# Twist formulas for one-row colored $A_2$ webs and $\mathfrak{sl}_3$ tails of $(2,2m)$-torus links

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## 1 Introduction

The author made tools to calculate the quantum invariant of knots and links obtained from $\mathfrak{sl}_3$ by using the Kuperberg's linear skein theory. In this article, we will introduce one of these tools, a full twist formula, which is useful to compute explicitly the $\mathfrak{sl}_3$ colored Jones polynomial. As an application, we give explicit formula of the one-row colored $\mathfrak{sl}_3$ tail for a $(2,2m)$-torus link.

The quantum invariants of knots are obtained through a functor from the category of framed oriented tangles to the representation category of a quantum group of a simple Lie algebra. Thus, one can define an invariant $J_V^{\mathfrak{g}}(K)$ of a knot $K$ for each simple Lie algebra $\mathfrak{g}$ and its representation $V$. For example, the colored Jones polynomial $J_n(K) = J_{V_{n+1}}^{\mathfrak{sl}_2}$ is obtained from the $(n+1)$-dimensional irreducible representation $V_2$ of $\mathfrak{sl}_2$ and one can compute it by applying the Kauffman bracket skein relation to a knot diagram colored by the Jones-Wenzl projector. In this case, $J_n(K)$ is explicitly calculated for many knots and there are many useful formulas which decompose a tangle diagram to the linear sum of the web basis by the skein relation. This method to calculate quantum invariants from knot diagrams is called the linear skein theory, see for example [Lic97]. The linear skein theory is constructed for some other $\mathfrak{g}$ than $\mathfrak{sl}_2$. We will treat the $\mathfrak{sl}_3$ colored Jones polynomial $J_{(n,0)}^{\mathfrak{sl}_3}(K)$ which is obtained from the irreducible representation with the highest weight $(n,0)$ of $\mathfrak{sl}_3$. The linear skein theory for $\mathfrak{sl}_3$ with a general irreducible representation $(m,n)$ was constructed by Kuperberg [Kup94, Kup96]. However, there are few non-trivial examples of explicit formulas of $J_{(m,n)}^{\mathfrak{sl}_3}(K)$. For example, Lawrence [Law03] calculated $J_{(m,n)}^{\mathfrak{sl}_3}(K)$ for the trefoil knot, more generally, In [GMV13, GV17] for the $(2,2m+1)$- and $(4,5)$-torus knots by using a representation theoretical method. In [Yua17], the author calculated $J_{(n,0)}^{\mathfrak{sl}_3}(K)$ for the two-bridge links $K$ with one-row coloring $(n,0)$ by the Kuperberg's linear skein theory. The full twist formula for one-row colored anti-parallel $A_2$ webs was used in this calculation, and this full formula plays an important role in the study of $\mathfrak{sl}_3$ tails of knots and links.

The tail of a knot $K$ is a $q$-series which is a limit of the colored Jones polynomials $\{J_n(K;q)\}_n$. Independently, the existence of the tails for alternating knots was shown in [DL06, DL07] and [GL15], more generally, for adequate links in [Arm13]. In [GL15], Garoufalidis and Lê showed a more general stability, the existence of the tail is the zero-stability, for alternating knots. Some explicit descriptions of tails are known for $T(2,2m+$

1) in [AD11], for $T(2,2m)$ in [Haj16], for a pretzel knot $P(2k+1,2,2l+1)$ in [EH17], for knots with small crossing numbers in [KO16], [BO17], and [GL15]. Especially, the tail of $T(2,2m+1)$ is given by the theta series and one of $T(2,2m)$ is the false theta series.

One can consider a tail for the one-row colored $\mathfrak{sl}_3$ colored Jones polynomial $\{J^{\mathfrak{sl}_3}_{(n,0)}(K)\}$. We call it the one-row colored $\mathfrak{sl}_3$ tail of $K$. As a case study, the author gave an explicit formula of the one-row colored $\mathfrak{sl}_3$ tail for a $(2,2m)$-torus link $T_{\leftrightarrows}(2,2m)$ with anti-parallel orientation by using the full twist formula. This $\mathfrak{sl}_3$ tail can be considered a special type of the $\mathfrak{sl}_3$ false theta series. In fact, Bringmann-Kaszian-Milas [BKM19] commented that the $\mathfrak{sl}_3$ tail of $T_{\leftrightarrows}(2,2m)$ coincides with the diagonal part of the $\mathfrak{sl}_3$ false theta series defined through the study of vertex operator algebras in [BM15, BM17, CM14, CM17]. For a parallel $(2,2m)$-torus link $T_{\rightrightarrows}(2,2m)$, the author obtained the one-row colored $\mathfrak{sl}_3$ tail of it in [Yua20].

## 2 Preliminaries

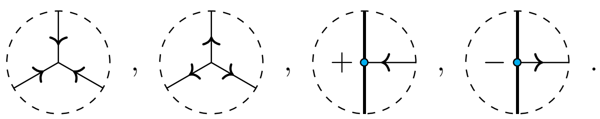
We use the following notation.

- $[n]=\frac{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}{q^{\frac{1}{2}}-q^{\frac{1}{2}}}$ is a *quantum integer* for $n\in\mathbb{Z}_{\geq 0}$.
- $\begin{bmatrix} n \\ k \end{bmatrix}=\frac{[n]}{[k][n-k]}$ for $0\leq k\leq n$ and $\begin{bmatrix} n \\ k \end{bmatrix}=0$ for $k>n$.
- $(q)_n=\prod_{i=1}^{n}(1-q^i)$ is a *q-Pochhammer symbol.*
- $\binom{n}{k}_q=\frac{(q)_n}{(q)_k(q)_{n-k}}$ for $0\leq k\leq n$ and $\binom{n}{k}_q=0$ for $k>n$.
- $\binom{n}{k_1,k_2,\ldots,k_m}_q=\frac{(q)_n}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_m}}$ for positive integers $k_i$'s such that $\sum_{i=1}^{m}k_i=n$.

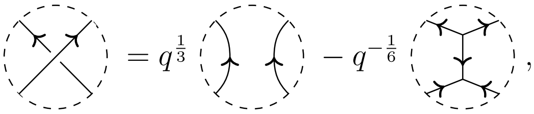
Let us define $A_2$ web spaces based on [Kup96]. We consider a disk $D$ with signed marked points $(P,\epsilon)$ on its boundary where $P\subset\partial D$ is a finite set and $\epsilon\colon P\to\{+,-\}$ a map.

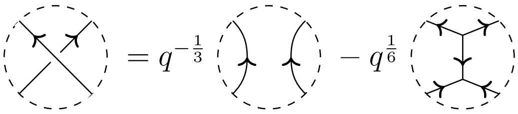
A *tangled bipartite uni-trivalent graph* on $D$ is an immersion of a directed graph into $D$ satisfying (1) – (4):

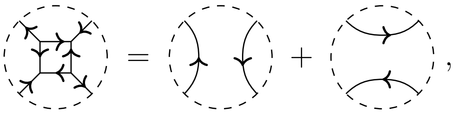
1. the valency of a vertex of underlying graph is 1 or 3,
2. all crossing points are transversal double points of two edges with under/over information,
3. the set of univalent vertices coincides with $P$,
4. a neighborhood of a vertex is one of the followings:

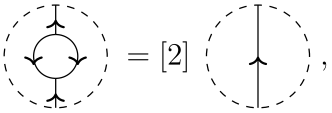


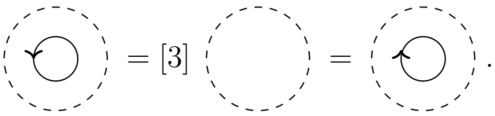
**Definition 2.1** ($A_2$ web space [Kup96])**.** Let $G(\epsilon;D)$ be the set of boundary fixing isotopy classes of tangled trivalent graphs on $D$. The $A_2$ web space $W(\epsilon;D)$ is the quotient of the $\mathbb{C}(q^{\frac{1}{6}})$-vector space on $G(\epsilon;D)$ by the following $A_2$ *skein relation*:

- 


- 

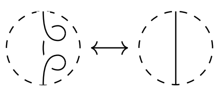
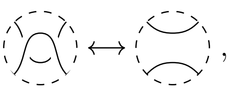

- 

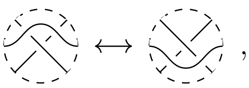
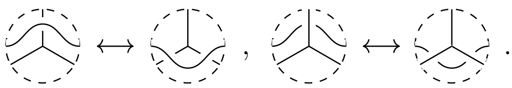

- 


- 


An element in $W(\epsilon; D)$ is called *web* and an element in $G(\epsilon; D)$ without crossings which has no internal 0-, 2-, 4-gons a *basis web*. Any web is described as the sum of basis webs.

The $A_2$ skein relation realize the *Reidemeister moves* (R1) – (R4), that is, we can show that webs represent diagrams in the left side and right side is the same web in $W(D; \epsilon)$.

(R1)  (R2) 

(R3)  (R4) 

We review a diagrammatic definition of an $A_2$ clasp introduced in [Kup96, OY97, Kim07] and its properties. The $A_2$ clasp gives a coloring of strands in a web by pairs of non-negative integers. It plays an important role as is the case with the Jones-Wenzl projector.

We construct a projector called the $A_2$ clasp of type $(n, m)$ in a special web space $\mathsf{TL}^{A_2}(+^n-^m, +^n-^m)$.

**Definition 2.2** (the Temperley-Lieb category for $A_2$)**.** Let $D = [0,1] \times [0,1]$ and $\mathbf{n}$ denotes a set of $n$ points on $I = [0,1]$ dividing it into $n+1$ equal parts. The *Temperley-Lieb category* $\mathsf{TL}^{A_2}$ is a linear category over $\mathbb{C}(q^{\frac{1}{6}})$ is defined as follows:

- an object is a word (finite sequence) over $\{+, -\}$,
- the tensor product of two words is defined by the product (concatenation),
- the space of morphisms $\mathsf{TL}^{A_2}(\alpha, \beta)$ is the web space $W(\bar{\alpha} \sqcup \beta; D)$ where $\bar{\alpha}$ is the word consisting of opposite signs of $\alpha$.

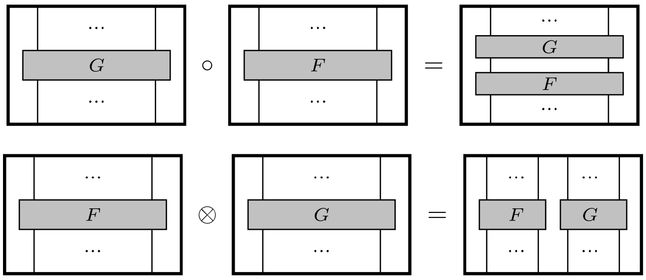


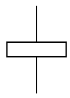

Figure 2.1: the composition and the tensor product in $\mathsf{TL}^{A_2}$

We identify an object $\alpha$ with length $n$ as a map $\mathbf{n} \to \{+,-\}$ by using the order on $\mathbf{n}$. A map $\bar{\alpha} \sqcup \beta$ means that the domain $\mathbf{n} \subset I$ of the map $\bar{\alpha}$ is identified with the top edge $[0,1] \times \{0\}$ and $\beta$ identified with the bottom edge $[0,1] \times \{1\}$.

- The composition $GF \in G(\bar{\alpha} \sqcup \gamma; D)$ of $F \in G(\bar{\alpha} \sqcup \beta; D)$ and $G \in G(\bar{\beta} \sqcup \gamma; D)$ is given by gluing the top edge of $F$ and the bottom edge of $G$.
- The "tensor product" $F \otimes G \in G(\bar{\alpha}_1\bar{\alpha}_2, \beta_1\beta_2; D)$ of $F \in G(\bar{\alpha}_1 \sqcup \beta_1; D)$ and $G \in G(\alpha_2 \sqcup \beta_2; D)$ by gluing the right edge $\{1\} \times [0,1]$ of $F$ and the left edge $\{0\} \times [0,1]$ of $G$.

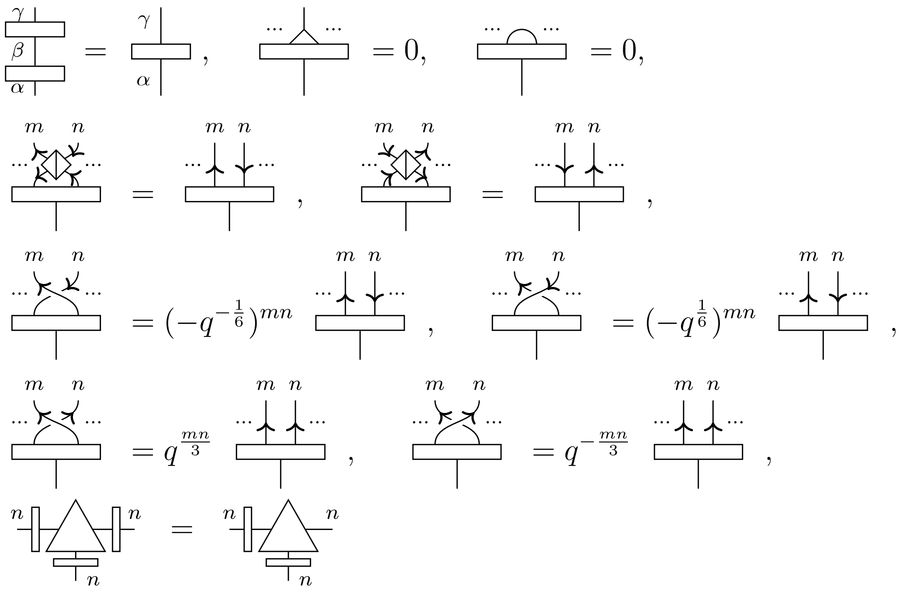
They define the composition and the tensor product on the space of morphisms by linearization. The diagrammatic description of them is in Figure 2.1.

One can define the $A_2$ clasp $JW_{\alpha}^{\beta} \in \mathsf{TL}^{A_2}(\alpha, \beta)$ where $\alpha$ and $\beta$ are obtained by rearranging the order of $+^m-^n$. We describe the $A_2$ clasp by a white box as follows:

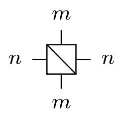


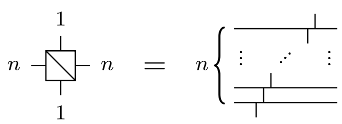
The $A_2$ clasp has the following properties.

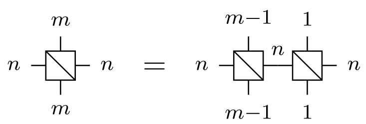
**Lemma 2.3.**



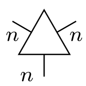
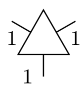
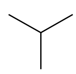

The "stair-step" and "triangle" webs appear in the above are defined by the following. These webs also appear in [Kim06, Kim07, Yua17, FS20].

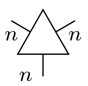
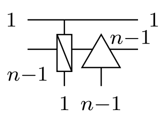
**Definition 2.4.** For positive integers $n$ and $m$,  is defined by

 and

 for $m > 1$.

Specifying a direction on an edge around the box determine the all directions of edges in the box.

**Definition 2.5.** For positive integer $n$,  is defined by  $=$  and

 $=$  for $n > 1$. We obtain a web by specifying a direction of an edge around the triangle.

## 3 Twist formula and $q$-binomial coefficient

In this section, we show new twist formulas derived from a combinatorial method related to the generating function of lattice paths. We firstly explain this method which is used in the proof of full twist formulas in [Yua17].

Let us consider a set of web $\{\sigma_n(k,l) \mid 0 \leq k, l \leq n\}$ such that

1. $\sigma_n(k,l)$ is basis web if $k+l=n$,
2. $\sigma_n(k,l) = X(k,l)\sigma_n(k+1,l) + Y(k,l)\sigma_n(k,l+1)$,

where $X(k,l), Y(k,l) \in \mathbb{C}(q^{\frac{1}{6}})$. The web $\sigma_n(k,l)$ corresponds to a lattice point $(k,l)$. We denote a lattice path from $(0,0)$ to $(k,l)$ by a sequence of $x$ and $y$ where $x$ (resp. $y$) means $(1,0)$-shifting (resp. $(0,1)$-shifting). Let us define a *weight* $\mathrm{wt}(e)$ of edge $e$ from $(k,l)$ to $(k+1,l)$ (resp. $(k,l+1)$ by $X(k,l)$ (resp. $Y(k,l)$). A weight of a path is the product of weights of edges on the path. Then, the coefficient of a basis web $\sigma_n(k,l)$ in the expansion of $\sigma_n(0,0)$ is the summation of weights of all paths from $(0,0)$ to $(k,l)$. If we assume

$$\mathrm{wt}(axyb) = q\,\mathrm{wt}(ayxb)$$

for any sequences $a$ and $b$ of $x,y$. One can describe the coefficient of $\sigma_n(k,l)$ by using the generating function of the number of lattice paths:

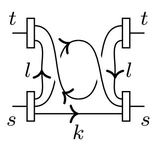
$$\mathrm{wt}(y^{l-1}x^{k-1})\binom{k+l}{k}_q.$$

Thus, one can obtain the following.

**Proposition 3.1.**

$$\sigma_n(0,0) = \sum_{k+l=n}\prod_{j=0}^{l-1} Y(0,j)\prod_{i=0}^{k-1} X(i,l)\binom{n}{k}_q \sigma_n(k,l).$$

In [Yua17], the author obtained an expansion formula of anti-parallel one-row colored $A_2$ webs by setting $n = d = \min\{s,t\}$ and

$$\sigma_n(k,l) =$$ 

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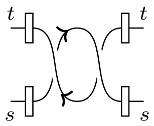
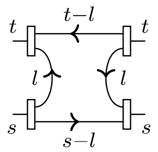
This $A_2$ web satisfies the condition of Proposition 3.1. Thus, one can obtain the following.

**Theorem 3.2** (full twist formula for one-row colored anti-parallel $A_2$ webs [Yua17]). Let

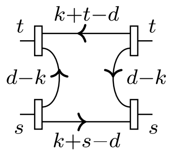
$d = \min\{s,t\}$ and $\delta = |s-t|$.




$$= q^{\frac{st}{3}} \sum_{l=0}^{\infty} q^{l^2-l} q^{-(s+t)l} (q)_l \binom{s}{l}_q \binom{t}{l}_q$$




$$= q^{-\frac{2}{3}d(d+\delta)-d} \sum_{k=0}^{d} q^{k(k+\delta)+k} \frac{(q)_{d+\delta}}{(q)_{k+\delta}} \binom{d}{k}_q$$

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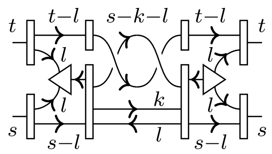
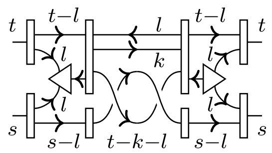
In [Yua20], the author obtained a full twist formula for one-row colored parallel $A_2$ webs by setting $n = d = \min\{s,t\}$ and

$$\sigma_d(k,l) =$$




if $d = s$,

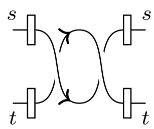
$$\sigma_d(k,l) =$$



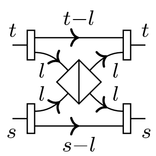

if $d = t$.

These $A_2$ webs also satisfy the condition of Proposition 3.1. Thus, one can obtain the followings.

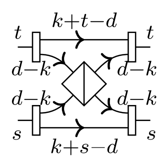
**Theorem 3.3** (full twist formula for one-row colored parallel $A_2$ webs [Yua20])**.** Let $d = \min\{s,t\}$ and $\delta = |s-t|$.




$$= q^{\frac{2}{3}st} \sum_{l=0}^{\infty} q^{l^2-\frac{l}{2}} q^{-(s+t)l} (q)_l \binom{s}{l}_q \binom{t}{l}_q$$




$$= q^{-\frac{d(d+\delta)}{3}-\frac{d}{2}} \sum_{k=0}^{d} q^{k(k+\delta)+\frac{k}{2}} \frac{(q)_{d+\delta}}{(q)_{k+\delta}} \binom{d}{k}_q$$

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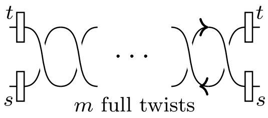
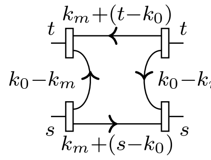
From these formulas, the author obtained an $m$-full twist formula for one-row colored $A_2$ webs.

**Theorem 3.4** (anti-parallel $m$-full twist formula [Yua17])**.** Let $\underline{k} = (k_1, \ldots, k_m)$ be an

$m$-tuple of integers, $k_0 = d = \min\{s,t\}$, and $\delta = |s-t|$.




$$= q^{-\frac{2m}{3}k_0(k_0+\delta)-2mk_0} \sum_{k_0 \geq k_1 \geq \cdots \geq k_m \geq 0} C(\underline{k})$$

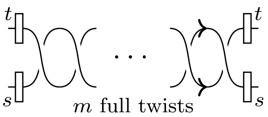



where

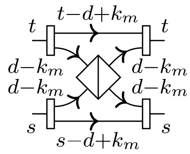
$$C(\underline{k}) = q^{\sum_{i=1}^{m} k_i(k_i+\delta)+2k_i} q^{k_0-k_m} \frac{(q)_{k_0+\delta}}{(q)_{k_m+\delta}} \binom{k_0}{k_1', k_2', \ldots, k_m', k_m}_q$$

and $k_{i+1}' = k_i - k_{i+1}$ for $i = 0, 1, \ldots, m-1$.

**Theorem 3.5** (parallel $m$-full twist formula)**.** Let $\underline{k} = (k_1, \ldots, k_m)$ be an $m$-tuple of integers, $k_0 = d = \min\{s,t\}$, and $\delta = |s-t|$.




$$= q^{-\frac{m}{3}k_0(k_0+\delta)-mk_0} \sum_{k_0 \geq k_1 \geq \cdots \geq k_m \geq 0} D(\underline{k})$$




where

$$D(\underline{k}) = q^{\sum_{i=1}^{m} k_i(k_i+\delta)+k_i} q^{\frac{1}{2}(k_0-k_m)} \frac{(q)_{k_0+\delta}}{(q)_{k_m+\delta}} \binom{d}{k_1', k_2', \ldots, k_m', k_m}_q$$

and $k_{i+1}' = k_i - k_{i+1}$ for $i = 0, 1, \ldots, m-1$.

# 4 The tail of one-row colored $\mathfrak{sl}_3$ Jones polynomial of a $(2,2m)$-torus link

We give a definition of the one-row colored $\mathfrak{sl}_3$ Jones polynomial of an oriented framed links via the $A_2$ web.

**Definition 4.1.** Let $L$ be a link diagram of a framed link whose framing is given by the blackboard framing. The *one-row colored* $\mathfrak{sl}_3$ *Jones polynomial of* $n$ *boxes* $J^{\mathfrak{sl}_3}_{L,n}(q)$ is defined by

$$\langle L^{(n)} \rangle = J^{\mathfrak{sl}_3}_{L,n}(q)\emptyset \in W(\emptyset; D) \cong \mathbb{Q}(q^{\pm\frac{1}{6}}).$$

$\langle L^{(n)} \rangle$ is the clasped $A_2$ web obtained by attaching white boxes corresponding to $A_2$ clasps to each link component of $L$. We also introduce a normalization $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$ obtained by $\pm q^{-\deg\langle L^{(n)}\rangle}$ such that constant term is positive. $\deg\langle L^{(n)} \rangle$ is the minimum degree of $\langle L^{(n)} \rangle$

**Remark 4.2.**
- In [Le00], Lê showed the integrality of the coefficients of a quantum $\mathfrak{g}$ invariant of links. It says that $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$ belongs to $\mathbb{Z}[q]$.
- $J^{\mathfrak{sl}_3}_{L,n}(q)$ is $J^{\mathfrak{sl}_3}_{(n,0)}(L)$ in section 1.

**Definition 4.3** (the one-row colored $\mathfrak{sl}_3$ tail of a link)**.** The one-row colored $\mathfrak{sl}_3$ colored Jones polynomial $\{\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)\}_n$ of a link $L$ is *zero stable* if there exists a formal power series $\mathcal{T}^{\mathfrak{sl}_3}_L(q)$ in $\mathbb{Z}[[q]]$ such that

$$\mathcal{T}^{\mathfrak{sl}_3}_L(q) - \hat{J}^{\mathfrak{sl}_3}_{L,n}(q) \in q^{n+1}\mathbb{Z}[[q]]$$

for all $n$. Then, we call $\mathcal{T}^{\mathfrak{sl}_3}_L(q)$ a *tail* of $\{\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)\}_n$ or the *one-row colored* $\mathfrak{sl}_3$ *tail* of $L$.

A $(2,2m)$-torus link is obtained by taking a closure of the $m$ full twist two strands. Thus, one can obtain the explicit formula of $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$ for the $(2,2m)$-torus link $L = T_{\leftrightarrows}(2,2m)$ and $T_{\rightrightarrows}(2,2m)$.

**Theorem 4.4.**

$$\begin{aligned}\hat{J}^{\mathfrak{sl}_3}_{T_{\leftrightarrows}(2,2m),n}(q) = &\sum_{n\geq k_1\geq\cdots\geq k_m\geq 0} q^{\sum_{i=1}^m {k_i}^2+2k_i} q^{-2k_m} \frac{(q)_n}{(q)_{k_m}} \frac{(q)_n}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}} \\ &\times \frac{(1-q)(1-q^2)}{(1-q^{n-k_m+1})(1-q^{n-k_m+2})}\Delta(n,0)^2\end{aligned}$$

$$\begin{aligned}\hat{J}^{\mathfrak{sl}_3}_{T_{\rightrightarrows}(2,2m),n}(q) = &\sum_{n\geq k_1\geq\cdots\geq k_m\geq 0} q^{\sum_{i=1}^m {k_i}^2+k_i} q^{-k_m} \frac{(q)_n}{(q)_{k_m}} \frac{(q)_n}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}} \\ &\times \frac{1-q^2}{1-q^{d-k_m+1}}\Delta(n,0)^2\end{aligned}$$

By the above explicit formulas, one can obtain explicit formulas of one-row colored $\mathfrak{sl}_3$ tails of $T_{\leftrightarrows}(2,2m)$ and $T_{\rightrightarrows}(2,2m)$.

**Theorem 4.5** (An explicit formula for the one-row colored $\mathfrak{sl}_3$ tail of a $(2,2m)$-torus link [Yua18, Yua20])**.**

$$\mathcal{T}^{\mathfrak{sl}_3}_{T_{\leftrightarrows}(2,2m)}(q) = \frac{(q)_\infty}{(1-q)^2(1-q^2)} \sum_{k_1\geq k_2\geq\cdots\geq k_m\geq 0} \frac{q^{-k_m} q^{\sum_{i=1}^m k_i^2+k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3}\cdots(q)_{k_{m-1}-k_m}(q)^2_{k_m}}$$

$$\mathcal{T}^{\mathfrak{sl}_3}_{T_{\rightrightarrows}(2,2m)}(q) = \frac{(q)_\infty}{(1-q)(1-q^2)} \sum_{k_1\geq k_2\geq\cdots\geq k_m\geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^m k_i^2+2k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3}\cdots(q)_{k_{m-1}-k_m}(q)^2_{k_m}}$$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