroff condition. Then μ is weak-regular.*

By theorem Theorem 10, we have

Corollary 20. *Let X be a metric space and $\mathcal{B}(X)$ a σ-field of all Borel subsets of X. Let $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ be a non-additive Borel multi measure on X which is p.g.p and satisfies weak-Egoroff condition. Assume that there exists $B\in\mathcal{F}$ with $\mu(B)=\{\theta\}$ and for μ-regulator $\{A_{m,n}\}$, $(\cup_{m=1}^{\infty}\cap_{n=1}^{\infty}A_{m,\,n})\cap B\neq\emptyset$ holds. Then μ is regular.*

We have the following.

Corollary 21. *Let X be a metric space and $\mathcal{B}(X)$ a σ-field of all Borel subsets of X. Let $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ be a non-additive Borel multi measure on X which is null-null-additive, continuous from above Type (I) and has property* (S). *We assume that $\mathcal{P}_{cl}(Y)$ has property* (EP). *Then μ is regular.*

7. Lusin's theorem

In this section, we shall further generalize well-known Lusin's theorem in classical measure theory to set-valued non-additive measure spaces in the case where the range space is an ordered topological vector space by using the results obtained in Sections 2-3. For the real valued fuzzy measure case, see [9, 10], and the Vector(Riesz space)-valued fuzzy measure case, see [11]. For the monotone set-valued measure case, see [28].

By Theorem 19, we have the following.

Theorem 22. *Let X be a metric space and $\mu:\mathcal{B}(X)\to E$ a non-additive Borel multi measure which is weak-p.g.p and satisfies the weak-Egoroff condition. If f is a Borel measurable real valued function on X, then there exists a sequence of closed set $\{F_n\}$ such that $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\left(X\setminus F_n\right)\ni\theta$ and f is continuous on each F_n.*

By theorem 10, we have the following.

Corollary 23. *Let X be a metric space and $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ a non-additive Borel multi measure on X which is p.g.p and satisfies weak-Egoroff condition. Assume that there exists $B\in\mathcal{B}(X)$ with $\mu(B)=\{\theta\}$ and for μ-regulator $\{A_{m,n}\}$,*

$$(\cup_{m=1}^{\infty}\cap_{n=1}^{\infty}A_{m,\,n})\cap B\neq\emptyset$$

holds. If f is a Borel measurable real valued function on X, then there exists a sequence of closed set $\{F_n\}$ such that $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\,(X\setminus F_n)=\{\theta\}$ *and f is continuous on each F_n.*

Corollary 24. *Let X be a metric space and $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ a non-additive Borel multi measure on X which is p.g.p and satisfies Egoroff condition. If f is a Borel measurable real valued function on X, then there exists a sequence of closed set $\{F_n\}$ such that* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\,(X\setminus F_n)=\{\theta\}$ *and f is continuous on each F_n.*

Theorem 25. *Let X be a metric space and $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ a non-additive Borel measure on X which is weak null-null-additive, continuous from above Type (I) and has property weak* (S). *We assume that $\mathcal{P}_{cl}(Y)$ has property weak* (EP). *If f is a Borel measurable real valued function on X, then there exists a sequence of closed set $\{F_n\}$ such that* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\,(X\setminus F_n)\ni\theta$ *and f is continuous on each F_n.*

We also have the following.

Theorem 26. *Let X be a metric space and $\mu:\mathcal{B}(X)\to\mathcal{P}_{cl}(Y)$ a non-additive Borel measure on X which is null-null-additive, continuous from above Type (I) and has property* (S). *We assume that $\mathcal{P}_{cl}(Y)$ has property* (EP). *If f is a Borel measurable real valued function on X, then there exists a sequence of closed set $\{F_n\}$ such that* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\,(X\setminus F_n)=\{\theta\}$ *and f is continuous on each F_n.*

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