TRAVELING WAVE SOLUTIONS OF FREE BOUNDARY PROBLEMS

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1. Introduction

The evolving interface are observed in various phenomena. Growth of snow crystal, cloud floating, surface of pond are typical examples in nature and cell locomotion, growth of plants, cloud dynamics of animals in life science. In general, the motion of interfaces are mainly determined by the energy from interface itself and from interior and exterior of the interface (driving force). To describe the motion of interface, some partial differential equations are proposed. The most simple example is *mean curvature flow* given by

$$V = -\kappa, \tag{1}$$

where κ is the mean-curvature of $\Gamma(t)$, which stands for an embedded hypersurface in $\mathbb{R}^N$ at the time t, and V is the normal velocity on $\Gamma(t)$. The equation (1) arises from the first variation of the surface area functional. The mean curvature flow (1) is also proposed by Mullins [19] as a mathematical model describing the motion of grain boundaries. According to the result of Gage-Hamilton[12], any convex closed curve shrinks to a single point in a finite time (see also [14]). From this, (1) does not have any stationary solutions and traveling waves $\Gamma(t)$ composed of Jordan curves. Here we call $\Gamma(t)$ a *traveling wave* if

$$\Gamma(t) = \Gamma_0 + t\mathbf{c}$$

for $t \geq 0$, where Γ_0 is a curve that is positively oriented (counterclockwise direction) and $\mathbf{c}$ is a vector in $\mathbb{R}^N$. However, if $\Gamma(t)$ is allowed to be a function defined on the whole space of $\mathbb{R}^2$, there exists a traveling wave called *Grim Reaper* (cf. see [1]). Indeed, if it is moving upward with speed c and is represented by the graph $y = \varphi(x) + ct$, then φ satisfies

$$c = \frac{\varphi_{xx}}{1 + \varphi_x^2}.$$

By solving the above equation, it is easily seen that

$$y = \varphi(x) + ct = -\frac{1}{c} \log \cos cx + ct.$$

Another example of curvature flows is *the curvature-eikonal flow* written as

$$V = A - \kappa, \tag{2}$$

where A is a positive constant. The curvature-eikonal flow (2) is derived from the singular limit of the Allen-Cahn-Nagumo equation or the FitzHugh-Nagumo equation (see [11]). We immediately confirm that $\Gamma(t)$ has a unique disk-shaped stationary solution with the radius $1/A$. This stationary solution is unstable, and we can regard this circle as a traveling wave with the speed $c = 0$. In addition, it was reported by Ninomiya-Taniguchi [25, 26] that there exists a traveling wave defined on the whole space, called a *V-shaped traveling wave* if the speed c is greater than A. More precisely, the curve of the V-shaped traveling wave $\Gamma(t) = \{(x, y) \mid x \in \mathbb{R},\ y = \varphi(x) + ct\}$ is represented by use of $\theta = \arctan \varphi_x(x)$ as follows:

$$\begin{cases} x(\theta; c) & := \dfrac{\theta}{c} + \dfrac{A}{c\sqrt{c^2 - A^2}} \log \left| \dfrac{1 + \sqrt{\dfrac{c+A}{c-A}} \tan \dfrac{\theta}{2}}{1 - \sqrt{\dfrac{c+A}{c-A}} \tan \dfrac{\theta}{2}} \right|, \\ y(\theta; c) & := -\dfrac{1}{c} \log \left(\dfrac{c \cos \theta - A}{c - A} \right) \end{cases}$$

for $\theta \in (-\theta_*, \theta_*)$, where $\theta_* = \arctan(\sqrt{c^2 - A^2}/A)$.

We can generalize the equation (2) to

$$\beta(\mathbf{n})V = \gamma(\mathbf{n}) - \alpha(\mathbf{n})\kappa, \tag{3}$$

where $\mathbf{n}$ denotes the outer normal vector of $\Gamma(t)$ and α, β, γ are given functions. One of the typical examples of (3) is the following anisotropic interface equation in $\mathbb{R}^2$ [13]:

$$b(\theta)V = A - \left(f(\theta) + f''(\theta) \right) \kappa, \tag{4}$$

where θ is the angle between the x-axis and $\mathbf{n}$ at $\Gamma(t)$, A is a positive constant describing the difference between the phases in terms of bulk energy, $b(\theta)$ is a positive function, and $f(\theta)$ is the interfacial energy. Marutani et al [18] discussed the V-shaped traveling wave of (4). The short-time existence and uniqueness of classical solutions to (3) has been demonstrated by Gage-Hamilton [12], Evans-Spruck [9, 10] and Huskin and Polden [15] (see also [7]). For more general setting, we refer the reader to Chen-Giga-Goto [5] and Brakke [2]. If we extend $\gamma(\mathbf{n})$ of (3) to a more general driving force $\gamma = \gamma(\mathbf{n}, u, \nabla u, C(t))$, that is,

$$\beta(\mathbf{n})V = \gamma(\mathbf{n}, u, \nabla u, C(t)) - \alpha(\mathbf{n})\kappa, \tag{5}$$

where $C(t)$ is a non-local function, e.g., the perimeter of $\Gamma(t)$ or the volume of the phase $\Omega(t)$, and u is an unknown function defined in the domain $\Omega(t)$ adjacent to $\Gamma(t)$ or in $\mathbb{R}^N$, (5) is relevant to many free boundary problems, interface equations and singular limit problems. For instance, the volume-preserving mean curvature flows, the Stefan condition, the modified Gibbs-Thomson relation in some FBPs such as the Stefan problem, the Mullins-Sekerka problem (or the Hele-shaw problem) [20], the incompressible viscous two-phase fluid flow [16], and also the crystal growth [3], laminar flame propagation [17], the FBP describing cell locomotion [6, 22, 23], and the singular limit problem derived from the Allen-Cahn-Nagumo equation [11], the Lotka-Volterra equations [8], FitzHugh-Nagumo equation [4].

In recent years, the localized patterns are observed in the phenomena related to (5) (see also [21, 27]) and it is important to study the traveling waves whose boundary are composed of Jordan curves in $\mathbb{R}^2$. We call such a traveling wave a *compact traveling wave*. However, in general, α, β and γ in (5) are complicated due to the coupling of u, which is often unknown function depending on the evolutionary equation in phase $\Omega(t)$. In this study, we start with a simple case (3). In addition, we assume that α and β are positive given functions to guarantee that (3) is well-posed. By dividing by α, the anisotropic curvature flow (3) can be simplified as the following curvature flow with a driving force $\gamma(\mathbf{n})$:

$$\beta(\mathbf{n})V = \gamma(\mathbf{n}) - \kappa. \tag{6}$$

We denote the angle between x-axis and the outer normal vector $\mathbf{n}(s)$ by $\theta = \theta(s)$, where s is the arc length of Γ_0. From this, it follows that $\mathbf{n}(s) = (\cos\theta, \sin\theta)$ directly. Thus β and γ are represented by

$$\beta(\mathbf{n}) = \tilde{\beta}(\theta) \quad \textit{and} \quad \gamma(\mathbf{n}) = \tilde{\gamma}(\theta),$$

respectively. For simplicity of notation, omitting a tilde, we consider

$$\beta(\theta)V = \gamma(\theta) - \kappa. \tag{7}$$

Throughout this paper, we always assume that Γ_0 is positively oriented (counterclockwise). Let $\Gamma(t)$ be a traveling wave of (7) along $\mathbf{e}_c = (\cos\eta, \sin\eta)$ where η can be chosen arbitrarily if $\mathbf{c} = \mathbf{0}$. Then we see that

$$V = c\,\mathbf{e}_c \cdot \mathbf{n} = c\cos(\theta - \eta).$$

Since the curvature κ of Γ_0 is given by θ_s, (7) is rewritten to

$$c\beta(\theta)\cos(\theta - \eta) = \gamma(\theta) - \theta_s.$$

Since Γ_0 is a Jordan curve which is represented by $\{(x(s), y(s)) \mid s \in [0, L)\}$, $(\theta(s), x(s), y(s))$ has to satisfy the boundary condition

$$(\theta(0), x(0), y(0)) = (\theta(L) + 2\pi, x(L), y(L)).$$

Summarizing these equations, we obtain

$$
(8) \qquad \begin{cases} \theta_s = \gamma(\theta) - c\beta(\theta)\cos(\theta - \eta) & \text{in } (0, L), \\ x_s = -\sin\theta & \text{in } (0, L), \\ y_s = \cos\theta & \text{in } (0, L), \\ \theta(0) = \theta_0,\ x(0) = x_0,\ y(0) = y_0, \\ \theta(L) = \theta_0 + 2\pi,\ x(L) = x_0,\ y(L) = y_0, \end{cases}
$$

where L is a perimeter of Γ_0 and θ_0, x_0, y_0 are initial data. Therefore our goal is to find a curve Γ_0 composed of $(\theta, x, y, L, c, \eta)$ satisfying (8).

2. Main results

Following to [24], we will state the main results. We prepare a few notations. We say that $x \in C^{m,1}([0, \ell])$ if and only if, for any $s \in [0, \ell]$, $x(s)$ is continuously differentiable for m-times and m-th order derivative $x^{(m)}(s)$ is Lipschitz continuous. In addition, Γ_0 belongs to $C^{2,1}$ if there exist functions $x, y \in C^{2,1}(-\ell, \ell)$ such that

$$
\{(x(s), y(s)) \in \mathbb{R}^2 \mid s \in (-\ell, \ell)\} = \Gamma_0 \cap U, \quad x(0) = x_0,\ y(0) = y_0\}
$$

for any $(x_0, y_0) \in \Gamma_0$, where U is a suitable open set in $\mathbb{R}^2$. We say that traveling wave $(\Gamma_0, \mathbf{c})$ of (6) is *strictly convex* (*concave*) if curvature κ of Γ_0 is positive (negative) for any point of Γ_0. Any continuous function γ can be classified into one of the following three cases : We state that γ is *positive* (*non-positive*) if $\gamma(\theta) > 0$ ($\gamma(\theta) \leq 0$) for any θ; γ is *sign-changing* if there exist θ_1 and θ_2 such that $\gamma(\theta_1) > 0$ and $\gamma(\theta_2) \leq 0$. We note that the constant function $\gamma \equiv 0$ is regarded as non-positive one.

If β and γ are Lipschitz continuous with 2π-periodic, then we can obtain the following results (see [24] for the details).

(a) Every compact traveling wave of (6) is strictly convex and unstable.
(b) A compact traveling wave is unique if it exists.
(c) If γ is positive, then there exists a unique compact traveling wave of (6). In addition, velocity vector $\mathbf{c}$ satisfies that $\mathbf{c} \cdot \mathbf{e}_\gamma \leq 0$, where

$$
\mathbf{e}_\gamma := \left(\int_0^{2\pi} \frac{\cos\theta}{\gamma(\theta)}\, d\theta, \int_0^{2\pi} \frac{\sin\theta}{\gamma(\theta)}\, d\theta \right).
$$

In particular, if $\mathbf{e}_\gamma = \mathbf{0}$, then $\mathbf{c} = \mathbf{0}$.
(d) There is no compact traveling wave of (6) if γ is non-positive.
(e) If β and γ are symmetric to $\eta_* \in [0, \pi)$, then (6) has a unique compact traveling wave $(\Gamma_0, \mathbf{c})$ if and only if $\mathcal{S} \neq \emptyset$ defined by (9).

We remark that the uniqueness of a compact traveling wave is up to the shift. Actually, $\Gamma(t) + (x_1, y_1)$ is also a compact traveling wave for any $(x_1, y_1) \in \mathbb{R}^2$, if $\Gamma(t)$ is a compact traveling wave.

To show the existence and the uniqueness of compact traveling waves, we prepare the following set:

$$\mathcal{S} := \left\{ (c,\eta) \in [0,\infty) \times [0,2\pi) \;\middle|\; \inf_{\theta\in[0,2\pi)} \Big[\gamma(\theta) - c\beta(\theta)\cos(\theta-\eta)\Big] > 0 \right\}. \tag{9}$$

Note that $\mathcal{S}$ is non-empty if compact traveling wave exists, because the result (a) implies the positivity of $\gamma(\theta) - c\beta(\theta)\cos(\theta-\eta)$ due to (8). If γ is sign-changing, then a compact traveling wave is unique if it exists. In contrast with the positive case in (c), a compact traveling wave does not always exist for any sign-changing function γ. Indeed, (6) does not possess a compact traveling wave when

$$\gamma(\theta) = \cos(2\theta) + \frac{1}{2}$$

because $\mathcal{S}$ is empty. This example also implies that the positivity of total energy of driving force on $\Gamma(t)$ is not essential to construct a traveling wave in (6) because

$$\oint_{\Gamma_0} \gamma\, ds > 0.$$

Therefore, we need to impose a additional condition on $\gamma(\theta)$. It will be shown that (6) includes a compact traveling wave, if γ satisfies the *admissible condition* which is a geometrical condition for $\mathcal{S}$. See [24] for the details.

3. Applications

In this section, we discuss the topics related to compact traveling waves of (6).

3.1. **Translation invariance.**

Assume that $(\Gamma_0^*, c^*\mathbf{e}(\eta^*))$ is a compact traveling wave of (6) with γ. Then we can show that (6) with $\widetilde{\gamma}(\theta) := \gamma(\theta) + \nu\beta(\theta)\cos(\theta-\eta^*)$ for $\nu \in \mathbb{R}$ also has a compact traveling wave $(\Gamma_0, \mathbf{c})$ where

$$(\Gamma_0,\mathbf{c}) = \begin{cases} (\Gamma_0^*, (c^*+\nu)\mathbf{e}(\eta^*)) & \text{if } c^*+\nu \geq 0, \\ (\Gamma_0^*, |c^*+\nu|\mathbf{e}(\eta^*+\pi)) & \text{if } c^*+\nu < 0. \end{cases}$$

3.2. **The inverse problem.**

Here we consider the inverse problem. Consider the following question:

Q. *Is there a function γ such that* (6) *possesses the given compact traveling wave* $(\Gamma_0^*, \mathbf{c}^*)$ *?*

We have an affirmative answer as follows: Let Γ_0^* and $\mathbf{c}^*$ be an arbitrary strictly convex Jordan curve and vector, respectively. Then there exists γ such that $(\Gamma_0^*, \mathbf{c}^*)$ is a compact traveling wave of (6).

3.3. **The Wulff shape.** In this subsection, we consider the application of our theorems to a typical example (4). As introduced in Section 1, the anisotropic interface equation (4) is obtained by considering the interfacial energy and the difference between bulk energies. In (6), setting

$$\beta(\theta) = b(\theta)\left(f(\theta) + f''(\theta)\right)^{-1}, \quad \gamma(\theta) = A\left(f(\theta) + f''(\theta)\right)^{-1},$$

we see that (6) corresponds to (4). Recall that A is a positive constant. Let us consider that the interfacial energy $f(\theta)$ is *strictly stable* (see [13]), that is,

$$f(\theta) + f''(\theta) > 0.$$

Since γ is positive, we have the unique compact traveling wave $(\Gamma_0, \mathbf{c})$ to (4). Using the integration by parts, we easily check that

$$\int_0^{2\pi} \gamma(\theta)^{-1} e^{i\theta} d\theta = A^{-1} \int_0^{2\pi} (f(\theta) + f''(\theta)) e^{i\theta} d\theta = 0,$$

which implies that $\mathbf{c} = (0, 0)$ by the result (c). Due to the uniqueness of compact traveling waves, we can confirm the same result as in [13].

As $A = 1$, in particular, the closed domain whose boundary is the stationary solution Γ_0 is often called *the Wulff shape*. Thus we know that the convex traveling wave of (4) (namely, stationary solution) is represented by the extension of the Wulff shape.

3.4. **Non-convex compact traveling waves.** To construct an example of non-convex traveling waves, we need to violate the condition for γ. If the driving force γ depends not only on θ but also y, then we can construct a non-convex compact traveling wave $(\Gamma_0, \mathbf{c})$ with $\mathbf{c} = \mathbf{0}$. Let γ be a function of y with $\gamma(y) = \gamma(-y)$ and (θ, x, y) be a solution of

$$\begin{cases} \theta_s = \gamma(y) & \text{in } (0, \ell), \\ x_s = -\sin\theta & \text{in } (0, \ell), \\ y_s = \cos\theta & \text{in } (0, \ell), \\ \theta(0) = 0, \ \theta(\ell) = \pi/2, & \\ x(0) = y(0) = 0, & \end{cases}$$

for a constant $\ell > 0$. By the symmetry of γ, we can extend the solution (x, y, θ) to the interval $[0, 4\ell]$ by

$$x_s = -\sin\Theta, \ y_s = \cos\Theta, \ \Theta := \begin{cases} \theta(s) & \text{in } [0, \ell) \\ \pi - \theta(s - \ell) & \text{in } [\ell, 2\ell) \\ \pi + \theta(s - 2\ell) & \text{in } [2\ell, 3\ell) \\ 2\pi - \theta(s - 3\ell) & \text{in } [3\ell, 4\ell) \end{cases},$$

and we obtain a Jordan curve $\Gamma_0 = \{(x(s), y(s)) \mid s \in [0, L),\ L = 4\ell\}$. In order to show non-convexity of Γ_0, $\gamma(y(s))$ must be at least negative at some $s \in (0, \ell)$. Set

$$\gamma(y) := 1 + 2y^2 - \frac{11}{2}y^4 + 2y^6.$$

Then we see that $\gamma(1) = -1/2 < 0$. It is confirmed that there are two points y_1, y_2 such that $0 < y_1 < y_2 < y_*$, $\gamma(y_1) = \gamma(y_2) = 0$ and

$$\gamma = \begin{cases} > 0 & \text{in } (0, y_1) \cup (y_2, y_*), \\ < 0 & \text{in } (y_1, y_2), \end{cases}$$

where

$$y_* = \int_0^\ell \cos\theta \, ds.$$

Repeating the same argument as in Subsection 3.1, we also see the existence of a non-convex compact traveling wave with the velocity $\mathbf{c} = c\,\mathbf{e}(0)(\neq \mathbf{0})$ when γ is replaced by $\gamma(y) + c\cos\theta$ and $\beta \equiv 1$.

References

[1] Angenent, S.B.: On the formation of singularities in the curve shortening flow. J. Differ. Geom., 33, 601–633 (1991)

[2] Brakke, K.A.: The motion of a surface by its mean curvature. Mathematical Notes, 20. Princeton University Press, Princeton, New Jersey (1978)

[3] Burton, W.K., Cabrera, N., Frank, F.C.: The growth of crystals and the equilibrium structure of their surfaces. Philos. Trans. R. Soc. Lond. Ser. A. 243, 299–358 (1951)

[4] Chen, Y.Y., Kohsaka,Y., Ninomiya, H.: Traveling spots and traveling fingers in singular limit problems of reaction-diffusion systems. Discret. Contin. Dyn. Syst. Ser. B, 19, 697–714 (2014)

[5] Chen, Y.G., Giga, Y., Goto, S.: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differ. Geom., 33, 749–786 (1991)

[6] Choi, Y.S., Lui, R.: Existence of traveling domain solutions for a two-dimensional moving boundary problem. Trans. Am. Math. Soc., 361, 4027–4044 (2009)

[7] Chou, K.S., Zhu, X.P.: The Curvature Shortening Problem. Chapman and Hall/CRC, Washington, D.C. (2001)

[8] Ei, S.I., Yanagida, E.: Dynamics of interfaces in competition-diffusion systems. SIAM J. Appl. Math., 54, 1355–1373 (1994)

[9] Evans, L.C., Spruck, J.: Motion of level sets by mean curvature. I. J. Differ. Geom., 33, 635–681 (1991)

[10] Evans, L.C., Spruck, J.: Motion of level sets by mean curvature. II. Trans. Am. Math. Soc., 330, 321–332 (1992)

[11] Fife, P.C.: Dynamics of Internal Layers and Diffusive Interfaces. CBMS-NSF Reg. Conf. Ser. Appl. Math., 53, SIAM, Philadelphia (1988)

[12] Gage, M., Hamilton, R.S.: The heat equation shrinking convex plane curves. J. Differ. Geom., 23, 69–96 (1986)

[13] Gurtin, M.E.: Thermomechanics of Evolving Phase Boundaries in the Plane. Oxford Mathematical Monographs, Oxford University Press (1993)

[14] Grayson, M.A.: The heat equation shrinks embedded plane curves to round points. J. Differ. Geom., 26, 285–314 (1987)

[15] Huisken, G., Polden, A.: Geometric evolution equations for hypersurfaces. In: Hildebrandt, S., Struve, M.(Eds.), Calculus of variations and geometric evolution problems, pp. 45–84. Springer, Berlin (1999)
[16] Liu, C., Walkington, N. J.: An Eulerian description of fluids containing visco-elastic particles, Arch. Ration. Mech. Anal., 159, 229–252 (2001)
[17] Markstein, G.H.: Nonsteady Flame Propagation. Pergamon, New York (1964)
[18] Marutani, Y., Ninomiya, H., Weidenfeld, R.: Traveling curved fronts of anisotropic curvature flows. Jpn. J. Ind. Appl. Math., 23, 83–104 (2006)
[19] Mullins, W.W.: Solid surface morphologies governed by capillarity. In: Robertson, W.D., Gjostein, N.A. (eds.) Metal Surfaces: Structure, Energetics and Kinetics. pp. 17–66, Am. Soc. Met., Ohio (1963)
[20] Mullins, W.W., Sekarka, R.F.: Morphological stability of a particle growing by diffusion or heat flow. J. Appl. Phys., 34, 323–329 (1963)
[21] Mihaliuk, E., Sakurai, T., Chirila, F., Showalter, K.: Experimental and theoretical studies of feedback stabilization of propagating wave segments. Faraday Discuss., 120, 383–394 (2002)
[22] Monobe, H., Ninomiya, H.: Multiple existence of traveling waves of a free boundary problem describing cell motility. Discret. Contin. Dyn. Syst. Ser. B, 19 789–799 (2014)
[23] Monobe, H., Ninomiya, H.: Traveling wave solutions with convex domains for a free boundary problem. Discret. Contin. Dyn. Syst. Ser. A, 37, 905–914 (2017)
[24] Monobe, H., Ninomiya, H.: Compact traveling waves for anisotropic curvature flow with driving force. submitted to TAMS.
[25] Ninomiya, H., Taniguchi, M.: Traveling curved fronts of a mean curvature flow with constant driving force, In: Kenmochi, N. (ed.) Free boundary problems: Theory and Applications I, GAKUTO Internat. Ser., Math. Sci. Appl., 13, pp. 206–221. Gakkōtosho, Tokyo (2000)
[26] Ninomiya, H., Taniguchi, M.: Existence and global stability of traveling curved fronts in the Allen-Cahn equations. J. Differ. Equ., 213, 204–233 (2005)
[27] Ohta, T., Mimura, M., Kobayashi, R.: Higher-dimensional localized patterns in excitable media. Phys. D, 34, 115–144 (1989)