LIPSCHITZ BOUNDS

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Abstract. We review some recent Lipischitz regularity results for solutions to nonlinear elliptic equations and systems. In particular, we deal with minima of integral functionals. Emphasis is put on the nonuniformly elliptic case.

Contents

1. Introduction

In this paper we review some recent results on Lipschitz continuity of solutions to elliptic equations and systems of the type

$$-\operatorname{div} a(x,Du)=f \qquad \text{in } \Omega\subset\mathbb{R}^n\,, \tag{1.1}$$

where $a\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ is a continuous vector field, with $n\geq 2, N\geq 1$; the right-hand side data f can be in the most general case a measure. We shall assume in general that, for every choice of $x\in\Omega$. The notion of solution will be clarified in each situation; anyway, all solutions considered here are at least distributional and belong to some Sobolev spaces, i.e., we always have, at least, $u\in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$.

Ellipticity here means that the condition

$$\partial_{z_j^\beta}a_i^\alpha(x,z)\xi_i^\alpha\cdot\xi_j^\beta\equiv\partial_z a(x,z)\xi\cdot\xi\geq 0$$

is satisfied for every choice of $z,\xi\in\mathbb{R}^{N\times n}$ and $x\in\Omega$, provided $\partial_z a(x,z)$ exists. Such condition is obviously not sufficient to get regularity results for solutions to (1.1), but, when properly reinforced and supplemented by suitable growth conditions from below and above, that is, when providing suitable lower and upper bounds on the eigenvalues of $\partial_z a$, it can be used to prove that solutions are more regular.

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Equation (1.1) naturally connects to integral functional of the Calculus of Variations of the type

$$W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)\ni w\mapsto \mathcal{F}(w;\Omega):=\int_\Omega \left[F(x,Dw)-f\cdot w\right]\,dx\,. \tag{1.2}$$

Indeed, in case it is $\partial_z F(\cdot)=a(\cdot)$, the one in (1.1) is the Euler-Lagrange equation (actually a system when $N>1$) of the functional in (1.2). When considering functionals as in (1.2) we always assume that the vector field $f\colon\Omega\mapsto\mathbb{R}^N$ is at least L^n-integrable, i.e. $f\in L^n(\Omega;\mathbb{R}^N)$. The notion of local minimizer used in this paper is quite standard in the literature.

Definition 1. *A map $u\in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$ in (1.2) with $f\in L^n(\Omega;\mathbb{R}^N)$ if, for every open subset $\tilde\Omega\Subset\Omega$, we have $\mathcal{F}(u;\tilde\Omega)<\infty$ and $\mathcal{F}(u;\tilde\Omega)\le\mathcal{F}(w;\tilde\Omega)$ holds for every competitor $w\in u+W^{1,1}_0(\tilde\Omega;\mathbb{R}^N)$.*

From now on, we shall systematically abbreviate “local minimizer” simply by “minimizer”. In this note we want to discuss recent progresses about the following, basic:

Problem P. *Find minimal regularity assumptions on f and $x\mapsto F(x,\cdot), x\mapsto a(x,\cdot)$, guaranteeing local Lipschitz continuity of minima of the functional $\mathcal{F}$ in (1.2) and weak solutions to (1.1), respectively, provided this type of regularity holds when $f\equiv 0$ and no x-dependence occurs, i.e., $F(x,z)\equiv F(z)$ and $a(x,z)\equiv a(z)$.*

In other words, we want to discuss optimal dependence on *external ingredients*, that is coefficients dependence on x and data f. In the linear case, this is an example of classical Schauder theory. This claims that $W^{1,2}$-solutions to linear elliptic equation as $-\operatorname{div}(A(x)Du)=f$ are $C^{1,\alpha}$ are regular provided so are the entries of the uniformly elliptic matrix $A(\cdot)$, and $f\in L^q$ for some $q>n$ (see for instance [26, Chapter 10]).

We are mainly interested in the nonlinear case (1.1); for this, we take as a the classical p-Laplacean system with coefficients

$$-\operatorname{div}(\mathfrak{c}(x)|Du|^{p-2}Du)=f\,,\quad p>1\,,\quad 0<\nu\le\mathfrak{c}(\cdot)\le L\,. \tag{1.3}$$

For this we have [38, 40]

Nonlinear Stein Theorem (NST). *Let $u\in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a weak solution to (1.3). If $f\in L(n,1)(\Omega;\mathbb{R}^N)$, and $\mathfrak{c}(\cdot)$ is Dini continuous, then Du is continuous.*

We recall that f belongs to the Lorentz space $L(n,1)(\Omega;\mathbb{R}^N)$ iff

$$\|f\|_{L(n,1)(\Omega)}:=\int_0^\infty |\{x\in\Omega: |f(x)|>\lambda\}|^{1/n}\,d\lambda<\infty\,, \tag{1.4}$$

and also that $L^q\subset L(n,1)\subset L^n$ for every $q>n$. Moreover, denoting by $\omega(\cdot)$ the modulus of continuity of $\mathfrak{c}(\cdot)$, the Dini continuity of $\mathfrak{c}(\cdot)$ amounts to require that

$$\int_0 \omega(\varrho)\,\frac{d\varrho}{\varrho}<\infty \tag{1.5}$$

The linkage with the perturbation Problem P is clear: solutions to $-\operatorname{div}(|Du|^{p-2}Du)=0$ are known to be locally $C^{1,\alpha}$-regular by classical results of Uhlenbeck [55] and Uraltseva [56]. Therefore the above result gives sharp condition on coefficients and data allowing to preserve Lipschitz continuity from the unperturbed case.

The terminology bearing the name of Stein is motivated by the fact that, for $\mathfrak{c}(\cdot)\equiv 1$ and $p=2$, the last theorem is a classical result of Stein [54]. It is optimal both with respect to condition (1.4), as shown by Cianchi [10], and with respect to (1.5), as shown by Jin, Maz'ya & Van Schaftingen [43].

We immediately remark a relevant fact here. The conditions on f and $\mathfrak{c}(\cdot)$ implying local Lipschitz continuity are independent of the exponent p. We shall expand on this point in the following sections. In particular, throughout this note, we shall emphasize what is in some sense the universal role of the space $L(n,1)$, which appears independently of the particular operator considered, whenever we are considering divergence form equations and systems.

In the following, when considering an open ball $B \equiv B(x) \subset \mathbb{R}^n$ centered at x, we shall denote by $\mathtt{r}(B)$ its radius. We shall also denote

$$\fint_B g(y)\,dy := \frac{1}{|B|}\int_B g(y)\,dy$$

the componentwise average of an integrable map $g\colon B \to \mathbb{R}^k$.

2. The uniformly elliptic case

2.1. **Local estimates and Nonlinear Potential Theory.** The NST stated in Section 1 finds its room in the larger setting of Nonlinear Potential Theory of general equations of the type in (1.1). Let us show briefly how. For this, we consider the following general growth and ellipticity conditions for equations ($N=1$), originally considered by Ladyzhenskaya & Uraltseva [44]

$$\begin{cases} |a(x,z)| + (|z|^2+\mu^2)^{1/2}|a_z(x,z)| \le L(|z|^2+\mu^2)^{(p-1)/2} \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \le \partial_z a(x,z)\xi\cdot\xi \\ |a(x,z)-a(x_0,z)| \le L\omega\left(|x-x_0|\right)(|z|^2+\mu^2)^{(p-1)/2}\,, \end{cases} \tag{2.1}$$

whenever $x, x_0 \in \Omega$, $z,\xi \in \mathbb{R}^n$ where $0<\nu\le 1\le L$ and $\mu\in[0,1]$. The modulus of continuity $\omega(\cdot)$ is assumed to satisfy (1.5). Under such assumptions local estimates for the gradient of solutions hold via classical Riesz potentials. This is in the following:

Theorem 1. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation (1.1) under the assumptions (2.1) with $p > 2-1/n$ and $f \in L^1$. Then*

- *There exists a constant $c \equiv c(n,p,\nu,L)$, and a positive threshold radius $R_* \equiv R_*(n,p,\nu, L,\omega(\cdot))$, such that the inequality*

$$|Du(x)|^{p-1} \le c\int_B \frac{|f(y)|}{|y-x|^{n-1}}\,dy + c\left[\fint_B |Du|\,dy + s\right]^{p-1} \tag{2.2}$$

holds whenever $B \equiv B(x) \Subset \Omega$ is ball with $\mathtt{r}(B) \le R_*$. No limitation on $\mathtt{r}(B)$ occurs when $a(\cdot)$ is independent of x.
- *Moreover, if*

$$\lim_{\mathtt{r}(B)\to 0}\int_{B(x)} \frac{|f(y)|}{|y-x|^{n-1}}\,dy = 0 \quad \textit{locally uniformly in } \Omega \textit{ w.r.t. } x \tag{2.3}$$

then Du is continuous in Ω.
- *Finally, if the quantity in the right-hand side in (2.2) is finite, then the following limit exists and therefore defines the precise representative of Du at the point x:*

$$\lim_{\mathtt{r}(B)\to 0}(Du)_{B(x)} =: Du(x)\,.$$

Some remarks are in order.

- The fundamentals of Nonlinear Potential Theory have been laid down in [49]. Another important reference is [29]. The first quantity appearing in the right-hand side of (2.2) is the truncated Riesz potential. Recall that the 1-Riesz potential operator $I_1(f)$ is defined by

$$I_1(f)(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|y-x|^{n-1}}\,dy\,.$$

- Theorem 1 connects some of the facts originally proved in [22,32,33,37,51]; earlier results are in [23]. Analog statements for the parabolic case are available in [34–36]. Similar estimates, but this time for u rather than for Du, are available in the fundamental work [30].
- Theorem 1 extends to the vectorial case $N > 1$ when $a(x, Du) \equiv |Du|^{p-2}Du$ with $p \geq 2$; this can be found in [40].
- Letting $R \to \infty$ in (2.2), and assuming a suitable decay at infinity of Du, i.e.,

$$\lim_{\mathrm{r}(B)\to\infty} \fint_B |Du|\, dy = 0$$

yields

$$|Du(x)|^{p-1} \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|y-x|^{n-1}}\, dy \;. \tag{2.4}$$

In particular, when $p = 2$, we obtain [51]

$$|Du(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|y-x|^{n-1}}\, dy \;,$$

which is the usual estimate valid for the standard Poisson equation $-\triangle u = f$. As a matter of fact, Theorem 1 implies that any gradient estimate for solutions to (1.1) can be reduced to that of the Poisson equation via the analysis of the action of the Riesz potentential on the various function spaces one is interested in. In turn, such action is perfectly known. The ultimate aim of Theorem 1 is to reduce some large parts of gradient regularity theory for nonlinear elliptic equations in divergence form to that of the Poisson equation via estimates as (2.4).
- Theorem 1 is essentially given in the form of a priori estimate for more regular solutions. Indeed, as assuming that $u \in W^{1,p}(\Omega)$ implies that $f \in W^{-1,p'}(\Omega)$, there is no reason to have that general solutions to (1.1), with $f \in L^1$, belong to $W^{1,p}$, and therefore are so called energy solutions. On the other hand, Theorem 1 can be easily extended to solutions to general measure data problems (when f in a measure), via certain refined approximation methods. This ultimately yields to consider a special class of distributional solutions called SOLA (Solutions Obtained by Limits of Approximations). These are obtained as limits of solutions to more regular problems, where the assumptions of Theorem 1 are in force. In general they are not energy solutions and their gradient belongs to L^q for every $q < n(p-1)/(n-1)$. For this procedure we refer to the original paper [6] and to [33,37,40] for more details concerning this specific context.
- A relevant extension to Theorem 1 is given by Baroni in [1], and this is about divergence form operators with non-polynomial growth, but that are still uniformly elliptic.

Let us now explain the catch between the space $L(n,1)$ and the NST from Section 1 on one side, and Theorem 1 on the other. Indeed, it is easy to see that $f \in L(n,1)$ implies condition (2.3); therefore the gradient continuity follows (see again [37]). This immediately implies the NST for the scalar case $N = 1$ and $p > 2 - 1/n$. Anyway, the arguments given in [33,37] can also be used as a starting point to get the full result for $N > 1$ and $p > 1$ [38]. A parabolic analog of the NST is contained in [39]; again Lorentz spaces can be used to provide optimal conditions for regularity. We notice that the first appearance of the condition $f \in L(n,1)$ to prove local Lipschitz regularity is in [21] for local estimates, and in [11] for global ones (see also next section). These two papers leave out the two-dimensional case $n = 2$.

2.2. **Global estimates and rearrangements.** Those in Section 2.1 are local estimates, as well as those implied by the proof of the NST given in [38]. The question of global Lipschitz regularity has been addressed in a series of paper both by Cianchi & Maz'ya, who concentrated, amongst the other things, on problems of the type

$$\begin{cases} -\operatorname{div}(\tilde{a}(|Du|)Du) = f & \text{in} \quad \Omega \\ u \equiv 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.5}$$

under the uniformly elliptic assumption

$$\begin{cases} -1 < i_{\rm a} \leq \dfrac{\tilde{a}'(t)t}{\tilde{a}(t)} \leq s_{\rm a} < \infty \quad \text{for every } t > 0 \\ \tilde{a}\colon (0,\infty) \to [0,\infty) \text{ is of class } C^1_{\rm loc}(0,\infty)\,. \end{cases} \tag{2.6}$$

These problems are naturally well-posed in the Orlicz space $W^1L^A(\Omega)$ defined by the function

$$A(t) := \int_0^t \tilde{a}(s)s\,ds\,. \tag{2.7}$$

Under condition $(2.6)_1$, this is essentially the spaces of functions w whose distributional derivatives are such that $A(|Dw|) \in L^1(\Omega)$. Obviously, the p-Laplacean operator from (1.3) is obtained a special by taking $\tilde{a}(t) \equiv t^{p-2}$ for $t > 0$. The following result is taken from [11, 12]:

Theorem 2. *Let $u \in W^1L^A(\Omega;\mathbb{R}^N)$ be a weak solution to (2.5), under assumptions (2.6) and $n \geq 3$. If $f \in L(n,1)(\Omega;\mathbb{R}^N)$ and Ω is a convex and bounded domain, then $Du \in L^\infty(\Omega)$. The convexity of Ω can be replaced by assuming that $\partial\Omega \in W^2L(n-1,1)$.*

Notice that this result holds in the vectorial case too. The condition $\partial\Omega \in W^2L(n-1,1)$ means that Ω is locally the subgraph of a function of $n-1$ variables, whose second-order distributional derivatives belong to the Lorentz space $L(n-1,1)$. We mention that an analog of Theorem 1 for equations as in (2.5) has been obtained in [1].

2.3. **Extended role $\boldsymbol{L(n,1)}$.** In the NST the space $L(n,1)$ plays a role when looking at data, while, following a usual custom in the literature, a Dini-modulus of continuity is prescribed on coefficients. Here we give an alternative criterion, trading the rate of continuity of coefficients with differentiability. For this we need a few preliminaries. We recall that the space $L^2(\log L)^{\mathfrak{a}}(\Omega;\mathbb{R}^N)$ with $\mathfrak{a} \geq 0$, consists of all the measurable maps $f\colon \Omega \to \mathbb{R}^N$ such that

$$f \in L^2(\log L)^{\mathfrak{a}}(\Omega;\mathbb{R}^N) \Longleftrightarrow \int_\Omega |f|^2 \log^{\mathfrak{a}}\big(\mathrm{e} + |f|\big)\,dx < \infty\,.$$

We use this space as a replacement of $L(n,1)$ when $n = 2$. Specifically, we define $\mathfrak{X}(\Omega)$ by

$$|f| \in \mathfrak{X}(\Omega) = \begin{cases} L(n,1)(\Omega) & \text{if } n > 2 \\ L^2(\log L)^{\mathfrak{a}}(\Omega)\,,\ \mathfrak{a} > 2 & \text{if } n = 2\,. \end{cases} \tag{2.8}$$

We then have [19]

Theorem 3. *Let $u \in W^1L^A_{\rm loc}(\Omega;\mathbb{R}^N)$ be a weak solution to*

$$-\operatorname{div}(\mathfrak{c}(x)\tilde{a}(|Du|)Du) = f\,, \quad 0 < \nu \leq \mathfrak{c}(\cdot) \leq L\,,$$

where $A(\cdot)$ is defined in (2.7), under assumptions (2.6). If $|f|, |D\mathfrak{c}| \in \mathfrak{X}(\Omega)$, then $Du \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$. Moreover, there exists a positive radius $R_* \equiv R_*(n,N,i_{\rm a},s_{\rm a},\mathfrak{c}(\cdot)) \leq 1$ such that if $B \Subset \Omega$ is a ball with $\boldsymbol{r}(B) \leq R_*$, then

$$\|A(|Du|)\|_{L^\infty(sB)} \leq \frac{c}{(1-s)^n[\boldsymbol{r}(B)]^n}\|A(|Du|)\|_{L^1(B)} + c\|f\|_{\mathfrak{X}(B)}^{\frac{i_{\rm a}+2}{i_{\rm a}+1}} + c \tag{2.9}$$

holds for every $s \in (0,1)$, where $c \equiv c(n,N,\nu,L,i_{\rm a},s_{\rm a})$.

This applies in particular to (1.3) by taking $\tilde{a}(t) \equiv \mathfrak{c}(x)t^{p-2}$. Already when $f \equiv 0$, this provides a new regularity criterion, which goes beyond the known and classical one in (1.5). Indeed, $D\mathfrak{c} \in L(n,1)$ implies that $\mathfrak{c}(\cdot)$ is continuous [54], but not necessarily with a modulus of continuity $\omega(\cdot)$ satisfying (1.5). We refer to [19] for more general results. Specifically, we can consider an autonomous system of the type $-\operatorname{div}(\tilde{a}(x,|Du|)Du) = f$; the degree of regularity of coefficients is then prescribed by requiring that $\partial_x \tilde{a}(x,\cdot) \in \mathfrak{X}(\Omega)$, coupled with suitable growth conditions with respect to the gradient variable z. We refer to Section 5 below for more results in this direction.

2.4. **What do we call nonuniform ellipticity?** When considering elliptic equations and systems, the most important quantity intervening when performing integral estimates is the ratio between the highest and the lowest eigenvalue of the operator in question. Specifically, adopting a notation that we shall keep on using for the rest of this paper, when considering an autonomous system of the type

$$-\operatorname{div} a(Du) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n \,, \tag{2.10}$$

it is natural to quantify its ellipticity by two functions $g_1, g_2 \colon (0,\infty) \to (0,\infty)$ as follows:

$$g_1(|z|)\mathbb{I}_{\rm d} \lesssim \partial_z a(z) \lesssim g_2(|z|)\mathbb{I}_{\rm d} \tag{2.11}$$

for all $z \in \mathbb{R}^{N\times n}$ with $|z| > 0$. It is then natural to introduce the *ellipticity ratio*

$$\mathcal{R}_a(z) := \frac{\text{highest eigenvalue of } \partial_z a(z)}{\text{lowest eigenvalue of } \partial_z a(z)} \,. \tag{2.12}$$

which is then bounded as

$$\mathcal{R}_a(z) \leq \frac{g_2(|z|)}{g_1(|z|)} \,,$$

in view of (2.11). The uniformly elliptic case now occurs when $\mathcal{R}_a(z)$ remains bounded with respect to $|z|$. More specifically, to the aim of proving local Lipschitz continuity of solutions, uniformly ellipticity occurs when

$$\sup_{|z|\geq T} \mathcal{R}_a(z) < \infty \tag{2.13}$$

for a fixed number $T > 0$. When this does not happen, we are in the nonuniformly elliptic case. In particular, we shall consider the situation when

$$\limsup_{|z|\to\infty} \frac{g_2(|z|)}{g_1(|z|)} = \infty \,. \tag{2.14}$$

Now, looking at the operator considered in Theorem 1, by $(2.1)_{2,3}$ it follows that $\mathcal{R}_a(z) \lesssim L/\nu$ whenever $|z| > 0$, we are therefore in the realm of uniform ellipticity. Similarly, in the case of (2.5), by letting $a(z) := \tilde{a}(z)z$, we notice that

$$\begin{cases} |\partial_z a(z)| \leq \sqrt{Nn}\max\{1, s_{\rm a}+1\}\tilde{a}(z) \\ \min\{1, i_{\rm a}+1\}\tilde{a}(|z|)|\xi|^2 \leq \partial_z a(z)\xi\cdot\xi \end{cases}$$

hold for every choice of $z, \xi \in \mathbb{R}^{N\times n}$, $|z| \neq 0$, and therefore

$$\mathcal{R}_a(z) \lesssim \frac{\max\{1, s_{\rm a}+1\}}{\min\{1, i_{\rm a}+1\}}$$

so that, also the case of (2.5) falls in the realm of uniform ellipticity. In the following, when referring to a vector field $a(\cdot)$ occurring in a Euler-Lagrange equation, i.e., when $a(\cdot) \equiv \partial_z F(\cdot)$ for some integrand F, we shall denote $\mathcal{R}_a \equiv \mathcal{R}_{\partial_z F} \equiv \mathcal{R}_F$, as done also in [19].

3. Nonuniform ellipticity: examples.

3.1. **Autonomous problems.** We shall now present a few classical examples of nonuniformly elliptic problems, going beyond the classical ones related to the minimal surface operator [44,53]. For this, we shall concentrate on equations and systems stemming from variational integrals as in (1.2). In many cases, uniformly elliptic integrands satisfy p-polynomial growth conditions of the type $|z|^p \lesssim F(x,z) \lesssim |z|^p + 1$, for some $p > 1$. More flexible conditions are $|z|^p \lesssim F(x,z) \lesssim |z|^q + 1$, where $q \geq p > 1$. These are known as (p,q)-growth conditions, after the basic work of Marcellini [46, 47], who systematically treated regularity problems for minimizer under such an assumption. Such functionals typically generate nonuniformly elliptic equations. They often show up with similar conditions on the Hessian of F, i.e.,

$$|z|^{p-2}\mathbb{I}_{\mathrm{d}} \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_{\mathrm{d}} \qquad \text{for } |z| \text{ large} .$$

A typical example in this respect is for instance

$$w \mapsto \int_\Omega \left[|Dw|^p + \sum_{i=1}^{n} |D_i w|^{q_i} - fw \right] dx\,, \qquad 1 < p < q_1, \ldots, q_n\,, \tag{3.1}$$

first considered in the pioneering work of Urdaletova & Uraltseva [57]. In this case a simple computation shows that the best bound one can get for $\mathcal{R}_F$ defined in (2.12) is

$$\mathcal{R}_F(z) \lesssim |z|^{q-p} + 1\,, \tag{3.2}$$

where $q := \max\{q_k\}$, so that nonuniform ellipticity occurs as long as all the exponents are not equal. In general, this is the best bound one can use in presence of (p,q)-growth conditions; see also Theorem 5 below.

The functional in (3.1), although being nonuniformly elliptic, still exhibits polynomial growth conditions in the gradient. There are anyway other types of nonuniformly elliptic functionals, having faster growth conditions. A basic instance is

$$w \mapsto \int_\Omega \exp(|Dw|^p)\, dx\,, \qquad p \geq 1\,, \tag{3.3}$$

considered in [20,45,48], or for example

$$w \mapsto \int_\Omega \left[\exp(A_0|Dw|^p) + \sum_{i=1}^{n} \exp(A_i|D_i w|^p) - fw \right] dx\,,$$

where again it is $p \geq 1$ and $0 < A_0 \leq A_1 \leq \ldots \leq A_n$. This last one, in a sense, combines the features of the functionals in (3.1) and (3.3). Even faster growth conditions can be considered by allowing arbitrary compositions of exponentials, i.e.,

$$w \mapsto \int_\Omega \left[\exp(\exp(\ldots\exp(|Dw|^p)\ldots)) - fw \right] dx\,, \qquad p \geq 1\,, \tag{3.4}$$

When considering the functional in (3.4), we have that [4, (6.13)]

$$\mathcal{R}_F(z) \lesssim t^{p-1}\exp(\exp(\ldots\exp(|z|^p)\ldots)) + 1 \tag{3.5}$$

where, if $k \geq 1$ is the number of composing exponentials involved in (3.4), the number in (3.5) is $k-1$ (it is zero in the case of (3.3)). Not surprisingly, $\mathcal{R}_F(z)$ grows proportionally to the growth of the functional.

3.2. **Nonautonomous problems.** In the previous examples, the nonuniform ellipticity of the functional stems directly from the way the corresponding integrand depends on the gradient variable. As a matter of fact, we have always considered autonomous functionals. Now, it is a distintive feature of several uniformly elliptic problems the possibility to recover the regularity of solutions to nonautonomous problems via perturbation from the autonomous case. Classical theories as Schauder's and Calderón & Zygmund's are some of the most classical outcomes of this procedure. There are anyway classes of nonautonomous functionals, for which pertubation methods not always work, but whose integrand is still uniformly elliptic when "freezing coefficients". This occurs because these functionals reveal to be ultimately nonuniformly elliptic, once nonuniform ellipticity is understood in a proper sense. An example is given by the double phase functional

$$w \mapsto \int_\Omega H(x, Dw)\,dx := \int_\Omega [|Dw|^p + a(x)|Dw|^q]\,dx \tag{3.6}$$

with $1 < p \leq q$, $0 \leq a(\cdot) \in L^\infty(\Omega)$. This has been originally introduced by Zhikov [58, 59] in the setting of Homogeneization. The basic regularity theory (for $f \equiv 0$) has been established in [3, 13, 14]; more recent contributions are in [17, 18]. Notice that, fixing $x_0 \in \Omega$ and defining the frozen integrand $H_0(z) := H(x_0, z)$, we find a uniformly elliptic integrand in the sense of Section (3.1), i.e.,

$$\mathcal{R}_{H_0}(z) \approx \frac{\max\{1, q-1\}}{\max\{1, p-1\}} \frac{|z|^{p-2} + a(x_0)|z|^{q-2}}{|z|^{p-2} + a(x_0)|z|^{q-2}} \lesssim 1\,.$$

On the other hand, even assuming that $a(\cdot)$ is Hölder continuous, as in Schauder theory, does not imply in general that minimizers are continuous. Indeed, as shown in [25, 28], already when $f \equiv 0$, local minima fail to be continuous if the ratio q/p is too far from 1, in dependence on the rate of Hölder continuity α. Specifically, the condition

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n}\,, \qquad a(\cdot) \in C^{0,\alpha}(\Omega)\,, \qquad \alpha \in (0, 1] \tag{3.7}$$

is necessary [25, 28] and sufficient [3, 16] to get gradient local Hölder continuity. Condition in (3.7) is typical when nonuniform ellipticity is directly generated by the presence of the x-variable as in (3.6). Therefore, in general, plain perturbation arguments - i.e., fixing a point x_0 and making small variations around it - does not work unless there is no enough continuity with respect to the x-variable. A more drastic example occurs when nonautonomous integrands have fast growth, as for instance

$$w \mapsto \int_\Omega \left[\mathfrak{c}_1(x)\exp(\mathfrak{c}_2(x)|Dw|^p) - f \cdot w\right]\,dx\,, \qquad p > 1\,,$$

where $0 < \nu \leq \mathfrak{c}_1(\cdot), \mathfrak{c}_2(\cdot) \leq L$. Such integrands fail to satisfy the so-called $\triangle_2$-condition, i.e., $\tilde{F}(x, 2t) \not\lesssim \tilde{F}(x, t)$. Therefore there is a loss of related integrability properties on minimizers as soon as one tries to make perturbations. In other words, $\exp(\mathfrak{c}_2(\cdot)|Dw|^p) \in L^1$ does not necessarily imply $\exp(\mathfrak{c}_2(x_0)|Dw|^p) \in L^1$ (this also happens in the case of the double phase functional). Exponential type functionals are classical in the Calculus of Variations starting by the work of Duc & Eells [20] and Marcellini [48]. In the nonautonomous version, they are treated for instance in the setting of weak KAM-theory, but only under very special assumptions and boundary conditions [27].

3.3. **Different measures of nonuniform ellipticity.** The discussion on the double phase functional in (3.6) leads to a different notion of nonuniform ellipticity, which is more tailored to nonautonomous problems. Specifically, for any given ball $B \subset \Omega$, we consider the nonlocal quantity

$$\mathfrak{R}_F(z,B) := \frac{\sup_{x\in B} \text{highest eigenvalue of } \partial_{zz}F(x,z)}{\inf_{x\in B} \text{lowest eigenvalue of } \partial_{zz}F(x,z)} \,, \tag{3.8}$$

which is more efficient to quantify nonuniform ellipticity than for instance the obvious pointwise version of (2.12)

$$\mathfrak{R}_F(x,z) := \frac{\text{highest eigenvalue of } \partial_{zz}F(x,z)}{\text{lowest eigenvalue of } \partial_{zz}F(x,z)} \,. \tag{3.9}$$

Notice that using this last quantity would indeed again qualify the functional in (3.6) as uniformly elliptic. As a matter of fact, it is the quantity in (3.8) which reveals to play a key role when performing integral estimates for nonautonomous problems, rather than the smaller one in (3.9). Looking back at (3.6), and considering a ball B such that $B \cap \{a(x) = 0\} \neq 0$, we have that $\mathfrak{R}_H(z,B) \approx |B|^{1/n-1/d}|z|^{q-p} + 1$, therefore $\mathfrak{R}_H(z,B) \to \infty$ when $|z| \to \infty$, so that, the functional in (3.6) is back to the realm of nonuniformly elliptic integrals. Moreover, condition (5.22) corrects the growth of $\mathfrak{R}_H(z,B)$ with respect to $|z|$ with the smallness of $|B|$. This explains the occurrence of the bound in (3.7): when considering small balls touching $\{a(x) = 0\}$ the largeness of α helps compensating the growth of $\mathfrak{R}_H(z,B$ with respect to $|z|$.

The functional in (3.6) is not the only one for which nonuniform ellipticity directly connects to presence of coefficients. Another classical example is given by the variable growth exponent functional

$$w \mapsto \int_\Omega |Dw|^{p(x)}\,dx\,, \qquad p(x) > 1\,, \tag{3.10}$$

for which a wide literature has now been developed starting by the fundamental papers of Zhikov [58, 59]. Here it is

$$\mathfrak{R}_F(z,B) \approx |z|^{p_M(B)-p_m(B)}\,,$$

where $p_M(B)$ and $p_m(B)$ are the sup and the inf of $p(\cdot)$ in the ball B, respectively. In this case the balancing condition implying that minimizers are not discontinuous looks like

$$\lim_{\mathtt{r}(B)\to 0} [p_M(B) - p_m(B)] \log\left(\frac{1}{\mathtt{r}(B)}\right) < \infty\,.$$

This condition plays the role of (3.7) when dealing with the variable growth case: again, small oscillations of $p(\cdot)$ help balancing the growth of the ellipticity ratio with respect to $|z|$. It is a weaker condition in fact, and also reflects in that the transition from one growth exponent to another is sharper in (3.6) than in (3.10). More details on this can be found in [50] for the regularity viewpoint. See also [15, 41, 52] for function spaces related to such functionals.

The analogies between the functionals in (3.6) and (3.10) from the point of view of regularity theory have ben noted in [2], and a general approach has been suggested in [3]. More recently, a unified approach to regularity for certain classes of nonautonomous functionals has been proposed in the interesting paper [42]; this catches up both the functionals in (3.6) and (3.10), as also other model cases considered in the literature.

4. Nonuniform ellipticity: regularity in the scalar case.

In this section we concentrate on scalar functionals of the type

$$w \mapsto \int_\Omega \left[F(Dw) - f\cdot w\right]\,dx\,, \tag{4.1}$$

therefore considering an autonomous integrand $F(\cdot)$. We shall present a general result taken from [4], with a few consequences. The standing assumption will be that $F(\cdot)$ is convex on $\mathbb{R}^n$ and it is locally C^2-regular in $\{|z| \geq T\} \subset \mathbb{R}^n$, where $T > 0$ is a fixed number. We next pass to describe the minimal assumptions qualifying $F(\cdot)$ as elliptic. For this, as in (2.11), we use two locally bounded, measurable functions $g_1, g_2\colon (0,\infty) \to [0,\infty)$, aimed at controlling the lower and the upper eigenvalues of $\partial^2 F(z)$, respectively, when $|z| \geq T$. They are continuous on $[T,\infty)$ and such that $g_1(T), g_2(T) > 0$. Ellipticity is now quantified by assuming that

$$|\partial_{zz} F(z)| \leq g_2(|z|) \quad \text{and} \quad g_1(|z|)|\xi|^2 \leq \partial_{zz} F(z)\xi \cdot \xi \tag{4.2}$$

hold for every $\xi \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ such that $|z| \geq T$. As a minimal requirement on g_1 and g_2, we assume that $[T,\infty) \ni t \mapsto g_2(t)/g_1(t)$ and $t \mapsto g_1(t)t$ are almost non-decreasing and non-decreasing, respectively. This means that

$$T \leq s \leq t \quad \Longrightarrow \quad \frac{g_2(s)}{g_1(s)} \leq c_{\rm a} \frac{g_2(t)}{g_1(t)} \quad \text{and} \quad g_1(s)s \leq g_1(t)t\,, \tag{4.3}$$

hold for some constant $c_{\rm a} \geq 1$. Finally, we assume that

$$\begin{cases} \nu(t^2+\mu^2)^{\tau/2} \leq g_1(t) \quad \text{for } t \geq T\,, \text{ for some } \tau > -1 \\[2mm] \displaystyle\int_T^{|z|} g_1(s)s\,ds \leq F(z) \quad \text{for } |z| \geq T\,, \end{cases} \tag{4.4}$$

where $\nu > 0$ and $0 \leq \mu \leq 1$ are fixed constants. We then have the following [4]:

Theorem 4. *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a minimizer of the functional in (4.1) under the assumptions (4.2)-(4.4). Assume that $f \in \mathfrak{X}(\Omega)$ and that the inequality*

$$\frac{g_2(t)}{g_1(t)} \leq c_{\rm b} \min\left\{ \left(\int_T^t g_1(s)s\,ds\right)^{\frac{2-\sigma}{n}}, \left(\frac{1}{t^{1/\beta_1}}\int_T^t g_1(s)s\,ds\right)^{\frac{4\beta_1}{n-2}} \right\} + c_{\rm b} \tag{4.5}$$

holds for every $t \geq T$, for some σ with $0 < \sigma \leq 2$, and some fixed positive constants $\beta_1 < 1$, $c_{\rm b} \geq 1$. Then Du is locally bounded in Ω. Moreover, when $n > 2$, the estimate

$$\begin{aligned} \int_T^{\|Du\|_{L^\infty(B/2)}} g_1(s)s\,ds &\leq c\left(\fint_B F(Du)\,dx + \|f\|_{L^n(B)}^{\frac{\gamma}{\gamma-1}} + T^\gamma + 1\right)^{2/\sigma} \\ &\quad + c\|f\|_{L(n,1)(B)}^{\frac{\tau+2}{\tau+1}} + c\|f\|_{L(n,1)(B)}^{\frac{1}{1-\beta_1}} + c(T+\mu)\|f\|_{L(n,1)(B)} \end{aligned} \tag{4.6}$$

holds for every ball $B \Subset \Omega$, for a constant c depending only on $n, \nu, \tau, c_{\rm a}, c_{\rm b}, \sigma, \beta_1$, but otherwise independent of T. Finally, when either $f \equiv 0$ or $n = 2$, assumption (4.5) can be replaced by

$$\frac{g_2(t)}{g_1(t)} \leq c_{\rm b}\left(\int_T^t g_1(s)s\,ds\right)^{\frac{2-\sigma}{n}} + c_{\rm b} \qquad \forall\ t \geq T\,. \tag{4.7}$$

Again for $n = 2$, an estimate similar to (4.6) holds replacing $\|f\|_{L(n,1)(B)}$ by $\|f\|_{L^2(\log L)^{\mathfrak{a}}(B)}$.

Theorem 4 offers a unified approach to nonunifornly elliptic problems, catching, for instance, both functionals with (p,q)-growth and those with exponential one. For instance, we report a corollary concerning the first type; in this case, we consider an integrand $F\colon \mathbb{R}^n \to \mathbb{R}$ which is a convex function and locally C^2-regular in $\mathbb{R}^n \setminus \{0\}$. The crucial growth and ellipticity properties of F are described as follows:

$$\begin{cases} \nu(|z|^2+\mu^2)^{p/2} \leq F(z) \leq L(|z|^2+\mu^2)^{q/2} + L(|z|^2+\mu^2)^{p/2} \\ (|z|^2+\mu^2)|\partial^2 F(z)| \leq L(|z|^2+\mu^2)^{q/2} + L(|z|^2+\mu^2)^{p/2} \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \leq \partial^2 F(z)\xi\cdot\xi\,, \end{cases} \tag{4.8}$$

for every choice of $z, \xi \in \mathbb{R}^n$ with $z \neq 0$ and for exponents $1 < p \leq q$. Here $0 < \nu \leq 1 \leq L$ are fixed ellipticity constants and $\mu \in [0,1]$ serves to distinguish the so-called degenerate case

($\mu = 0$) and non-degenerate case ($\mu > 0$). Applied to this setting, with the choice

$$\begin{cases} g_1(t) \approx (t^2+\mu^2)^{(p-2)/2} \\ g_2(t) \approx (t^2+\mu^2)^{(q-2)/2} + (t^2+\mu^2)^{(p-2)/2} \end{cases} \tag{4.9}$$

Theorem 4 then yields the following:

Theorem 5. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional in (4.1) under assumptions (4.8) with $1 < p \leq q$ and $n > 2$. Assume*

$$\frac{q}{p} < 1 + \min\left\{\frac{2}{n}, \frac{4(p-1)}{p(n-2)}\right\} \qquad \text{and} \qquad f \in L(n,1)(\Omega)\,. \tag{4.10}$$

Then Du is locally bounded in Ω. Moreover, the local a priori estimate

$$\begin{aligned} \|Du\|_{L^\infty(B/2)} \quad &\leq \quad c\left(\fint_B F(Du)\,dx + \|f\|^{\frac{p}{p-1}}_{L(n,1)(B)} + 1\right)^{\frac{2}{(n+2)p-nq}} \\ &\qquad + c\|f\|^{\frac{4}{4(p-1)-(n-2)(q-p)}}_{L(n,1)(B)} \end{aligned} \tag{4.11}$$

holds for a constant $c \equiv c(n,p,q,\nu,L)$, whenever $B \Subset \Omega$ is a ball. When $p \geq 2 - 4/(n+2)$ or when $f \equiv 0$, condition (4.10) can be replaced by

$$\frac{q}{p} < 1 + \frac{2}{n}\,. \tag{4.12}$$

For the two-dimensional case, we instead have

Theorem 6. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional in (1.2) under assumptions (5.15) with $1 < p \leq q$ and $n = 2$. Assume that $q < 2p$ and that $f \in \mathrm{L}^2(\mathrm{Log\ L})^{\mathfrak{a}}(\Omega)$ holds form some $\alpha > 2$. Then Du is locally bounded in Ω.*

Again, we list some remarks.

- Considering for simplicity the case $f \equiv 0$, we see that in Theorem 4 the crucial assumption is (4.7). In view of the definition of $\mathcal{R}_F$ in (2.12), (4.7) implies

$$\mathcal{R}_F(z) \lesssim \left(\int_T^{|z|} g_1(s)s\,ds\right)^{\frac{2-\sigma}{n}} + 1 \qquad \forall\, t \geq T\,. \tag{4.13}$$

  On the right-hand side in the above display there appears the energy function bounded by the integrand (see $(4.4)_2$), that is

$$G(t) := \int_T^{\max\{t,T\}} g_1(s)s\,ds\,,$$

  and therefore conditions as (4.7) and (4.13) propose a balance between the coercivity and the ellipticity ratio. We shall see that what it really matters to get regularity is the ratio

$$R(z) := \frac{\mathcal{R}_F(z)}{G(|z|)+1} \tag{4.14}$$

  for $|z|$ large. In particular, analysing this quantity will explain why, although exhibiting a larger ellipticity ratio, exponential type functionals require less stringent assumptions in the nonautonomous case. See Remark 1 below.

- Condition (4.12) used in the setting of (p,q)-growth conditions, and therefore with the choice in (4.9), reads as
$$t^{q-p} \approx \frac{g_2(t)}{g_1(t)} \lesssim \left(\int_T^t g_1(s)s\,ds\right)^{\frac{2-\sigma}{n}} \approx t^{\frac{p(2-\sigma)}{n}}\,.$$
This leads to the bound in (4.12) as σ is allowed to be arbitrarily small. In turn, this is the same bound originally found by Marcellini [47] for $f \equiv 0$. It is worth mentioning that, still in the purely autonomous case, Bella & Schäffner [5] have recently proposed a method to scale (4.12) of one dimension, that is, they are able to use condition
$$\frac{q}{p} < 1 + \frac{2}{n-1} \qquad \text{for } n \geq 3\,.$$
- Theorem 4 applied to the p-Laplacean equation $-\operatorname{div}(|Du|^{p-2}Du) = f$ (take $p = q$) gives the local estimate
$$\|Du\|_{L^\infty(B/2)} \lesssim \left(\fint_B |Du|^p\,dx\right)^{\frac{1}{p}} + \|f\|_{L(n,1)(B)}^{\frac{1}{p-1}}\,,$$
see also [4, Section 6.1] for the derivation. This is nothing but the classical $L^\infty - L^p$ estimate for p-harmonic functions [37] when $f \equiv 0$. For $f' \neq 0$, it is the estimate derived in [21, 33, 40].
- In the genuine (p,q)-case, for $f \equiv 0$ estimate (4.11) becomes
$$\|Du\|_{L^\infty(B/2)} \lesssim \left(\fint_B F(Du)\,dx\right)^{\frac{2}{(n+2)p-nq}} + 1\,.$$
which is exactly the local estimate found by Marcellini in [47].
- Assumptions (4.2) prescribe ellipticity only outside the set $\{|z| \geq T\}$. This allows to deal with problems that are completely degenerate on $\{|z| < T\}$, where, in fact, ellipticity properties becomes irrelevant. Only convexity of F is assumed globally.

Theorem 4 applies to functionals with fast growth conditions as well. For instance, it implies that minimizers of functionals as (3.3) and (3.4) are locally Lipschitz. It also applies to the functional in (3.5), where, for $f \in \mathfrak{X}(\Omega)$, conditions in (4.7) translate in
$$\frac{A_n}{A_0} < 1 + \frac{2}{n}\,.$$
We are going to give more room to fast growth functionals in Section 5.2 below.

5. Nonautonomous functionals and the vectorial setting

Here we concentrate on the vectorial case $N > 1$ and on general functionals of the type in (1.2). We shall present some results from [19]. The standing assumption will be that F has radial structure, i.e., there exists $\tilde{F} : \Omega \times [0,\infty) \to [0,\infty)$ such that

$$\begin{cases} F(x,z) = \tilde{F}(x,|z|) \quad \text{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n} \\ t \mapsto \tilde{F}(x,t) \in C^1_{\rm loc}[0,\infty) \cap C^2_{\rm loc}(0,\infty) \quad \text{for all } x \in \Omega \\ x \mapsto \tilde{F}'(x,t) \in W^{1,n}(\Omega) \quad \text{for every } t > 0\,. \end{cases} \tag{5.1}$$

This particular structure, whose effective use in the setting of regularity goes back to the work of Uhlenbeck [55], is an additional assumption typical for the vectorial case. It serves to rule out singularities otherwise occurring no matter the degree of smoothness is assumed on F. No difference in the autonomous case. For this see for instance [50]. Assumption $(5.1)_3$ is one of the main peculiarities here. As a way to prescribe a reinforced Hölder continuity on coefficients, we impose various types of differentiability assumptions, in turn implying Hölder continuity by Sobolev-Morrey embedding theorem. As already done in Section 4, in order to describe the ellipticity and growth properties of the integrand F, we use three locally bounded functions $g_1, g_2, g_3 \colon \Omega \times (0,\infty) \to [0,\infty)$. The first two g_1, g_2 are continuous and, as in Section 4, bound the lowest and the largest eigenvalues of $\partial_{zz}F$, respectively. The last one, g_3, is Carathéodory regular and controls the growth of mixed derivatives. Therefore, similarly to Theorem 4, we assume that there exists $T > 0$ such that

$$\begin{cases} z \mapsto F(x,z) & \text{is convex for all } x \in \Omega \\ |\partial_{zz}F(x,z)| \le g_2(x,|z|) & \text{for all } x \in \Omega \text{ on } \{|z| \ge T\} \\ g_1(x,|z|)|\xi|^2 \le \partial_{zz}F(x,z)\xi \cdot \xi & \text{on } \{|z| \ge T\} \text{ and for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{N\times n} \\ |\partial_x \tilde{F}'(x,t)| \le h(x) g_3(x,t) & \text{for } (x,t) \in \Omega \times (0,\infty) \ , \end{cases} \tag{5.2}$$

where $0 \le h(\cdot) \in L^n(\Omega)$ and we assume also that $\inf_{x\in\Omega} g_1(x,T) > 0$. With

$$\tilde{a}(x,t) := \frac{\tilde{F}'(x,t)}{t} \qquad (x,t) \in \Omega \times (0,\infty), \tag{5.3}$$

we assume, for fixed numbers $\gamma > 1$, $\mu \in [0,1]$, and for every $x \in \Omega$, the minimal γ-superlinear growth of the lowest eigenvalue of $\partial_{zz}F(x,\cdot)$, that is

$$t \mapsto \frac{\tilde{a}(x,t)}{(t^2+\mu^2)^{\frac{\gamma-2}{2}}} \quad \text{and} \quad t \mapsto \frac{g_1(x,t)}{(t^2+\mu^2)^{\frac{\gamma-2}{2}}} \quad \text{are non-decreasing on } (0,\infty)\,. \tag{5.4}$$

We finally assume that the ratio g_2/g_1 is almost non-decreasing with respect to t, i.e.,

$$T \le s < t \Longrightarrow \frac{g_2(x,s)}{g_1(x,s)} \le c_{\mathrm{a}} \frac{g_2(x,t)}{g_1(x,t)} \tag{5.5}$$

holds for all $x \in \Omega$ and some fixed constant $c_{\mathrm{a}} \ge 1$. For every $(x,t) \in \Omega \times (0,\infty)$, we set

$$\bar{G}(x,t) := \int_T^{\max\{T,t\}} g_1(x,s)s\,ds + (T^2+1)^{\gamma/2}\,.$$

Then, we consider numbers $d, \sigma, \hat{\sigma} \ge 0$ such that

$$h(\cdot) \in L^d(\Omega)\,, \qquad d > n \tag{5.6}$$

and

$$\sigma + \hat{\sigma} < \begin{cases} \min\left\{\frac{1}{n} - \frac{1}{d}, \frac{4}{\vartheta(n-2)}\left(1 - \frac{1}{\gamma}\right)\right\} & \text{if} \quad n > 2 \\ \min\left\{\frac{1}{2} - \frac{1}{d}, \frac{2}{\vartheta}\left(1 - \frac{1}{\gamma}\right)\right\} & \text{if} \quad n = 2\,. \end{cases} \tag{5.7}$$

Here it is $\vartheta \equiv \vartheta(\gamma) = 1$ when $\gamma \ge 2$, and $\vartheta = 2$ otherwise. We assume that $x \mapsto g_1(x,t) \in W^{1,d}(\Omega)$ for all $t \ge T$ and that $\partial_x g_1(\cdot)$ is Carathéodory regular on $\Omega \times [T,\infty)$. Finally, for every $(x,t) \in \Omega \times [T,\infty)$, and for a fixed constant $c_{\mathrm{b}} \ge 1$, we assume that

$$|\partial_x g_1(x,t)| \le h(x)[\bar{G}(x,t)]^{\hat{\sigma}} g_1(x,t)\ , \tag{5.8}$$

$$g_3(x,t)\sqrt{t^2+\mu^2} \le c_{\mathrm{b}}[\bar{G}(x,t)]^{1+\sigma} \quad \text{and} \quad \frac{[g_3(x,t)]^2}{g_1(x,t)} \le c_{\mathrm{b}}[\bar{G}(x,t)]^{1+2\sigma}\ , \tag{5.9}$$

$$\frac{g_2(x,t)}{g_1(x,t)} \le c_{\mathrm{b}}[\bar{G}(x,t)]^{\sigma}\ . \tag{5.10}$$

Gathering the various parameters as in `data` $:= (n, N, \nu, \gamma, T, c_{\mathrm{a}}, c_{\mathrm{b}}, d, \sigma, \hat{\sigma}, \mathfrak{a})$, we have

Theorem 7. *Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of $\mathcal{F}$ in (1.2), under assumptions (5.1)-(5.5) and (5.6)-(5.10). If $f \in \mathfrak{X}(\Omega)$, then $Du \in L^{\infty}_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$. Moreover, there exists a positive radius $R_* \equiv R_*(\texttt{data}, f(\cdot)) \le 1$ such that if $B \Subset \Omega$ is a ball with $\mathbf{r}(B) \le R_*$, then*

$$\sup_{x\in sB} \int_T^{\max\{T,|Du(x)|\}} g_1(x,s)s\,ds \le \frac{c}{(1-s)^{\beta}[\mathbf{r}(B)]^{\beta}}\left[\|F(\cdot,Du)\|^{\theta}_{L^1(B)} + \|f\|^{\theta}_{\mathfrak{X}(B)} + 1\right] \tag{5.11}$$

holds for every $s \in (0,1)$, where $c \equiv c(\texttt{data}, \|h\|_{L^d(B)}) \ge 1$, $\beta,\theta \equiv \beta,\theta(n,\gamma,d,\sigma,\hat\sigma) > 0$. When either $\gamma \ge 2$ or $f \equiv 0$, the upper bound on $\sigma+\hat\sigma$ in (5.7) can be replaced by $\sigma + \hat\sigma < 1/n - 1/d$.

Looking back at the discussion made in Section 3.3, we see that (5.10) serves to bound the quantity in (3.9) as

$$\mathfrak{R}_F(x,z) \lesssim \frac{g_2(x,t)}{g_1(x,t)} \lesssim [\bar G(x,t)]^{\sigma} \,. \tag{5.12}$$

This, together with (5.8)-(5.9), allows in fact to get an indirect control on the quantity $\mathfrak{R}_F(z,B)$ in (3.8). Despite the technicalities involved in its formulation, Theorem 7 allows to treat, in a unified way, very different classes of nonautonomous functionals, covering essentially all the main model cases appearing in the literature under sharp assumptions. This is due to the flexibility of conditions (5.8)-(5.10) and the possibility of playing with two different parameters σ and $\tilde\sigma$ to control various types of nonuniform ellipticity. An example of this phenomenon is in the next general result, still from [4]:

Theorem 8. *Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of $\mathcal{F}$ in (1.2), under assumptions (5.1)-(5.5), (5.6) and (5.8)-(5.9), with*

$$\sigma+\hat\sigma < \begin{cases} 1/n-1/d & \text{if } \ n>2 \\ \min\{1/2-1/d, \gamma-1\} & \text{if } \ n=2 \end{cases} \qquad \text{and} \qquad \frac{g_2(x,t)}{g_1(x,t)} \le c_{\rm b}\,. \tag{5.13}$$

If $f \in \mathfrak{X}(\Omega)$, then $Du \in L^{\infty}_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$ and (5.11) holds.

Notice that the second condition in (5.13) implies

$$\mathfrak{R}_F(x,z) \lesssim 1\,, \tag{5.14}$$

and therefore we are assuming some sort of uniform ellipticity in a pointwise sense. This does not exclude that the quantity $\mathfrak{R}_F(z,B)$ defined in (3.8) might blow-up, and its control stems from a combination of (5.14) and (5.8)-(5.9). This implies a better bound (5.13) and it is exactly the situation of the double phase functional; see Theorem 9 below.

5.1. **Corollaries in the polynomial growth case.** We again consider functionals of the type (1.2), this time with (p,q)-growth conditions. Specifically, we assume that

$$\begin{cases} F(x,z) = \tilde F(x,|z|) \quad \text{for all } (x,z) \in \Omega\times\mathbb{R}^{N\times n} \\ \nu(|z|^2+\mu^2)^{p/2} \le F(x,z) \le \Lambda(|z|^2+\mu^2)^{q/2} + \Lambda(|z|^2+\mu^2)^{p/2} \\ (|z|^2+\mu^2)|\partial_{zz}F(x,z)| \le \Lambda(|z|^2+\mu^2)^{q/2} + \Lambda(|z|^2+\mu^2)^{p/2} \\ \nu(|z|^2+\mu^2)^{(p-2)/2}|\xi|^2 \le \partial_{zz}F(x,z)\xi\cdot\xi\,, \end{cases} \tag{5.15}$$

hold for every choice of $z,\xi \in \mathbb{R}^n$ such that $|z| \neq 0$. Here $0 < \nu \le 1 \le \Lambda$ and $\mu \in [0,1]$ are fixed constants, and $t \mapsto \tilde F(x,t) \in C^1_{\rm loc}[0,\infty) \cap C^2_{\rm loc}(0,\infty)$ for $x \in \Omega$. We moreover assume that

$$t \mapsto \frac{\tilde F'(x,t)}{(t^2+\mu^2)^{(p-2)/2}t} \quad \text{is non-decreasing} \tag{5.16}$$

for every $x \in \Omega$. As for the crucial dependence on x, we assume that for every $t \geq 0$ it is $x \mapsto \tilde{F}'(x,t) \in W^{1,d}(\Omega)$ and that

$$|\partial_x \tilde{F}'(x,t)| \leq h(x)\left[(t^2+\mu^2)^{(q-1)/2}+(t^2+\mu^2)^{(p-1)/2}\right], \quad h(\cdot) \in L^d(\Omega)\,, \; d>n \tag{5.17}$$

holds for $x \in \Omega$ and $t \geq 0$. These conditions allow to verify the assumptions of Theorem 7 with the same choice made in (4.9) and $g_3(t) \approx (t^2+\mu^2)^{(q-1)/2}+(t^2+\mu^2)^{(p-1)/2}$ (yes, they are still x-independent) provided we assume the bound

$$\frac{q}{p} < 1+\min\left\{\frac{1}{n}-\frac{1}{d}, \mathfrak{m}_p\right\} \quad \text{with} \;\; \mathfrak{m}_p := \begin{cases} \frac{4(p-1)}{\vartheta p(n-2)} & \text{if} \quad n \geq 3 \\ \frac{2(p-1)}{\vartheta p} & \text{if} \quad n=2\,, \end{cases} \tag{5.18}$$

where $\vartheta = 1$ if $p \geq 2$ and $\vartheta = 2$ otherwise. Then Theorem 7 provides

Theorem 9. *Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of $\mathcal{F}$ in (1.2), under assumptions (5.15)-(5.18). Then $Du \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$. When either $p \geq 2$ or $f \equiv 0$, (5.18) is replaced by*

$$q/p \leq 1+1/n-1/d\,. \tag{5.19}$$

In particular, when considering splitting integrands as

$$w \mapsto \int_\Omega \big[\mathfrak{c}(x)F(Dw)-f\cdot w\big]\,dx\,, \qquad 0<\nu\leq \mathfrak{c}(\cdot)\leq L\,, \tag{5.20}$$

Theorem 9 gives that minimizers are locally Lipschitz continuous provided $\mathfrak{c}(\cdot) \in W^{1,d}(\Omega)$ for some $d>n$ and such that (5.18) is satisfied, and with F satisfying (5.15) and (5.16).

Similarly to (5.20), for double phase functionals

$$w \mapsto \int_\Omega [|Dw|^p + a(x)|Dw|^q - f\cdot w]\,dx\,, \tag{5.21}$$

condition (5.17) again amounts to assume that $a(\cdot)$ is Sobolev regular:

Theorem 10. *Let $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional in (5.21), such that $0 \leq a(\cdot) \in W^{1,d}(\Omega)$ and that (2.8) holds together with*

$$q/p \leq 1+1/n-1/d\,, \;\; \text{if } n\geq 2 \;\; \text{and, when } n=2, \text{ also } q/p<p\,. \tag{5.22}$$

Then $Du \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times n})$.

Let's test the sharpness of assumptions (5.17) and (5.19). Sobolev-Morrey embedding yields $a(\cdot) \in C^{0,\alpha}$ with $\alpha = 1-n/d$. By this we find that conditions (3.7) and (5.19) coincide. In turn, (3.7) is sharp by the counterexamples in [25, 28]. Therefore, (5.17) is the sharp differentiable version of (3.7), which is stronger than (3.7), but weaker than assuming that $a(\cdot)$ is Lipschitz, as it is usually done in the literature.

We remark that using Sobolev regularity on coefficients is a natural approach that has also been considered elsewhere. See for instance the paper [31] in the case of uniformly elliptic integrals. As for the nonuniformly elliptic setting, this approach has been used for the first time in [24]. Indeed, the main result of [24] is covered by Theorem 9.

5.2. **Corollaries in the fast growth case.** Here we report another corollary of Theorem 7 that this time covers functionals (3.3) and (3.4). For this, we fix sequences of exponent functions $\{p_k(\cdot)\}$ and coefficients$\{\mathfrak{c}_k(\cdot)\}$, all defined on the open subset $\Omega \subset \mathbb{R}^n$, such that

$$\begin{cases} 1<p_{\mathfrak{m}}\leq p_0(\cdot)\leq p_M\,, & 0<p_m\leq p_k(\cdot)\leq p_M\,, \quad \text{for } k\geq 1 \\ 0<\nu\leq \mathfrak{c}_k(\cdot)\leq L\,, & p_k(\cdot), \mathfrak{c}_k(\cdot) \in W^{1,d}(\Omega)\,, \quad d>n\,, \quad \text{for } k \geq 0\,. \end{cases} \tag{5.23}$$

We then inductively define, for every $k \in \mathbb{N}$, the functions $\mathbf{e}_k \colon \Omega \times [0,\infty) \to \mathbb{R}$ as

$$\begin{cases} \mathbf{e}_{k+1}(x,t) & := \exp\left(\mathfrak{c}_{k+1}(x)\left[\mathbf{e}_k(t)\right]^{p_{k+1}(x)}\right) \\ \mathbf{e}_0(x,t) & := \exp\left(\mathfrak{c}_0(x)t^{p_0(x)}\right) \,, \end{cases}$$

and consider functionals defined by

$$w \mapsto \int_\Omega \left[\mathbf{e}_k(x,|Dw|) - fw\right]\, dx \,. \tag{5.24}$$

These kinds of variational integrals have been considered at length in the literature and they provide a chief case study to test how much it is possible to relax the standard uniform ellipticity assumptions; see [4, 48] and related references. Again as a corollary of Theorem 7, we then have [19]

Theorem 11. *Let $u \in W^{1,1}_{\operatorname{loc}}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional in (5.24) for some $k \in \mathbb{N}$, under assumptions (5.23) and such that f satisfies (2.8). Then $Du \in L^\infty_{\operatorname{loc}}(\Omega;\mathbb{R}^{N\times n})$.*

Remark 1 (Renormalization of the ellipticity ratio)**.** Comparing (3.2) and (3.5), one would expect that requirements on the integrability of $\partial_{zx}F$ would strengthen in the exponential case, with respect to the (p,q)-growth case. Here it is actually the opposite. In the exponential case we require $\partial_{zx}F \in L^q$ for any $q > n$, while in the second one q must be large enough in order to meet at least condition (5.19). This counterintuitive fact is actually linked to a relevant quantity implicitly appearing in the estimates. This is the renormalized ratio already introduced in (4.14), that in our case looks like

$$R(z,B) := \sup_{x\in B} \frac{\mathcal{R}_F(x,z)}{\bar{G}(x,|z|)}$$

for a fixed ball $B \subset \Omega$, where $\mathcal{R}_F(x,z)$ has been defined in (3.9). Now, an assumption as (5.10), and therefore (5.12), serves to make $R(z,B) \to 0$ fast enough, when $|z| \to \infty$. At this point, the rate of convergence is exponential in the case of fast growth functionals as (5.10), while is only polynomial in the case of (p,q)-growth conditions. This explains the stronger requirement on the integrability of coefficients in the polynomial growth case. In a way, comparing this situation to what happens in Theorem 3, the case of functionals with exponential growth reveals to be closer to that of uniformly elliptic functionals.

6. Obstacle problems

The approach developed in [19] allows to get new and sharp results on obstacle problems. We consider the functional

$$\mathcal{F}_0(w;\Omega) := \int_\Omega F(x,Dw)\, dx \tag{6.1}$$

defined on $W^{1,1}(\Omega)$, where F is for instance one of the integrands considered in Theorems 9-3; here we of course consider the scalar case $N = 1$. Next we consider a measurable function $\psi \colon \Omega \to \mathbb{R}$ and the convex set $\mathcal{K}_\psi(\Omega) := \{w \in W^{1,1}_{\operatorname{loc}}(\Omega) \colon w(x) \geq \psi(x) \text{ for a.e. } x \in \Omega\}$. We then say that a function $u \in W^{1,1}_{\operatorname{loc}}(\Omega) \cap \mathcal{K}_\psi(\Omega)$ is a constrained local minimizer of $\mathcal{F}_0$ if, for every open subset $\tilde{\Omega} \Subset \Omega$, we have $\mathcal{F}_0(u;\tilde{\Omega}) < \infty$ and if $\mathcal{F}_0(u;\tilde{\Omega}) \leq \mathcal{F}_0(w;\tilde{\Omega})$ holds for every competitor $w \in u + W^{1,1}_0(\tilde{\Omega})$ such that $w \in \mathcal{K}_\psi(\Omega)$. We then have

Theorem 12. *Let $u \in W^{1,1}_{\operatorname{loc}}(\Omega)$ be a constrained local minimizer of $\mathcal{F}_0$ in (6.1), where $F \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is one of the integrands from Theorems 9 and 11 (with $p \geq 2$ in the setting of Theorem 9). It follows that $\psi \in W^{2,1}_{\operatorname{loc}}(\Omega)$ and $|D^2\psi| \in \mathfrak{X}(\Omega)$ imply $Du \in L^\infty_{\operatorname{loc}}(\Omega;\mathbb{R}^n)$.*

An analogous result holds in the setting of Theorem 3 (with $i_{\rm a} \geq 0$) and

$$\tilde{F}(x,t) := \mathfrak{c}(x) \int_0^t \tilde{a}(s)s\,ds\,, \qquad 0 < \nu \leq \mathfrak{c}(\cdot) \leq L\,. \tag{6.2}$$

This implies completely new results already in the case of uniformly elliptic functionals (6.2), and even in the one involving the classical p-Laplacean functional. In this last case Theorem 12 offers a new sharp criterion for Lipschitz continuity, which is alternative to those given in [8, 9], where it is assumed that $D\psi$ is locally Hölder continuous. Here we trade this last condition with $|D^2\psi| \in \mathfrak{X}(\Omega)$, that in turn implies the mere continuity of $D\psi$ by [54]. This is essentially the same phenomenon seen in Theorem 3, where the condition $|D\mathfrak{c}| \in \mathfrak{X}(\Omega)$ replaces the Dini-continuity of $\mathfrak{c}(\cdot)$.

On the other hand, we consider for the first time obstacle problems involving functionals with exponential growth, where no result was known, even in the case of smooth obstacles. We refer to [19] for further results.

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