

A remark on the asymptotic behavior of solutions of the Allen-Cahn equation

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1 Introduction

We consider the reaction-diffusion equation

$$u_t = \Delta u + (u + a)(1 - u^2), \quad (x, y) \in \mathbb{R}^2, \quad (1.1)$$

where $u = u(x, y, t)$ is unknown and $a \in (0, 1)$ is a constant. Our interest is the asymptotic behavior of solutions of (1.1).

It is well-known that (1.1) has one-dimensional traveling fronts connecting stable equilibria 1 and -1 . One of the fronts is given by $u_*(x, y, t) = \Phi(y - kt)$, where $\Phi(\eta) := -\tanh(\eta/\sqrt{2})$ and $k = \sqrt{2}a$. We note that the front propagates in y -direction with speed k and that Φ solves

$$\Phi'' + k\Phi' + (\Phi + a)(1 - \Phi^2) = 0. \quad (1.2)$$

It is shown that u_* is stable in appropriate senses (see for instance [3, 4, 8]), and therefore u_* plays an important role to understand front propagation in (1.1). To see this, let us observe the asymptotic behavior of a radially symmetric solution $u(x, y, t) = u(r, t)$, $r = \sqrt{x^2 + y^2}$. According to [1, 5, 6, 7, 10, 11, 12], the following hold. For any $\varepsilon > 0$, there exists $r_0 > 0$ such that if

$$\min_{r \leq r_0} u(r, 0) \geq -a + \varepsilon, \quad \limsup_{r \rightarrow \infty} u(r, 0) < -a, \quad \liminf_{r \rightarrow \infty} u(r, 0) > -\frac{1}{\varepsilon},$$

then for some function h satisfying $h(t) \rightarrow \infty$ ($t \rightarrow \infty$),

$$u(r, t) - \Phi(r - h(t)) \rightarrow 0 \quad \text{uniformly for } r > 0 \text{ as } t \rightarrow \infty. \quad (1.3)$$

Furthermore, the asymptotic behavior of h is given by

$$h(t) = kt - \frac{1}{k} \log t + O(1) \quad (t \rightarrow \infty). \quad (1.4)$$

From (1.3), we see that the profile of u locally looks like a one-dimensional traveling front. Indeed, one finds that

$$u(x, \eta + kt + \tilde{h}(t), t) \rightarrow \Phi(\eta) \quad \text{locally uniformly for } (x, \eta) \in \mathbb{R}^2 \text{ as } t \rightarrow \infty, \quad (1.5)$$

where $\tilde{h}(t) = h(t) - kt$. (1.4) and (1.5) mean that the difference of the position of a level set between u and u_* grows logarithmically while u converges locally to u_* . The natural question arises whether it is possible to find a solution which satisfies (1.5) for some $\tilde{h}(t)$ growing polynomially. Our main result is concerned with the existence of such solutions.

Theorem 1. *Let $0 < \beta < 1/2$ and $b > 0$. Then there exists a solution u of (1.1) such that for some function h satisfying $h(t) = kt + bt^\beta + o(t^\beta)$ ($t \rightarrow \infty$) and some time sequence $\{t_i\}$ with $t_1 < t_2 < \dots \rightarrow \infty$,*

$$u(x, \eta + h(t_i), t_i) \rightarrow \Phi(\eta) \quad \text{locally uniformly for } (x, \eta) \in \mathbb{R}^2 \text{ as } i \rightarrow \infty. \quad (1.6)$$

Remark 2. The theorem should hold for $\beta < 1$ and without taking a time sequence.

2 Sketch of the proof of Theorem 1

In this section we give a procedure to prove Theorem 1.

2.1 Supersolutions and subsolutions

We find the desired solution by constructing a supersolution and a subsolution. We look for a supersolution in the form

$$u^+(x, y, t) = \Phi \left(\frac{y - \phi(x, t)}{\sqrt{1 + \phi_x(x, t)^2}} - p(t) \right) + q(t),$$

where the functions ϕ , p and q are determined later. A supersolution of this type was first used in [9] to construct conical traveling wave solutions. Put $f(u) = (u + a)(1 - u^2)$ and $\mathcal{F}[u] = u_t - \Delta u - f(u)$. By a direct calculation, we have

$$\begin{aligned} \mathcal{F}[u^+] &= -\frac{\Phi' \circ \zeta}{\sqrt{1 + \phi_x^2}} \left(\phi_t - \phi_{xx} - k\sqrt{1 + \phi_x^2} \right) \\ &\quad - \frac{\Phi' \circ \zeta}{1 + \phi_x^2} \left[\frac{2\phi_x^2 \phi_{xx}}{\sqrt{1 + \phi_x^2}} + (\zeta + p) \left\{ \phi_x(\phi_t - \phi_{xx})_x - \frac{(1 - 2\phi_x^2)\phi_{xx}^2}{1 + \phi_x^2} \right\} \right] \\ &\quad - \frac{(\zeta + p)(\Phi'' \circ \zeta)\phi_x^2 \phi_{xx}}{(1 + \phi_x^2)^{3/2}} \left\{ 2 + \frac{(\zeta + p)\phi_{xx}}{\sqrt{1 + \phi_x^2}} \right\} \\ &\quad - (\Phi' \circ \zeta)p_t + f(\Phi \circ \zeta) - f(\Phi \circ \zeta + q) + q_t, \end{aligned}$$

where we have used (1.2) and put

$$\zeta(x, y, t) = \frac{y - \phi(x, t)}{\sqrt{1 + \phi_x(x, t)^2}} - p(t).$$

Since we are looking for a solution converging locally to a one-dimensional traveling front, the derivatives of ϕ with respect to x should decay as $t \rightarrow \infty$. For this reason $\phi_t - \phi_{xx} - k\sqrt{1 + \phi_x^2}$ would dominate other terms. Hence it is appropriate to choose ϕ as a solution of the equation

$$\phi_t = \phi_{xx} + k + \frac{k}{2}\phi_x^2, \quad x \in \mathbb{R}. \quad (2.1)$$

As an initial value of ϕ , we take a function with sublinear growth, that is,

$$\phi(x, 0) = A|x|^\alpha, \quad x \in \mathbb{R}, \quad (2.2)$$

where $A > 0$ and $\alpha \in (0, 1)$. Then the derivatives ϕ_x and ϕ_{xx} indeed decay.

Lemma 3. *Let ϕ be a solution of (2.1)–(2.2). Then there is a constant $C > 0$ such that*

$$\|\phi_x(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq Ct^{-\frac{1-\alpha}{2-\alpha}}, \quad \|\phi_{xx}(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq Ct^{-\frac{2(1-\alpha)}{2-\alpha}}$$

for all $t \geq 1$.

We omit the proof of this Lemma. Suppose that ϕ satisfies (2.1)–(2.2). Then $\mathcal{F}[u^+]$ is written as

$$\mathcal{F}[u^+] = -(\Phi' \circ \zeta)p_t + f(\Phi \circ \zeta) - f(\Phi \circ \zeta + q) + q_t + R,$$

where

$$\begin{aligned} R = & \frac{k(\Phi' \circ \zeta)}{\sqrt{1 + \phi_x^2}} \left(\sqrt{1 + \phi_x^2} - 1 - \frac{1}{2}\phi_x^2 \right) \\ & - \frac{\Phi' \circ \zeta}{1 + \phi_x^2} \left[\frac{2\phi_x^2\phi_{xx}}{\sqrt{1 + \phi_x^2}} + (\zeta + p) \left\{ k\phi_x^2\phi_{xx} - \frac{(1 - 2\phi_x^2)\phi_{xx}^2}{1 + \phi_x^2} \right\} \right] \\ & - \frac{(\zeta + p)(\Phi'' \circ \zeta)\phi_x^2\phi_{xx}}{(1 + \phi_x^2)^{3/2}} \left\{ 2 + \frac{(\zeta + p)\phi_{xx}}{\sqrt{1 + \phi_x^2}} \right\}. \end{aligned}$$

Note that $\eta\Phi'(\eta)$ and $\eta^2\Phi''(\eta)$ are bounded in \mathbb{R} . From this fact and Lemma 3, we deduce that

$$|R| \leq K(1 + p^2)t^{-\gamma} \quad (2.3)$$

for $t \geq 1$, where $K > 0$ is a constant and $\gamma = 4(1 - \alpha)/(2 - \alpha)$. For positive constants p_0, q_0 and t_0 , we put

$$p(t) = Kp_0(t_0^{-\gamma+1} - t^{-\gamma+1}), \quad q(t) = Kq_0t^{-\gamma}.$$

We check that under the condition

$$\alpha < \frac{2}{3}, \quad (2.4)$$

u^+ satisfies $\mathcal{F}[u^+] \geq 0$ for $t \geq t_0$ provided that p_0, q_0 and t_0 are chosen appropriately. We first note that (2.4) implies $\gamma > 1$. It is easy to see that

$$\left(\min_{|s| \geq 1-2\delta} (-f'(s)) \right) > 0$$

for some $\delta > 0$. Then we take $q_0 > 0$ such that

$$\left(\min_{|s| \geq 1-2\delta} (-f'(s)) \right) q_0 \geq 3. \quad (2.5)$$

Since $\Phi(\eta) \rightarrow \pm 1$ as $\eta \rightarrow \mp\infty$, one can pick up $\eta_0 > 0$ such that

$$|\Phi(\eta)| \geq 1 - \delta \quad \text{for all } |\eta| \geq \eta_0. \quad (2.6)$$

The constant $p_0 > 0$ is then chosen so that

$$\left(\min_{|\eta| \leq \eta_0} (-\Phi'(\eta)) \right) (\gamma - 1)p_0 - \left(\max_{s \in \mathbb{R}} f'(s) \right) q_0 \geq 3. \quad (2.7)$$

Finally we choose $t_0 \geq 1$ so large that

$$\max \{ K p_0 t_0^{-\gamma+1}, \gamma q_0 t_0^{-1}, \delta^{-1} K q_0 t_0^{-\gamma} \} \leq 1. \quad (2.8)$$

This particularly gives $0 \leq p \leq K p_0 t_0^{-\gamma+1} \leq 1$ for $t \geq t_0$. From (2.3), (2.8) and the fact that $0 \leq p \leq 1$, we have

$$\begin{aligned} & \mathcal{F}[u^+] \\ & \geq K t^{-\gamma} \left\{ -(\Phi' \circ \zeta)(\gamma - 1)p_0 + \left(\int_0^1 -f'(\Phi \circ \zeta + \theta q) d\theta \right) q_0 - \gamma q_0 t^{-1} - (1 + p^2) \right\} \\ & = K t^{-\gamma} \left\{ -(\Phi' \circ \zeta)(\gamma - 1)p_0 + \left(\int_0^1 -f'(\Phi \circ \zeta + \theta q) d\theta \right) q_0 - 3 \right\} \end{aligned}$$

for $t \geq t_0$.

Let us verify $\mathcal{F}[u^+] \geq 0$. We consider the case $|\zeta| \geq \eta_0$. We see from (2.6) and (2.8) that

$$|\Phi \circ \zeta + \theta q| \geq |\Phi \circ \zeta| - q \geq 1 - \delta - K q_0 t_0^{-\gamma} \geq 1 - 2\delta$$

for $\theta \in [0, 1]$. Hence, by (2.5) and the fact that $\Phi' < 0$,

$$\mathcal{F}[u^+] \geq K t^{-\gamma} \left\{ \left(\min_{|s| \geq 1-2\delta} (-f'(s)) \right) q_0 - 3 \right\} \geq 0.$$

Moreover if $|\zeta| \leq \eta_0$, then (2.7) gives

$$\mathcal{F}[u^+] \geq Kt^{-\gamma} \left\{ \left(\min_{|\eta| \leq \eta_0} (-\Phi'(\eta)) \right) (\gamma - 1)p_0 - \left(\max_{s \in \mathbb{R}} f'(s) \right) q_0 - 3 \right\} \geq 0.$$

Thus we conclude that $\mathcal{F}[u^+] \geq 0$.

Let ϕ , p and q are chosen as above. Then from a similar computation we see that

$$u^-(x, y, t) = \Phi \left(\frac{y - \phi(x, t)}{\sqrt{1 + \phi_x(x, t)^2}} + p(t) \right) - q(t)$$

satisfies $\mathcal{F}[u^-] \leq 0$ provided $t \geq t_0$. By the monotonicity of Φ , we have $u^- \leq u^+$. Consequently the comparison principle shows that there is a solution u of (1.1) satisfying $u^- \leq u \leq u^+$.

2.2 Asymptotic behavior of ϕ

To determine the asymptotic profile of u^+ and u^- , we need to examine the precise behavior of ϕ . It is well-known that by the putting $v = e^{\phi - kt}$, the problem (2.1)–(2.2) is transformed into the heat equation

$$\begin{cases} v_t = v_{xx}, & x \in \mathbb{R}, t > 0, \\ v(x, 0) = \exp(A|x|^\alpha), & x \in \mathbb{R}. \end{cases}$$

Hence ϕ is given by

$$\begin{aligned} \phi(x, t) &= kt + \log v(x, t), \\ v(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp \left(-\frac{(x-y)^2}{4t} + A|y|^\alpha \right) dy. \end{aligned}$$

Splitting the interval of integration into $[0, \infty)$ and $(-\infty, 0)$ gives

$$\begin{aligned} v(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \exp \left(-\frac{(x-y)^2}{4t} + Ay^\alpha \right) dy \\ &\quad + \frac{1}{\sqrt{4\pi t}} \int_0^\infty \exp \left(-\frac{(x+y)^2}{4t} + Ay^\alpha \right) dy. \end{aligned}$$

By the change of variables $y = t^{1/(2-\alpha)}Y$, we obtain

$$v(x, t) = I \left(t^{-\frac{1}{2-\alpha}}x, t \right) + I \left(-t^{-\frac{1}{2-\alpha}}x, t \right), \quad (2.9)$$

where

$$I(X, t) := \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \int_0^\infty \exp\left(t^{\frac{\alpha}{2-\alpha}} g(Y)\right) dY,$$

$$g(Y) = g(Y; X) := -\frac{1}{4}(Y - X)^2 + AY^\alpha.$$

It is easily seen that $g(Y)$ has a unique critical point $Y_* = Y_*(X) > 0$. Then we define $M = M(X)$ and $K = K(X)$ by

$$M := g(Y_*) = -\frac{1}{4}(Y_* - X)^2 + AY_*^\alpha,$$

$$K := \frac{1}{\sqrt{-2g''(Y_*)}} = \frac{1}{\sqrt{1 + 2\alpha(1 - \alpha)AY_*^{\alpha-2}}}.$$

It is elementary to show that

$$g''(Y) < -\frac{1}{2}, \quad (2.10)$$

$$Y_*(X), M(X) \text{ and } K(X) \text{ are increasing.} \quad (2.11)$$

We set

$$U(X, t) := K(X) \exp\left(t^{\frac{\alpha}{2-\alpha}} M(X)\right).$$

The asymptotic behavior of ϕ is described by means of U .

Proposition 4. *The solution ϕ of (2.1)–(2.2) satisfies*

$$\left| \phi(x, t) - kt - \log\left(U\left(t^{-\frac{1}{2-\alpha}}x, t\right) + U\left(-t^{-\frac{1}{2-\alpha}}x, t\right)\right) \right| \leq Ct^{-\frac{\alpha}{2(2-\alpha)}}$$

for all $x \in \mathbb{R}$ and $t \geq 1$ with some constant $C > 0$.

To prove the proposition we consider the estimate of I .

Lemma 5. *There is a constant $C > 0$ such that*

$$\left| \frac{I(X, t)}{U(X, t)} - 1 \right| \leq Ct^{-\frac{\alpha}{2(2-\alpha)}}$$

for all $X \geq -1$ and $t \geq 1$.

Proof. Let $X \geq -1$. In the proof, constants in big O notation do not depend on X . Put

$$Y_0 := Y_*(-1), \quad C_1 := \frac{1}{6}g'''\left(\frac{Y_0}{2}\right) = \frac{2^{2-\alpha}}{3}\alpha(1-\alpha)(2-\alpha)AY_0^{\alpha-3},$$

and define $\delta > 0$ by

$$\delta := \min\left\{\frac{Y_0}{2}, \frac{1}{8C_1}\right\}.$$

From (2.11), we see that $0 < \delta < Y_*(X)$. We rewrite I as

$$\begin{aligned} I &= \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \int_0^\infty \exp\left(t^{\frac{\alpha}{2-\alpha}} (g(Y) - M)\right) dY \\ &= \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \left(\int_{|Y-Y_*|>\delta, Y \geq 0} + \int_{|Y-Y_*| \leq \delta} \right) \\ &=: \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) (I_1 + I_2). \end{aligned}$$

Let us estimate I_1 . From (2.10) and the fact that $g'(Y_*) = 0$,

$$g(Y) - M = \left\{ \int_0^1 (1-\theta) g''(Y_* + \theta(Y - Y_*)) d\theta \right\} (Y - Y_*)^2 \leq -\frac{1}{4} (Y - Y_*)^2.$$

For $a, b > 0$,

$$\int_{|x|>a} e^{-b|x|^2} dx = \int_{|x|>a} \frac{1}{|x|} \cdot |x| e^{-b|x|^2} dx \leq \frac{1}{a} \int_{\mathbb{R}} |x| e^{-b|x|^2} dx = \frac{1}{ab}. \quad (2.12)$$

Hence

$$|I_1| \leq \int_{|Y-Y_*|>\delta} \exp\left(-\frac{1}{4} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY \leq \frac{4}{\delta} t^{-\frac{\alpha}{2-\alpha}}. \quad (2.13)$$

I_2 is handled as follows. We rewrite I_2 as $I_2 = I_{21} + I_{22} + I_{23}$, where

$$\begin{aligned} I_{21} &:= \int_{\mathbb{R}} \exp\left(\frac{1}{2} g''(Y_*) t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY, \\ I_{22} &:= \int_{|Y-Y_*|>\delta, Y \geq 0} \exp\left(\frac{1}{2} g''(Y_*) t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY, \\ I_{23} &:= \int_{|Y-Y_*| \leq \delta} \left(\exp\left(t^{\frac{\alpha}{2-\alpha}} R(Y)\right) - 1 \right) \exp\left(\frac{1}{2} g''(Y_*) t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY, \\ R(Y) &:= g(Y) - M - \frac{1}{2} g''(Y_*) (Y - Y_*)^2. \end{aligned}$$

By a direct computation,

$$I_{21} = \sqrt{4\pi} K t^{-\frac{\alpha}{2(2-\alpha)}}.$$

From (2.10) and (2.12), I_{22} is estimated as

$$|I_{22}| \leq \int_{|Y-Y_*|>\delta} \exp\left(-\frac{1}{4} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY \leq \frac{4}{\delta} t^{-\frac{\alpha}{2-\alpha}}.$$

We consider I_{23} . Let $|Y - Y_*| \leq \delta$. Then, by (2.11) and the definition of δ , we have $Y_* + \theta(Y - Y_*) \geq Y_0 - \delta \geq Y_0/2$ for all $\theta \in [0, 1]$. This together with the fact that g''' is

positive and decreasing yields

$$|g'''(Y_* + \theta(Y - Y_*))| \leq g''' \left(\frac{Y_0}{2} \right) = 6C_1.$$

Therefore

$$|R(Y)| = \frac{1}{2} \left| \int_0^1 (1 - \theta)^2 g'''(Y_* + \theta(Y - Y_*)) d\theta \right| |Y - Y_*|^3 \leq C_1 |Y - Y_*|^3. \quad (2.14)$$

Furthermore,

$$C_1 |Y - Y_*| \leq C_1 \delta \leq \frac{1}{8}. \quad (2.15)$$

Combining (2.10), (2.14), (2.15) and the inequality $|e^a - 1| \leq |a|e^{|a|}$ ($a \in \mathbb{R}$), we deduce that

$$\begin{aligned} |I_{23}| &\leq t^{\frac{\alpha}{2-\alpha}} \int_{|Y-Y_*| \leq \delta} |R(Y)| \exp \left(t^{\frac{\alpha}{2-\alpha}} \left\{ -\frac{1}{4}(Y - Y_*)^2 + |R(Y)| \right\} \right) dY \\ &\leq C_1 t^{\frac{\alpha}{2-\alpha}} \int_{|Y-Y_*| \leq \delta} |Y - Y_*|^3 \exp \left(t^{\frac{\alpha}{2-\alpha}} \left(-\frac{1}{4} + C_1 |Y - Y_*| \right) (Y - Y_*)^2 \right) dY \\ &\leq C_1 t^{\frac{\alpha}{2-\alpha}} \int_{\mathbb{R}} |Y - Y_*|^3 \exp \left(-\frac{1}{8} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2 \right) dY \\ &= 64C_1 t^{-\frac{\alpha}{2-\alpha}}. \end{aligned}$$

From the computations for I_{21} , I_{22} and I_{23} , we obtain

$$I_2 = \sqrt{4\pi} K t^{-\frac{\alpha}{2(2-\alpha)}} \left(1 + O \left(t^{-\frac{\alpha}{2(2-\alpha)}} \right) \right) \quad (t \rightarrow \infty), \quad (2.16)$$

where we have used the fact that

$$K(X) \geq K(-1). \quad (2.17)$$

By (2.13), (2.16) and (2.17), we conclude that

$$I(X, t) = \left(1 + O \left(t^{-\frac{\alpha}{2(2-\alpha)}} \right) \right) U(X, t),$$

as claimed. □

Lemma 6. *There is a constant $C > 0$ such that*

$$I(X, t) \leq C t^{-\frac{\alpha}{2(2-\alpha)}} U(-X, t)$$

for all $X < -1$ and $t \geq 1$.

Proof. By (2.10),

$$\begin{aligned} I &= \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \int_{\mathbb{R}} \exp\left(t^{\frac{\alpha}{2-\alpha}} (g(Y) - M)\right) dY \\ &\leq \frac{1}{\sqrt{4\pi}} t^{\frac{\alpha}{2(2-\alpha)}} \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{4} t^{\frac{\alpha}{2-\alpha}} (Y - Y_*)^2\right) dY \\ &= \exp\left(t^{\frac{\alpha}{2-\alpha}} M\right). \end{aligned}$$

Since $X < -1 < 1 < -X$, we see from (2.11) that

$$\begin{aligned} \exp\left(t^{\frac{\alpha}{2-\alpha}} M(X)\right) &= \frac{1}{K(-X)} \exp\left(-t^{\frac{\alpha}{2-\alpha}} (M(-X) - M(X))\right) U(-X, t) \\ &\leq \frac{1}{K(1)} \exp\left(-t^{\frac{\alpha}{2-\alpha}} (M(1) - M(-1))\right) U(-X, t). \end{aligned}$$

It is elementary to show that for any fixed constant $c > 0$,

$$\exp\left(-ct^{\frac{\alpha}{2-\alpha}}\right) = O\left(t^{-\frac{\alpha}{2(2-\alpha)}}\right) \quad (t \rightarrow \infty). \quad (2.18)$$

Hence the lemma follows. \square

Proof of Proposition 4. In the proof, constants in big O notation do not depend on x . We set $X = t^{-1/(2-\alpha)}x$. Since $\phi(x, t) = kt + \log(I(X, t) + I(-X, t))$, the proof is completed by showing that

$$I(X, t) + I(-X, t) = \left(1 + O\left(t^{-\frac{\alpha}{2(2-\alpha)}}\right)\right) (U(X, t) + U(-X, t)) \quad (t \rightarrow \infty). \quad (2.19)$$

In the case $|X| \leq 1$, this is immediately verified by applying Lemma 5 to $I(X, t)$ and $I(-X, t)$.

Assume now that $X < -1$. In this case we use Lemma 5 for $I(-X, t)$ and Lemma 6 for $I(X, t)$ to obtain

$$I(X, t) + I(-X, t) = \left(1 + O\left(t^{-\frac{\alpha}{2(2-\alpha)}}\right)\right) U(-X, t) \quad (t \rightarrow \infty).$$

Since (2.11) and (2.18) show that

$$\begin{aligned} U(X, t) &\leq \frac{K(-1)}{K(1)} \exp\left(-t^{\frac{\alpha}{2-\alpha}} (M(1) - M(-1))\right) U(-X, t) \\ &\leq Ct^{-\frac{\alpha}{2(2-\alpha)}} U(-X, t) \end{aligned}$$

for some constant $C > 0$, we obtain (2.19). The case $X > 1$ can be handled in the same way as in the case $X < -1$, and therefore the proof is complete. \square

2.3 Proof of (1.6)

Finally we show that the solution u constructed above satisfies (1.6). We easily see that as $X \rightarrow 0$,

$$\begin{aligned} M(X) &= M_0 + O(X), & K(X) &= K_0 + O(X), \\ M_0 &= (2 - \alpha)2^{-\frac{1-\alpha}{2-\alpha}}\alpha^{\frac{\alpha}{2-\alpha}}A^{\frac{2}{2-\alpha}}, & K_0 &:= (2 - \alpha)^{-\frac{1}{2}}. \end{aligned}$$

From this we see that for each $x \in \mathbb{R}$,

$$\begin{aligned} & \left(U \left(t^{-\frac{1}{2-\alpha}}x, t \right) + U \left(-t^{-\frac{1}{2-\alpha}}x, t \right) \right) \exp \left(-t^{\frac{\alpha}{2-\alpha}}M_0 \right) \\ &= K \left(t^{-\frac{1}{2-\alpha}}x \right) \exp \left(t^{\frac{\alpha}{2-\alpha}} \left(M \left(t^{-\frac{1}{2-\alpha}}x \right) - M_0 \right) \right) \\ & \quad + K \left(-t^{-\frac{1}{2-\alpha}}x \right) \exp \left(t^{\frac{\alpha}{2-\alpha}} \left(M \left(-t^{-\frac{1}{2-\alpha}}x \right) - M_0 \right) \right) \\ & \rightarrow 2K_0 \quad (t \rightarrow \infty). \end{aligned}$$

Therefore, by Proposition 4,

$$\lim_{t \rightarrow \infty} \left(\phi(x, t) - kt - M_0 t^{\frac{\alpha}{2-\alpha}} \right) = \log(2K_0).$$

This together with Lemma 3 implies that for $(x, \eta) \in \mathbb{R}^2$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} u(x, \eta + h_0(t), t) &\leq \lim_{t \rightarrow \infty} u^+(x, \eta + h_0(t), t) = \Phi(\eta - \eta_1), \\ \liminf_{t \rightarrow \infty} u(x, \eta + h_0(t), t) &\geq \lim_{t \rightarrow \infty} u^-(x, \eta + h_0(t), t) = \Phi(\eta + \eta_1), \end{aligned} \tag{2.20}$$

where $h_0(t) = kt + M_0 t^{\frac{\alpha}{2-\alpha}} + \log(2K_0)$ and $\eta_1 = Kp_0 t_0^{-\gamma+1}$.

Now we discuss the convergence of the function $w(x, \eta, t) := u(x, \eta + h_0(t), t)$. It is seen that w satisfies

$$w_t = \Delta w + \left(k + \frac{\alpha}{2-\alpha} t^{-\frac{2(1-\alpha)}{2-\alpha}} \right) w_\eta + f(w).$$

The fact that $u_- \leq u \leq u_+$ shows that $w(\cdot, t)$ is uniformly bounded. By the regularity theory for parabolic partial differential equations and compact embeddings for Sobolev and Hölder spaces, we can take a sequence $\{t_i\}$, $t_1 < t_2 < \dots \rightarrow \infty$ such that $w(\cdot, t_i)$ converges locally uniformly to a solution $W = W(x, \eta)$ of the equation

$$\Delta W + kW_\eta + f(W) = 0 \tag{2.21}$$

as $i \rightarrow \infty$. From (2.20), we have

$$\Phi(\eta + \eta_1) \leq W(x, \eta) \leq \Phi(\eta - \eta_1). \tag{2.22}$$

[2, Theorem 3.1] shows that the function W satisfying (2.21) and (2.22) coincides with $\Phi(\eta - \eta_2)$ for some $\eta_2 \in \mathbb{R}$. Thus we conclude that (1.6) holds with

$$\beta = \frac{\alpha}{2 - \alpha}, \quad b = M_0 = (2 - \alpha)2^{-\frac{1-\alpha}{2-\alpha}}\alpha^{\frac{\alpha}{2-\alpha}}A^{\frac{2}{2-\alpha}}, \quad h(t) = h_0(t) + \eta_2.$$

The restriction $\beta < 1/2$ comes from the condition (2.4), and $b > 0$ can be chosen arbitrarily since $A > 0$ is arbitrary.

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