# On the one-peak stationary solutions for the Schnakenberg model with heterogeneity

東京都立大学理学研究科　石井　裕太
Yuta Ishii
Department of Mathematical Sciences,
Tokyo Metropolitan University

## 1 Introduction

In this paper, based on a recent work [4], we present our study on the existence and linear stability of one-peak stationary solutions for the following Schnakenberg model with heterogeneity:

$$\begin{cases} u_t - \varepsilon^2 u_{xx} = -u + g(x)u^2 v, & x \in (-1,1),\ t > 0, \\ \varepsilon v_t - D v_{xx} = \frac{1}{2} - \frac{c}{\varepsilon} g(x) u^2 v, & x \in (-1,1),\ t > 0, \\ u_x(\pm 1, t) = v_x(\pm 1, t) = 0, \end{cases} \tag{1}$$

where $c$ is a positive constant and $g(x)$ is a positive function on the interval $(-1,1)$. Moreover, $u(x,t)$ and $v(x,t)$ represent the density of two chemical substance at $t \geq 0$ and $x \in (-1,1)$, and $\varepsilon^2 > 0$ and $D > 0$ are diffusion constants of $u$ and $v$, respectively. This model, which describes an autocatalytic chemical reaction, was proposed by Schnakenberg [6], and is well-known as a model in pattern formation. $g(x)$ represent the reaction speed of the chemical reaction at $x \in (-1,1)$ and may vary on the location $x$, for example by the effect of temperature. Here, we note that the standard Schnakenberg model [6] is the case $g(x) = 1$.

There are huge works on the study of the Schnakenberg model (see e.g. [8] and the reference therein.) Since we are interested in the study of spiky solutions of (1), we first mention the work of Iron, Wei, and Winter [2] which studied the non-heterogeneity case, i.e., $g(x) = 1$. They gave the results of the existence and stability of multi-peak symmetric solutions in details. In particular, it was shown that a one-peak solution, which concentrate at $x = 0$, is stable for any $D < +\infty$. The model, which has a heterogeneity term, was studied in [5, 3, 4, 1]. We also mention the related work [7] on $N$-spike cluster solutions for the one-dimensional Gierer-Meinhardt system with heterogeneity.

## 2 Main results

We need several preliminaries to explain our main results in details. First, let $w$ be the unique solution of the following problem:

$$\begin{cases} w'' - w + w^2 = 0, \quad y \in \mathbb{R}, \\ w > 0,\ w(0) = \max_{\mathbb{R}} w,\ \lim_{|y|\to\infty} w(y) = 0. \end{cases} \tag{2}$$

For the unique solution $w(y)$ above, the following facts is known:

$$w(y) = \frac{3}{2}\Big(\cosh\frac{y}{2}\Big)^{-2}, \qquad \int_{\mathbb{R}} w^2 dy = 6.$$

Let $\chi$ be a cut-off function satisfying the following properties:

$$\chi \in C_0^\infty(\mathbb{R}),\ 0 \le \chi \le 1,\ \chi(x) = 1\ \Big(|x| < \frac{1}{4}\Big),\ \chi(x) = 0\ \Big(|x| > \frac{1}{2}\Big). \tag{3}$$

Next, we introduce the following function spaces:

$$H_N^2(-a,a) := \big\{u \in H^2(-a,a)\ \big|\ u'(\pm a) = 0\big\}, \quad a > 0. \tag{4}$$

Let $I := (-1,1)$ and $I_\varepsilon := (-\varepsilon^{-1}, \varepsilon^{-1})$ for $\varepsilon > 0$. For a function $u : I \to \mathbb{R}$, we define the following rescaling notation: $\overline{u}(y) := u(\varepsilon y)$ for $\in I_\varepsilon$.

Let us explain our main results. For $t \in (-1,1)$, we define the notations $F(t)$ and $\xi(t)$ as follows:

$$F(t) := \frac{t^2}{24cD} + \frac{1}{g(t)}, \quad \frac{6c}{g(t)\xi(t)} = 1. \tag{5}$$

For the existence, we assume the following condition:
**(A)**: Assume that $g \in C^3(I)$ and $g(x) > 0$. Moreover, there exists a point $t_0 \in I$ such that

$$F'(t_0) = 0, \quad F''(t_0) \neq 0. \tag{6}$$

We state the main result on the existence of a one-peak solution.

**Theorem 1** *Assume the assumption **(A)**. Then, for $\varepsilon > 0$ sufficiently small, (1) admits a one-peak stationary solution $(u_\varepsilon(x), v_\varepsilon(x)) \in H_N^2(I) \times H_N^2(I)$ which satisfies the following:1*

*(1) $u_\varepsilon(x)$ concentrates at some point $x = t_\varepsilon \in B(\varepsilon^{3/4}, t_0) := \{t \in I \mid |t - t_0| \le \varepsilon^{3/4}\}$.*

*(2) $u_\varepsilon(x)$ takes the following asymptotic form:*

$$u_\varepsilon(x) = w_{\varepsilon,t_\varepsilon}(x) + \phi_{\varepsilon,t_\varepsilon}(x), \tag{7}$$

*where*

$$w_{\varepsilon,t_\varepsilon}(x) := \frac{1}{g(t_\varepsilon)\xi(t_\varepsilon)} w\Big(\frac{x - t_\varepsilon}{\varepsilon}\Big)\chi\Big(\frac{x - t_\varepsilon}{r_0}\Big), \quad r_0 := \frac{1}{10}\min\{t_0 + 1, 1 - t_0\}, \tag{8}$$

*and $\phi_{\varepsilon,t_\varepsilon}(x)$ is a remainder term, namely $\phi_{\varepsilon,t_\varepsilon} \in H_N^2(I)$ such that*

$$\|\overline{\phi_{\varepsilon,t_\varepsilon}}\|_{H^2(I_\varepsilon)} \le C_0\varepsilon \tag{9}$$

*holds for some constant $C_0 > 0$ independent of $\varepsilon > 0$.*

*(3) $v_\varepsilon(x)$ satisfies*

$$v_\varepsilon(t_\varepsilon) = \xi(t_\varepsilon) + O(\varepsilon) \text{ as } \varepsilon \to 0. \tag{10}$$

Next, we study the linear stability of the solution $(u_\varepsilon, v_\varepsilon)$ given in Theorem 1. We linearize the system (1) at $(u_\varepsilon, v_\varepsilon)$ and obtain the following eigenvalue problem:

$$\begin{cases} \varepsilon^2\varphi_\varepsilon'' - \varphi_\varepsilon + 2g(x)u_\varepsilon v_\varepsilon\varphi_\varepsilon + g(x)u_\varepsilon^2\psi_\varepsilon = \lambda_\varepsilon\varphi_\varepsilon, & x \in (-1,1), \\ D\psi_\varepsilon'' - \frac{2c}{\varepsilon}g(x)u_\varepsilon v_\varepsilon\varphi_\varepsilon - \frac{c}{\varepsilon}g(x)u_\varepsilon^2\psi_\varepsilon = \varepsilon\lambda_\varepsilon\psi_\varepsilon, & x \in (-1,1), \\ \varphi_\varepsilon'(\pm1) = \psi_\varepsilon'(\pm1) = 0, \end{cases} \tag{11}$$

where $\lambda_\varepsilon$ is an eigenvalue and $(\varphi_\varepsilon, \psi_\varepsilon) \neq (0,0)$ is an eigenfunction. Now, we state the main result on the stability.

**Theorem 2** *Let $\varepsilon > 0$ be sufficiently small. We assume that $(u_\varepsilon, v_\varepsilon)$ is the solution given in Theorem 1. Then, we have the following result for large eigenvalues, namely $\lambda_\varepsilon \to \lambda_0 \neq 0$:*

*(1) We have $\mathrm{Re}(\lambda_\varepsilon) < 0$. Thus, $(u_\varepsilon, v_\varepsilon)$ is stable for any $D < +\infty$.*

*For small eigenvalues, namely $\lambda_\varepsilon \to 0$, we have the following results:*

*(2) It holds that*

$$\lambda_\varepsilon = -\varepsilon^2\frac{g(t_\varepsilon)\int_{\mathbb{R}} w^3dy}{3\int_{\mathbb{R}}(w')^2dy}F''(t_0) + o(\varepsilon^2) \text{ as } \varepsilon \to 0. \tag{12}$$

*(3) If $F''(t_0) > 0$, then $(u_\varepsilon, v_\varepsilon)$ is stable. If $F''(t_0) < 0$, then $(u_\varepsilon, v_\varepsilon)$ is unstable.*

**Remark 1** *We note that we can actually show that eigenvalues $\lambda_\varepsilon$ satisfying $\mathrm{Re}(\lambda_\varepsilon) \ge -4^{-1}$ are bounded. So we may assume that $\lambda_\varepsilon$ has a limit, up to a subsequence.*

For Theorem 1, we construct one-peak solutions which concentrate at $t_0 \in (-1,1)$ given by **(A)** by using the Liapunov-Schmidt reduction method. In particular, concentration points $t_0$ and amplitudes of one-peak solutions are determined by the interaction of the heterogeneity with the geometry of the domain, represented by Neumann Green function. For Theorem 2, we consider two cases: (i) The large eigenvalue case, namely $\lambda_\varepsilon \to \lambda_0 \neq 0$. (ii) The small eigenvalue case, namely $\lambda_\varepsilon \to 0$. For the large eigenvalue, by using the lemma of Wei and Winter ([2, 8]) for non-local eigenvalue problems, we can show $\mathrm{Re}(\lambda_0) < 0$ for any $D < +\infty$. Thus, for sufficiently small $\varepsilon > 0$, the large eigenvalue $\lambda_\varepsilon$ is a stable eigenvalue. For the small eigenvalue, by using several technical lemma, we show that the leading term of $\lambda_\varepsilon$ is given by (12).

## 3 Remark on the generalized system

Finally, we refer to one-peak solutions for the generalized system. We can generalize the system (1), for example to the following system:

$$\begin{cases} u_t - \varepsilon^2 u_{xx} = -u + g_1(x)u^2 v, & x \in (-1,1),\ t > 0, \\ \varepsilon v_t - D v_{xx} = \frac{1}{2} - \frac{c}{\varepsilon} g_2(x) u^2 v, & x \in (-1,1),\ t > 0, \\ u_x(\pm 1, t) = v_x(\pm 1, t) = 0, \end{cases} \tag{13}$$

where $g_1(x)$ and $g_2(x)$, respectively, are positive and $C^3$ functions. By using the same way of [4], under the suitable assumption, we can construct one-peak solutions and obtain its stability results. For the system above, the assumption **(A)** become
**(A′)**: There exists a point $t_0 \in I$ such that

$$\widetilde{F}(t_0) = 0, \quad \widetilde{F}'(t_0) \neq 0, \quad \widetilde{F}(t) := \frac{t}{12cD} - \frac{g_1'(t) g_2(t)}{g_1(t)^3}. \tag{14}$$

The amplitudes $\widetilde{\xi}(t)$ of the solutions are given by

$$\frac{6cg_2(t)}{g_1(t)^2 \widetilde{\xi}(t)} = 1. \tag{15}$$

Moreover, the stability of the solution, which is constructed under the assumption **(A′)**, is decided by the sign of $\widetilde{F}'(t_0)$. For the analysis in details, see [4]. In particular, we can conclude that the heterogeneity $g_1(x)$ of the equation of $u$, namely the first equation of (13), has a stronger effect than $g_2(x)$ on the spike position and the stability.

**Acknowledgements.** This work was supported by JSPS KAKENHI Grant Number 20J12212. This work was supported by the Research Institute for Mathematical Science, a Joint Usage/Research Center located in Kyoto University.

## References

[1] W. Ao and C. Liu, The Schnakenberg model with precursors, Discrete Contin. Dyn. Syst., 39(4) (2019) 1923-1955.

[2] D. Iron, J. Wei, and M. Winter, Stability analysis of Turing patterns generated by the Schnakenberg model, J.Math. Biol. 49(2004), 358-390.

[3] Y. Ishii, Stability of multi-peak symmetric stationary solutions for the Schnakenberg model with periodic heterogeneity, Commun. Pure Appl. Anal., 19(6) (2020) 2965–3031.

[4] Y. Ishii, The effect of heterogeneity on one-peak stationary solutions to the Schnakenberg model, submitted (2019).

[5] Y. Ishii and K. Kurata, Existence and stability of one-peak symmetric stationary solutions for the Schnakenberg model with heterogeneity, Discrete Contin. Dyn. Syst., 39(5) (2019) 2807-2875.

[6] J. Schnakenberg, Simple chemical reaction system with limit cycle behaviour, J. Theor. Biol., 81 (1979) 389–400

[7] J. Wei and M. Winter, Stable spike clusters for the one-dimensional Gierer-Meinhardt system, Eur. J. Appl. Math., 28 (2017), 576-635.

[8] J. Wei, M. Winter, *Mathematical Aspects of pattern formation in Biological Systems, Applied Math. Sci. vol.189(2014), Springer.*

Department of Mathematical Sciences, Tokyo Metropolitan Univeristy
Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397
JAPAN
E-mail address:
Yuta Ishii (6112007@alumni.tus.ac.jp)