QUESTION AND HOMOMORPHISMS ON ARCHIPELAGO GROUPS

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Abstract. The classical archipelago group is a quotient group of the fundamental group of the Hawaiian earring by the normal closure of the free group of countable rank, which is denoted by $\mathcal{A}(\mathbb{Z})$. Since the fundamental group of the Hawaiian earring is expressed by the free σ-product $\circledast_\omega \mathbb{Z}$, we obtain an archipelago group $\mathcal{A}(G)$ by replacing $\mathbb{Z}$ with G. In [1] the authors asserted that $\mathcal{A}(\mathbb{Z})$ and $\mathcal{A}(\mathbb{Z}/k\mathbb{Z})$ are isomorphic for $k \geq 3$. We clarify a gap in their proof and show that there are surjective homomorphisms between $\mathcal{A}(\mathbb{Z}/k\mathbb{Z})$'s and $\mathcal{A}(\mathbb{Z})$ for $k \geq 2$.

Finally we state our conjecture and some direction showing the conjecture.

1. Introduction and definitions

The main purpose of this note is to state the main question about archipelago groups and to investigate the homomorphisms defined in [1]. We also point out a gap in their proof of the main result in [1] by showing a certain property of the homomorphisms and also state a conjecture. For future developments, we define many things again and somewhat differently from [1]. Archipelago groups are the fundamental groups of so-called archipelagos, which are objects in wild algebraic topology. The reader is refered to [1] for the background.

We intend explicit presentations, but words are also used to express elements of free σ-products. For basical notions we refer to [2]. First we define archipelago groups. Let G_i $(i < \omega)$ be groups. Define $\mathcal{A}(G_i : i < \omega)$ to be the quotient group of the free σ-product $\circledast_{i<\omega} G_i$ factored by $N(*_{i<\omega} G_i)$, which is the normal closure of the free product $*_{i<\omega} G_i$. We simply write $\mathcal{A}(G)$ for $\mathcal{A}(G_i : i < \omega)$ in case $G_i = G$.

Let $\sigma_G : \circledast_{i<\omega} G_i \to \circledast_{i<\omega} G_i / N(*_{i<\omega} G_i)$ and $\sigma_H : \circledast_{i<\omega} H_i \to \circledast_{i<\omega} H_i / N(*_{i<\omega} H_i)$ be the quotient homomorphisms.

Next we introduce interesting homomorphisms in [1]. Let $\varphi_i : G_i \to H_i$ for $i < \omega$ be maps which preserve the inverses, i.e. $\varphi_i(x^{-1}) =$

2010 *Mathematics Subject Classification.* Primary 20E06, 55Q20 Secondary:20F05.

$\varphi_i(x)^{-1}$. We define $\varphi : \mathcal{W}(G_i : i < \omega) \to \mathcal{W}(H_i : i < \omega)$ by: $\overline{\varphi(W)} = \{\alpha \in \overline{W} \mid \varphi_i(W(\alpha) \neq e \text{ where } W(\alpha) \in G_i\}$ and

$$\varphi(W)(\alpha) = \varphi_i(W(\alpha)), \text{ if } W(\alpha) \in G_i.$$

Then, we define $\overline{\varphi} : \circledast_{i<\omega} G_i \to \circledast_{i<\omega} H_i$ by $\overline{\varphi}(W) = \varphi(W)$ for reduced words W. Since W is restricted to reduced words, $\overline{\varphi}$ is well-defined.

Finally we define $\overline{\overline{\varphi}} : \mathcal{A}(G_i : i < \omega) \to \mathcal{A}(H_i : i < \omega)$ by: $\overline{\overline{\varphi}} \circ \sigma_G = \sigma_H \circ \overline{\varphi}$, where the well-defined-ness is assured by the fundamental homomorphism theorem.

2. Results and proofs

A main part of the following theorem is contained in [1].

Theorem 2.1. [1] *Let φ_i be an inverse preserving map for each $i < \omega$. Then, $\overline{\overline{\varphi}}$ is a homomorphism and the non-triviality of $\overline{\overline{\varphi}}$ is equivalent to the existence of infinitely many i for which there exists an $x \in G_i$ such that $x \neq e$ and $\varphi_i(x) \neq e$.*

Proof. First we show that $\sigma_H \circ \overline{\varphi}$ is a homomorphism. Let $U, V \in \mathcal{W}(G_i : i < \omega)$ be reduced words and $W \in \mathcal{W}(G_i : i < \omega)$ be the reduced word such that $W = UV$. Then, there exists a reduced word W_0 such that

(1) $U \equiv U_0 W_0$, $V \equiv W_0^- V_0$ and $U_0 V_0$ is reduced; or
(2) $U \equiv U_0 a W_0$, $V \equiv W_0^- b V_0$ for some $a, b \in G_i$ satisfying $ab \neq e$ and $U_0(ab)V_0$ is reduced.

Therefore $W \equiv U_0 V_0$ or $W \equiv U_0(ab)V_0$ and hence $\overline{\varphi}(W) = \varphi(U_0)\varphi(V_0)$ or $\overline{\varphi}(W) = \varphi(U_0)\varphi_i(ab)\varphi(V_0)$.

Since $\varphi(W_0^-) \equiv \varphi(W_0)^-$ by preservation of the inverses,

$$\overline{\varphi}(U)\overline{\varphi}(V) = \varphi(U_0)\varphi(W_0)\varphi(W_0^-)\varphi(V_0) = \varphi(U_0)\varphi(V_0)$$

or

$$\begin{aligned} \overline{\varphi}(U)\overline{\varphi}(V) &= \varphi(U_0)\varphi_i(a)\varphi(W_0))\varphi(W_0^-)\varphi_i(b)\varphi(V_0) \\ &= \varphi(U_0)\varphi_i(a)\varphi_i(b)\varphi(V_0) \end{aligned}$$

Now, in the both bases we have

$$\sigma_H(\overline{\varphi}(U)\overline{\varphi}(V)) = \sigma_H(\varphi(U_0)\varphi(V_0)) = \sigma_H(\varphi(W))$$

and we have shown $\sigma_H \circ \overline{\varphi}$ is a homomorphism.

If there exist $x_i \in G_i$ for infinitely many i such that $x_i \neq e$ and $\varphi_i(x_i) \neq e$, the non-triviality of the map follows from considering a word obtained by ordering x_i in a natural way. Since a reduced word consists of nontrivial elements of groups G_i, the negation of the condition implies that $\varphi(W) \in *_{i<\omega} H_i$ for any reduced word $W \in \mathcal{W}(G_i : i < \omega)$, which implies $\overline{\varphi}(W) = e$. $\square$

Since $\sigma_H \circ \overline{\varphi}(*_{i<\omega} G_i) = \{e\}$, we have a homomorphism $\overline{\overline{\varphi}} : \circledast_{i<\omega} G_i / N(*_{i<\omega} G_i) \to \circledast_{i<\omega} H_i / N(*_{i<\omega} H_i)$ such that $\sigma_H \circ \overline{\varphi} = \overline{\overline{\varphi}} \circ \sigma_G$.

An element of $\circledast_{i<\omega} G_i / N(*_{i<\omega} G_i)$ is expressed as $\sigma_G(W)$ for a word $W \in \mathcal{W}(G_i : i < \omega)$. In particular we may restrict W to be a reduced one.

Lemma 2.2. *A word W is reduced, if $W \,|\, (\alpha, \beta) \neq e$ for each pair $\alpha < \beta \in \overline{W}$ satisfying that $W(\alpha), W(\beta) \in G_{i_0}$ and no letter in G_{i_0} appears in $W \,|\, (\alpha, \beta)$ for some i_0.*

Proof. Observe that $\circledast_{i<\omega} G_i \cong G_{i_0} * \circledast_{i \neq i_0} G_i$, we see every occurrence of a letter in W remains in the reduced word of W. $\square$

Lemma 2.3. *If $h : G \to H$ is an inverse-preserving surjective map which is not a homomorphism, then*

1. *there exist $a, b, c \in G$ which are not the identity such that $abc \neq e$ and $h(a)h(b)h(c) = e$; or*
2. *there exist $a, b \in G$ which are not the identity such that $ab \neq e$ and $h(a)h(b) = e$.*

Proof. In case $h(e) \neq e$, we have $a \in G$ such that $a \neq e$ and $h(a) = e$. Since $h(a^{-1}) = e^{-1} = e$, we have $a^2 = e$. Setting $b = c = a$ are desired ones for (1).

Otherwise, i.e. $h(e) = e$. Then, $h(uv) \neq h(u)h(v)$ implies $u \neq e$ and $v \neq e$ and also $uv \neq e$. Choose w so that $h(w) = h(u)h(v)$. If $w \neq e$, $a = u, b = v, c = w^{-1}$ are desired ones for (1). Otherwise, i.e. $w = e$, $a = u$ and $b = v$ are desired ones for (2). $\square$

To define domains of words, we introduce some notions. The empty sequence is denoted by $(\,)$ and let $n = \{0, \cdots, n-1\}$ for $n < \omega$. A finite sequence is denoted by $(i_0, \cdots, i_k)$ whose length is $k+1$. For a finite sequence $s = (i_0, \cdots, i_{k-1})$, let $s * (j) = (i_0, \cdots, i_{k-1}, j)$.

Theorem 2.4. *Suppose that $\varphi_i : G_i \to H_i$ is an inverse preserving surjective map for every $i < \omega$. If there exist infinitely many i such that φ_i are not homomorphisms, then $\overline{\overline{\varphi}}$ is never injective.*

Proof. Let J be the subset of ω consisting of all i such that φ_i are not homomorphisms. Enumerate J increasingly, i.e. $\{j_k \,|\, k < \omega\} = J$ and $j_k < j_{k+1}$.

Let $a_{j_k}, b_{j_k} \in G_{j_k}$ or $a_{j_k}, b_{j_k}, c_{j_k} \in G_{j_k}$ which satisfy the required properties (2) or (1) in Lemma 2.3 respectively. We define $\overline{W_\alpha} \subseteq Seq(3)$

inductively as the domain of W which is a tree with lexicographical ordering.

In the 0-step, if (2) in Lemma 2.3 holds for φ_{j_0}, then define $W((0)) = a_{j_0}, W((1)) = b_{j_0}$, and otherwise, define $W((0)) = a_{j_0}, W((1)) = b_{j_0}, W((2)) = c_{j_0}$.

Suppose that $W(s)$ is defined. Let $m = lh(s)$. As in the 0-step, if (2) in Lemma 2.3 holds for φ_{j_m}, then define $W(s*(0)) = a_{j_m}, W((s*(1)) = b_{j_m}$, and otherwise, define $W(s*(0)) = a_{j_m}, W(s*(1)) = b_{j_m}, W(s*(2)) = c_{j_m}$.

We can see that W is reduced and $\varphi(W) = e$ as follows. Since for each pair of letters indexed j_k appearing in W there appear $a_{j_{k+1}}, b_{j_{k+1}}$ between them and $a_{j_{k+1}} b_{j_{k+1}} = e$, or $a_{j_{k+1}}, b_{j_{k+1}}, c_{j_{k+1}}$ between them and $a_{j_{k+1}} b_{j_{k+1}} c_{j_{k+1}} = e$. Hence non-empty subwords of W is not equal to e. On the other hand, for every finite subset F of ω consider the projection to $*_{i\in F} H_i$ and letters indexed by the largest element j_k in F. We see $\varphi_{j_k}(a_{j_k}), \varphi_{j_k}(b_{j_k})$ or $\varphi_{j_k}(a_{j_k}), \varphi_{j_k}(b_{j_k}), \varphi_{j_k}(c_{j_k})$ appear contiguously. Since $\varphi_{j_k}(a_{j_k})\varphi_{j_k}(b_{j_k}) = e$, or $\varphi_{j_k}(a_{j_k})\varphi_{j_k}(b_{j_k})\varphi_{j_k}(c_{j_k}) = e$, we can cancel them and so on and we conclude the projectum is equal to e, which implies $\varphi(W) = e$.

Since W is a reduced word and there appear infinitely many letters, $\sigma_G(W)$ is not the identity. Since $\overline{\varphi}(W) = \varphi(W)$, $\overline{\overline{\varphi}}(\sigma_G(W)) = \sigma_H(\varphi(W)) = e$. We have shown that $\overline{\overline{\varphi}}$ is not injective.

□

Lemma 2.5. *Suppose that $\varphi_i : G_i \to H_i$ are surjective homomorphisms. Let $V \in \mathcal{W}(H_i : i < \omega)$ be a reduced word. Then, there exists a reduced word $U \in \mathcal{W}(G_i : i < \omega)$ such that $\varphi(U) \equiv V$.*

Proof. By the surjectivity of φ_i, we have $U \in \mathcal{W}(G_i : i < \omega)$ such that $\overline{U} = \overline{V}$ and $\varphi_i(U(\alpha)) = V(\alpha)$ for each $\alpha \in \overline{V}$, where $V(\alpha) \in H_i$. To show that U is reduced by contradiction, suppose that there exists a non-empty subword W of U such that $W = e$. For any $F \Subset \omega$, $W_F = e$ where W_F is a finite word such that $\overline{W_F} = \{\alpha \in \overline{W} \mid W(\alpha) \in \bigcup_{i\in F} G_i \setminus \{e\}\}$. Since φ_i is a homomorphism for each i, $\varphi(W)_F = e$, which implies V is not reduced. Now, we see that U is reduced. □

Theorem 2.6. *Suppose that $\varphi_i : G_i \to H_i$ is an inverse preserving surjective map for every $i < \omega$. Then $\overline{\overline{\varphi}}$ is surjective.*

Proof. If almost all φ_i are homomorphisms, by ignoring finitely many G_i and H_i we may assume that all φ_i are homomorphisms. Then, $\overline{\varphi}$ is surjective by Lemma 2.5. So we deal with the case that infinitely many φ_i are not homomorphisms.

For a given reduced word V, we consider $\varphi^{-1}(V)$. We cannot say it is a reduced word in $\mathcal{W}(G_i : i \in I)$ and even $\varphi^{-1}(V) \in \mathcal{W}(G_i : i \in I)$, since there may appear e in this sequence. When $V(\alpha) \in H_i$ and $\varphi_i(e) = V(\alpha)$, we replace e by letters u_i, v_i such that $u_i, v_i \neq e$ and $\varphi_(u_i)\varphi_i(v_i) = V(\alpha)$. This is done by the additional condition. Let U be the obtained one. Since such α appear only finitely many times for each i, $U \in \mathcal{W}(G_i : i \in I)$ and $\varphi(U) = V$. We claim the existence of a reduced word $U_0 \in \mathcal{W}(G_i : i \in I)$ such that $\varphi(U_0) = \varphi(U)$. Since $\varphi(U) = V$, we have $\overline{\varphi}(U_0) = V$ and hence $\overline{\overline{\varphi}}(\sigma_G(U_0)) = \sigma_H(V)$.

Actually we show the following:

> Suppose that $\varphi(U) = V$ for $U \in \mathcal{W}(G_i : i < \omega)$ and $V \in \mathcal{W}(H_i : i < \omega)$. Then, there exists a reduced word $U_0 \in \mathcal{W}(G_i : i < \omega)$ such that $\varphi(U_0) = V$.

We keep Lemma 2.2 in our mind and inserting reduced words W satisfying $\varphi(W) = e$ to U. We will define $W_\alpha \in \mathcal{W}(G_n : n \in J\}$ for each $\alpha \in \overline{U}$ such that $\varphi(W_\alpha) = e$. To state our proof rigorously we introduce some notions. Recall $3 = \{0, 1, 2\}$ and $5 = \{0, 1, 2, 3, 4\}$. We construct trees consisting of finite sequence of members of 5 whose lengths are nonzero. Enumerate $J \backslash \{0\}$ increasingly, i.e. $\{j_k \,|\, k < \omega\} = J \backslash \{0\}$ and $j_k < j_{k+1}$. Let $a_{j_k}, b_{j_k}, c_{j_k} \in G_{j_k}$ which satisfy the required propertied assured by Lemma 2.3.

In the first step, i.e. the 0-th step, we consider $\alpha, \beta \in \overline{U}$ such that $U(\alpha), U(\beta) \in G_0$ and $\alpha < \beta$ are contiguous, i.e. $\alpha < \gamma < \beta$ implies $U(\gamma) \notin G_0$. We admit $\beta = \infty$. We construct $W_\alpha \in \mathcal{W}(G_j : j \in J)$ similarly to W in (2), using $a, b, c \in G_j$ satisfying $abc \neq e$ and $\varphi_j(a)\varphi_j(b)\varphi_j(c) = e$. We define $\overline{W_\alpha}$ as a tree with lexicographical ordering. In the 0-substep, let u be the result of multiplications of elements of G_{j_0} appearing in the subword $U(\alpha, \beta)$ of U. We define $W_\alpha((0)) = a, W_\alpha((1)) = b, W_\alpha((2)) = c$, if $abcu \neq e$ and also $W_\alpha((3)) = a, W_\alpha((4)) = b, W_\alpha((5)) = c$ if $abcu = e$. We move β to the place of the leftmost appearance of a letter of G_{j_0} in U, if such a letter appears, and make β stay at the previous β otherwise.

Generally in the k-th substep, we let u to be the result of multiplications of letters of G appearing in $U|(\alpha, \beta)$ and define $W_\alpha(s * (0)) = a_{j_k}, W_\alpha(s * (1)) = b_{j_k}, W_\alpha(s * (2)) = c_{j_k}$ for s satisfying $lh(s) = k$. In addition if $a_{j_k}b_{j_k}c_{j_k}u = e$, we define $W_\alpha(s * (3)) = a_{j_k}, W_\alpha(s * (4)) = b_{j_k}, W_\alpha(s * (5)) = c_{j_k}$ for s which is the largest element in $\overline{W}_\alpha$ satisfying $lh(s) = k$. Then, we move β to the position of the leftmost appearance among letters whose multiplication is u in $\overline{U}$. If u does not exist, then we make β stay at the previous position. In this way we define W_α. If no letters of G_0 appear in U, we do not define anything.

Now in the m-step we consider the word obtained Y deleting all letters which do not belong to $\bigcup_{i=0}^{m} G_i$ from U, i.e. picking letters in $\bigcup_{i=0}^{m} G_i$ and order in the same way as in U. We define W_α for α satisfying $U(\alpha) \in G_m$ by letting $\beta \in \overline{U}$ to correspond to the next letter in the word in $\mathcal{W}(\bigcup_{i=0}^{m} G_i)$. We replace j_0 by j_m and j_k by j_{m+k}.

Our attaching W_α are done after the whole construction. Let $\overline{U_0} = \{(\alpha, s) \mid \alpha \in \overline{U}, s \in \overline{W_\alpha} \text{ or } s = \langle\,\rangle\}$ with the lexicographical ordering and $U_0(\alpha, \langle\,\rangle) = U(\alpha)$ and $U_0(\alpha, s) = W_\alpha(s)$ for $s \in \overline{W_\alpha}$.

The fact that $\varphi(U_0) = V$ follows from $\varphi(W_\alpha) = e$. To see that U_0 is reduced, let Y be a non-empty subword of U_0. Choose m be the least natural number such that a letter of G_m appears in Y. If there is only one letter of G_m which appears in Y, it implies $Y \neq e$. Let $\lambda, \mu \in \overline{Y}$ such that $\lambda < \mu$ and $Y(\lambda), Y(\mu)$ are contigous letters in G_m, i.e. $Y(\lambda), Y(\mu) \in G_m$ and $X(\nu) \notin G_m$ for $\lambda < \nu < \mu$.

(1) If the both appear as of form $U_0(\gamma, \langle\,\rangle)$ for some γ, then $Y(\lambda)$ and $Y(\mu)$ are considered in the m-th step. We remark that no letters of $\bigcup_{i=0}^{m-1} G_i$ appear in Y. According to considering letters in G_{j_m} in the substep 0 for W_α we conclude $Y|(\lambda, \mu) \neq e$.

(2) If $Y(\lambda)$ appears as of form $U_0(\gamma, s)$ for some γ and $s \in \overline{W_\gamma}$ and $Y(\mu)$ appears as of form $U_0(\delta, \langle\,\rangle)$ for some δ. We need to consider the remaining three cases where $Y(\lambda)$ appears as $U_0(\gamma, s)$ for some γ and $s \in W_\gamma$ and $Y(\mu)$ appears as $U_0(\delta, \langle\,\rangle)$ for some δ. There exists $k < m$ such that $U(\gamma) \in G_k$. By the minimality of m, no letter in $\bigcup_{i=0}^{m-1} G_i$ appears in Y. Hence β in the initial stage of the construction of W_γ is located to the right hand side of $Y(\mu)$. Therefore, $m = j_{k+l}$ and β in the substep l for γ is $\mu \in \overline{Y}$ and by the setting for elements of $G_{j_{k+l+1}}$ we conclude $Y(\lambda, \mu) \neq e$.

(3) If $Y(\lambda)$ appears as of form $U_0(\gamma, \langle\,\rangle)$ for some γ and $Y(\mu)$ appears as of form $U_0(\delta, s)$ for some δ and $s \in \overline{W_\delta}$. There exists $k < m$ such that $U(\delta) \in G_k$. By the minimality of m, δ is located at the left hand side of α, i.e. $\delta < \alpha$ in $\overline{U}$. Since no letters in U appear between $U_0(\delta, \langle\,\rangle)$ and $U_0(\delta, s)$, a contradiction occurs, i.e. this case does not happen.

(4) If $Y(\lambda)$ appears as of form $U_0(\gamma, s)$ for some γ and $s \in \overline{W_\gamma}$. and $Y(\mu)$ appears as of form $U_0(\delta, t)$ for some δ and $t \in \overline{W_\delta}$. By the minimality of m we have $\gamma = \delta$. Since W_γ is a reduced word $Y \mid (\alpha, \beta) \neq e$.

Now we have shown that Y is reduced. □

Corollary 2.7. *Let G_i and H_i be at most countable non-trivial groups. Then, there exists a surjective homomorphism from $\mathcal{A}(G_i : i < \omega)$ to $\mathcal{A}(H_i : i < \omega)$.*

Proof. Since $G * G'$ is inifinite for non-trivial groups G and G' and $\circledast_{i<\omega}(G_{2i} * G_{2i+1}) \cong \circledast_{i<\omega} G_i$, we may assume that G_i and H_i are infinite. Therefore we have an inverse-preserving surjective map from G_i to H_i for each i and hence have the conclusion by Theorem 2.6. □

Now we have the following corollary.

Corollary 2.8. *Let G and H be groups $\mathbb{Z}$ and $\mathbb{Z}/k\mathbb{Z}$ for some $k \geq 2$. Then, there are surjections from $\mathcal{A}(G)$ to $\mathcal{A}(H)$ and from $\mathcal{A}(H)$ to $\mathcal{A}(G)$.*

Remark 2.9. (1) G. Conner informed me that the surjectivity of homomorphisms in the assumption of Theorem 2.4 is essential.

(2) If there are surjections between finite groups G and H, then G and H are obviously isomorphic. There are many infinite groups for which the statement does not hold. The author debts to M. Dugas, L. Fuchs and D. Herden for this.

3. Conjecture

First impression to this question should be negative. Here we first explain the reason. For short expressions, let $C_p = \mathbb{Z}/p\mathbb{Z}$. There are natural surjections from $\circledast_\omega \mathbb{Z}$ to $\mathbb{Z}^\omega$ and $\circledast_\omega C_p$ to C_p^ω respectively. These induce surjections from $\mathcal{A}(\mathbb{Z})$ to $\mathbb{Z}^\omega / \oplus_\omega \mathbb{Z}$ and from $\mathcal{A}(C_p)$ to $C_p^\omega / \oplus_\omega C_p$ respectively. Though $\mathbb{Z}^\omega / \oplus_\omega \mathbb{Z}$ is a torsionfree group, $C_p^\omega / \oplus_\omega C_p$ is a torsion group. These themselves do not imply the non-isomorphicness of $\mathcal{A}(\mathbb{Z})$ and $\mathcal{A}(C_p)$, but we extract a conjecture: $\mathcal{A}(\mathbb{Z})$ is not isomorphic to any $\mathcal{A}(C_p)$. Let $\sigma : \circledast_\omega \mathbb{Z}$ to $\mathbb{Z}^\omega$ be the natural surjection. From the preceding argument, we have a surjective homomorphism $h : \circledast_\omega \mathbb{Z} \to \mathcal{A}(C_p)$ such that $\mathcal{A}(C_p)/h(\mathrm{Ker}(\sigma)) \cong C_p^\omega / \oplus_\omega C_p$. We conjecture the non-existence of a surjective homomorphism $h : \circledast_\omega \mathbb{Z} \to \mathcal{A}(\mathbb{Z})$ such that $\mathcal{A}(\mathbb{Z})/h(\mathrm{Ker}(\sigma))$ is a torsion group.

References

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