Market impact and its decay*

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Abstract

Most of the price models used in optimal execution literatures focusing on the liquidity risk for institutional investors are given exogenously in consideration of statistical validity. In this study, under the condition that both patient sellers and impatient buyers submit orders with time-homogeneous arrival rates, we characterize the market impact function derived from the endogenous equilibrium price formation model using a one-sided limit order book (LOB), and explain the price model with linear impact and exponential resilience (decay) that is often used in optimal execution problems.

1 Introduction

Most of the traditional equity exchanges all over the world have transformed from an quote-driven market to an order-driven market. In such a market, a limit order book (LOB) play a crucial role and the study of the LOB has been paid attention as one of the key topic in market microstructure and optimal execution literatures. In the market microstructure literature, Biais, Hilliton, and Spatt [3] and Sandas [15] empirically showed the dependence on LOB by the order choice and claimed the necessity of the dynamic model. Moreover Foucault [5] and Parlour [12] showed the dynamic model where it was not taken asymmetric information into account, and provided the analyses of bid-ask spread. Parlour and Seppi [13] is a representative survey of the dynamic model of the LOB. Foucault, Kadan and Kandel [6] and Roşu [14] provided the dynamic model considering to the waiting cost for the liquidity traders. As for the optimal execution problem using the LOB, Obizhaeva and Wang firstly derived the optimal execution strategy with a block-shape LOB (linear price impact) and an exponential resilience in their working paper in 2005 (which was published in 2013 [11]). Although many theoretical studies have used linear market impact functions, many empirical literatures (e.g., [2] and [4]) report that the market impact function is non-linear in trading volume in general. Alfonsi, Fruth, and Schied [1] extended [11] and characterized the dynamic optimal execution strategy with general non-linear market impact using the exogenously determined LOB. Many studies in optimal execution with LOB are considering the exogenously given LOB model or Block-shape LOB model because of the model simplicity . On the other hand, Ma, Wang, and Zhang [10] verified that the value function satisfies the dynamic programming principle and is a viscosity solution to the corresponding HJB equation with the endogenously determined LOB via competitive equilibrium argument by Roşu [14].

In this paper, we derive the equilibrium market impact (equilibrium density) function which is shown as a convex form using a LOB dynamic model in Roşu [14]. Then using this price

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formation model, we explain the price model in Obizhaeva and Wang [11] with a linear impact and an exponential decay endogenously which widely used in academia and practice.

The remainder of this paper is organized as follows. Section 2 extends Roşu [14] model and derives the market impact function in the equilibrium price formation model using LOBs under the condition that the order arrival rate of the investors forming the LOB is time-homogeneous. In Section 3, we will construct a system equation to explain the Obizhaeva and Wang price model using the framework in Section 2, and explain linear impact and exponential decay in the framework of the equilibrium price formation model. Section 4 concludes the paper.

2 Dynamics of LOB

In this section, using the analogy in Foucault, Kadan and Kandel (2005) and Roşu (2009), we specify the market impact function for optimal execution problem endogenously. Throughout this paper, we consider the one-sided (ask-side) LOB for simplicity. For that, in order to consider the connection over multiperiod with transient impact, we then incorporate the information of the large buy order execution into the best bid price.

2.1 Preliminaries

We consider a LOB of single asset with no dividend. In order to characterize the LOB, we consider two types of economic agents. One is a single large trader who submits large market buy orders at each predetermined equidistant trading time $t(=1, 2, \cdots, T)$. Although we should be define that the large trader would be a risk averse investor in general, we does not specify her risk aversion because the purpose of this paper is not to derive her optimal execution strategy but the LOB shape. Others are many risk-neutral liquidity traders who trade continuously one unit at a time. Moreover since we consider only an ask side LOB, the liquidity traders are divided into two type of traders. One is the patient sellers who submit one limit order and wait until it is matched. The other is the impatient buyers who submit one market order and leave the LOB. We assume that a patient seller or a impatient buyer enters into the book one by one at random and a patient sellers are able to cancel and resubmit their order without a fee as well as [14]. Moreover we also assume that the maximum number of limit sellers who can submit their order and wait until their order are executed is M_t, which is fixed for each time interval. In the following, we will find that M_t is determined by the spread between the upper bound and the lower bound at time t.

A patient seller enters into the book with Poisson arrival rate λ_{PS} and submits one unit of sell limit order then waits. But his or her order is not executed immediately, then the waiting cost would occur. Therefore if a patient seller submits one unit limit sell order at time t and that order would be executed at (random) time τ, then the expected utility of such a patient seller u_t can be defined as

$$u_t := E_t \left[P_\tau - r(\tau - t) \right], \tag{2.1}$$

where r is a patient coefficient which has the same value for all patient sellers then $r(\tau - t)$ is regarded as the waiting cost for the patient sellers, and P_τ is the execution price for all patient sellers. On the other hand, the impatient buyer gets to the book with Poisson arrival rate λ_{IB} and submits one unit buy market order. The LOB shape is formed based on the patient seller's expected utility taken the waiting cost into account. On the other hand single large trader submits large buy market order q_t at time t and her market order consumes the liquidity in LOB. Then the ask price is lifted up immediately to $\hat{p}_t$ at time t^+. Thereafter, a patient seller or an impatient buyer enters into the book with one unit of limit sell or market buy order at random.

2.2 LOB Dynamics in Equilibrium

In the following, we explain the LOB dynamics as well as [14] in Markov perfect equilibrium framework from time t to $t+1$. A patient seller who arrives the LOB first time, hereafter called Seller 1, submits a unit sell limit order at a certain point and we denote this point as an upper bound A_t. That is,

$$u_1(t) := a_1(t) = A_t. \tag{2.2}$$

Later, we will consider about how to determine this upper bound. Nextly, if an impatient buyer enters into the book next time, return to the first state. However if a patient seller, hereafter Seller 2, enters into the book next time, the price competition between Seller 1 and Seller 2 is occurred because both sellers can cancel their order and resubmit it to the better price. In equilibrium which corresponds to sub-game perfect equilibrium, the best ask price $a_2(t)$ is determined to the level where the utilities of Seller 1 and Seller 2 are equal. In (Markov perfect) equilibrium, the seller who enters into this book at m^{th}, called Seller m, submits his or her limit order to the point at which the expected utilities of all sellers are the same. Regardless of buy market order or sell limit order, since we consider one unit trades, we can regard m as a state variable. As above, we can set the expected utility of the seller as Equation (2.1), and then waiting cost is

$$\frac{r}{\lambda_{PS} + \lambda_{IB}}. \tag{2.3}$$

$u_m(t)$ denotes the expected utility of seller m at time t. Then following recursive system equation is provided between time t and $t+1$,

$$u_m(t) = \frac{\lambda_{PS}}{\lambda_{PS} + \lambda_{IB}} u_{m+1}(t) + \frac{\lambda_{IB}}{\lambda_{PS} + \lambda_{IB}} u_{m-1}(t) - \frac{r}{\lambda_{PS} + \lambda_{IB}}. \tag{2.4}$$

As for the lower boundary, we denote the lower bound at time t as B_t. Since the maximum state is M_t, the patient seller (Seller M) who submits a limit sell order at the lowest price modifies to sell market order with the Poison arrival rate ν with random time $T(\sim \exp(\nu))$ or is matched with an impatient buyer who submits a buy market order before then and leave the book. Therefore,

$$\begin{aligned} u_M(t) &= u_{M-1}(t) - r\frac{1}{\lambda_{IB} + \nu} \\ u_M(t) &= B_{t+1} \end{aligned} \tag{2.5}$$

Then we get the recursive system equation between t and $t+1$,

$$\begin{cases} u_1(t) := A_t, \\ u_m(t) = \frac{1}{1+c} u_{m+1}(t) + \frac{c}{1+c} u_{m-1}(t) - \frac{r}{\lambda_{PS}+\lambda_{IB}}, \\ u_{M_t}(t) = u_{M_t-1}(t) - \frac{r}{\lambda_{IB}+\nu}, \\ u_{M_t}(t) := B_t, \end{cases} \tag{2.6}$$

where we define $c := \frac{\lambda_{IB}}{\lambda_{PS}}$ and λ_{PS},λ_{IB},r,ν, and ρ are fixed. According to [14], in equilibrium, we get

$$u_m(t) = A_t + C_t\left(c^m - 1\right) + \frac{r}{\lambda_{PS} - \lambda_{IB}} m, \tag{2.7}$$

where

$$C_t := \frac{r \frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu}}{(\lambda_{PS} - \lambda_{IB})(c^{M_t-1} - c^{M_t})}. \tag{2.8}$$

Then for $m(\leq M_t)$, we define the best ask price at time $t+1$,

$$a_m(t+1) \quad := \quad p_{t+1} = u_m(t), \tag{2.9}$$

$$a_M(t+1) \quad := \quad B_{t+1} + \frac{r}{\lambda_{IB}}. \tag{2.10}$$

The (expected) maximum state M_t is determined by the following relation,

$$E_t\left[\frac{A_t - B_t}{\frac{r}{\lambda_{PS}-\lambda_{IB}}}\right] = \frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu} \frac{\left(\frac{\lambda_{PS}}{\lambda_{IB}}\right)^{M_t} - 1}{\left(\frac{\lambda_{PS}}{\lambda_{IB}}\right) - 1} - M_t. \tag{2.11}$$

Then, if we could find the boundaries at time t, we can specify the maximum state from the time t^+ to $t+1$. Moreover if the market impact function is included in these boundaries, we could characterize this function.

2.3 Construction of Market Impact Function

The market impact represents the change in price per unit execution. In this subsection, we construct the equilibrium market impact function $\mu(t, x)$ which represents the change in price when x units are executed at time t.

Proposition 1 *Given the LOB at initial time. Assume $\lambda_{PS} \geq \lambda_{IB}$ that is $c \leq 1$, and m is fixed. Then the market impact function has convex form.*

Proof. For each patient seller From Equation (2.7), we can check for $t = k$

$$\begin{aligned} u_{m-1}(k) - u_m(k) &= C_k \left(\frac{\lambda_{IB}}{\lambda_{PS}}\right)^m \left(1 - \frac{\lambda_{IB}}{\lambda_{PS}}\right) - \frac{r}{\lambda_{PS} - \lambda_{IB}} \\ &= \frac{r}{\lambda_{PS}\lambda_{IB}} \left\{ \frac{\left(\frac{\lambda_{IB}}{\lambda_{PS}}\right)^m}{\left(\frac{\lambda_{IB}}{\lambda_{PS}}\right)^{M_k}} \frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu} - 1 \right\} \geq 0. \end{aligned} \tag{2.12}$$

Therefore, if M_k is given and the order book is consumed $x(< m)$ units by the large trader then we can get the market impact function at time $k+1$

$$\begin{aligned}
\mu(k+1,x) &:= u_{m-x}(k) - u_m(k) \\
&= C_k\left(c^{m-x} - c^m\right) - \frac{r}{\lambda_{PS}-\lambda_{IB}}x \\
&= \frac{r}{\lambda_{PS}-\lambda_{IB}}\left\{\frac{c^m\left(c^{-x}-1\right)}{c^{M_k}\left(c^{-1}-1\right)}\frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu} - x\right\} \\
&= \alpha(k)c^{-x} - \beta x - \gamma(k), \qquad (2.13)
\end{aligned}$$

where

$$\alpha(k) := \frac{r\left(\lambda_{PS}+\nu\right)c^m}{\left(\lambda_{PS}-\lambda_{IB}\right)\left(\lambda_{IB}+\nu\right)\left(c^{M_k}\left(c^{-1}-1\right)\right)}, \qquad (2.14)$$

$$\beta := \frac{r}{\lambda_{PS}-\lambda_{IB}}, \qquad (2.15)$$

$$\gamma(k) := C_k\left(\frac{\lambda_{IB}}{\lambda_{PS}}\right)^m\left(1-\frac{\lambda_{IB}}{\lambda_{PS}}\right) - \frac{r}{\lambda_{PS}-\lambda_{IB}}. \qquad (2.16)$$

Then, the backward induction can continue until $k = 1$, thus we get the convex impact function for all $t \in [1, T]$. □

Remark 1 For simplicity, we consider a unit trading interval for the large trader. Then we assume that the time of reaching new equilibrium is negligible and the (average) state m_t that the large trader submits new large buy market order q_{t+1} may be fixed. For more detailed, refer to [10].

Remark 2 In order to apply the framework in [14] to the optimal execution problem, we have a problem about how to determine the value of $M_.$. According to Equation (2.11), the upper bound $A_.$ and the lower bound $B_.$ play the crucial role. If the impact function is deterministic, it can be solved in principle.

In Roşu (2009), since the decay (resilience) function is linear because A_t and B_t are both constant ($A_t = A, B_t = B$) and we are considering a risk-neutral patient sellers and the constant arrival rate, then we have a linear decay on average.

Figure 1 shows the shape of the market impact function when $r = 0.005$, $\nu = 1000$, $\lambda_{PS} = 3040$, $\lambda_{IB} = 2975$, $m = 1000$, and $M = 1500$. Moreover, the right side of Figure 1 shows the shape of the LOB when the tick size is 50. Since the buy market order is taken into consideration, it can be seen that a lot of orders are accumulated near the best ask when considering the discrete tick size because we are considering the trade-off between patient wait time and execution price.

Figure 2 illustrates the differences in market impact function when $r = 0.005$ (solid line), $r = 0.001$ (rough dot line), $r = 0.0005$ (fine dotted line) and each line has $\nu = 1000$, $\lambda_{PS} = 3050$, $\lambda_{IB} = 2980$, $m = 1000$, and $M = 1500$. It can be seen that as the seller's patient coefficient in the market increases, the market impact increases correspondingly. However, these values are unstable, and even a small change in the value causes the impact function to behave explosively.

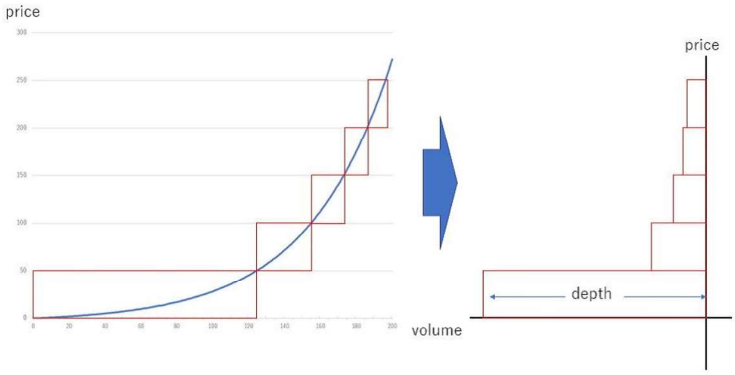

Figure 1: Market impact function and LOB shape

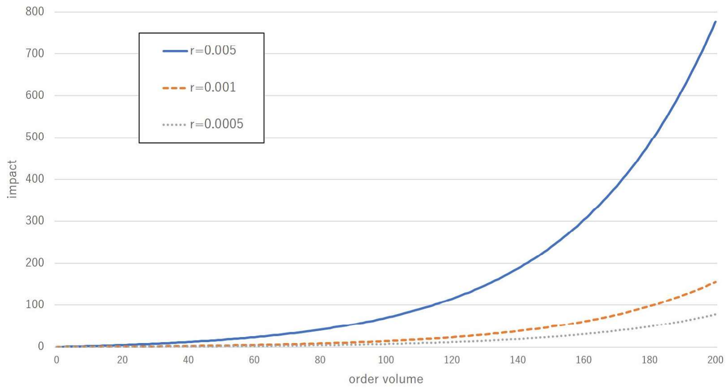

Figure 2: Differences in market impact function due to various patience coefficients

Also, when $c \to 1$, μ can take a negative value, which cannot be explained in reality. The validity of the numbers remains for future research.

3 Block-shaped LOB and Exponential Decay

In this section we consider an equilibrium model framework for having linear impact and exponential decay in Obizhaeva and Wang price model. For this purpose, we assume that the trade information of a large trader incorporate into the price exponentially as stated in [9] and [11]. Then we also denote the accumulated past trading information of the large trader at time t as S_t;

$$S_t := e^{-\rho t}(1 - e^{-\rho}) \sum_{k=1}^{t-1} \mu(k, q_k) e^{\rho k}, \tag{3.1}$$

where we define $\mu(t, x)$ as a market impact function, that is to say, if a large trader purchases q_t shares of a risky asset at time t, the price is lifted up for $\mu(t, q_t)$. Moreover, we assume that p_t^0 denoted as the fundamental price process, and $p_{t+1}^0 - p_t^0 := \epsilon_t \sim N(0, \sigma^2)$ and $\hat{p}_t$ denotes the execution price. Then if p_t is the best ask price at time t, the execution price by the purchasing

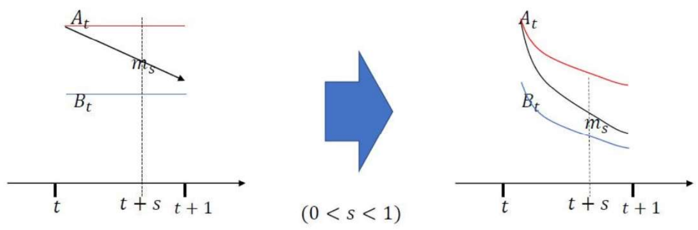

Figure 3: Price regression by movement of the upper bound and the lower bound

q_t shares is

$$\hat{p}_t = p_t + \mu(t, q_t). \tag{3.2}$$

Here, a general price model for optimal execution problems using Gatheral (2010) [7] framework is,

$$p_t = p_0 + \int_0^t f(\dot{q}_k)G(t-k)dk + \int_0^t \sigma dZ_k, \quad k < t, \tag{3.3}$$

where, f represents the market impact function and G is the decay kernel. The last dZ term is a random term. For the function f and G, Obizhaeva and Wang (2013) price model with linear impact and exponential decay is set as follows,

$$\begin{cases} f(\dot{q}_k) = \mu\dot{q}_k, \\ G(t-k) = \exp\{-\rho(t-k)\}. \end{cases} \tag{3.4}$$

On the other hand, to represent a Block-shaped LOB, it is enough if the following equation is satisfied.

$$u_{m-1}(t) - u_m(t) = u_m(t) - u_{m+1}(t) = D, \tag{3.5}$$

where $D = constant$. Then a sufficient condition of the Block-shaped LOB is $c = 0 (c \neq 1)$. That is, $\lambda_{IB} = 0 (c \neq 1)$. This means that only patient sellers can form the LOB. However, since this setting alone represents linear decay rather than exponential decay, we consider the a model in which the price also exponentially decays due to the exponential decay of the upper and the lower bound. Figure 3 illustrates how to revert the price by the movement of the upper and the lower bound. When the upper and the lower bound are both constant, the price regression is linear. On the other hand, by setting the entire LOB to exponentially decay, it is possible to express it as having a linear market impact and exponential decay function as shown in the right side of Figure 3. Since these boundaries need to decay exponentially, then we construct the system equation as follows.

$$\begin{cases} u_1(t) := A_t = p_{t-1} + \epsilon_t + e^{-\rho s}\mu, \\ u_m(t) = u_{m+1}(t) - \frac{r}{\lambda_{PS}}, \\ u_{M_t}(t) = u_{M_t-1}(t) - \frac{r}{\nu}, \\ u_{M_t}(t) := B_t = A_t - L, \end{cases} \tag{3.6}$$

where $L = constant$ and $\mu, \lambda_{PS}, r, \nu$, and ρ are fixed. In this case, the upper bound reflects the resilience (decay) effect, that is to say, the accumulated trading information and underlying

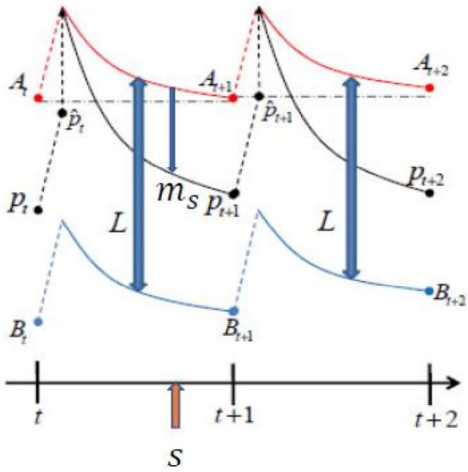

Figure 4: Average price movement satisfying the system equation

price process. The lower bound is determined by the constant difference with L. Therefore from Equation (2.11), the maximum state M is determined by the following relation,

$$A_t - Bt = \frac{r}{\lambda_{PS}} M_t \quad \Leftrightarrow \quad M_t = \frac{\lambda_{PS}}{r} L. \tag{3.7}$$

Therefore, $M_t = constant$ over trading time. Under the framework of the unit trading interval, since m and M could be fixed, then from Equation (2.14), (2.15), and (2.16) we find that if $c \to 0$,

$$\alpha(k) = \frac{r(\lambda_{PS} + \nu) c^{n-M+1}}{\lambda_{PS}(1-c)^2(\nu + c\lambda_{PS})} \to 0, \tag{3.8}$$

$$\beta \to \frac{r}{\lambda_{PS}}, \tag{3.9}$$

$$\gamma(k) = C_k c^m (1-c) - \frac{\frac{r}{\lambda_{PS}}}{1-c} \to -\frac{r}{\lambda_{PS}}. \tag{3.10}$$

So the market impact function is,

$$\mu(x) = -\beta x - \gamma. \tag{3.11}$$

This certainly represents a linear impact or block-shaped LOB. Nextly, the ask price is

$$p_{t+1} := u_{m_s}(t+1) = u_{m_s}(t) = A_t - \frac{r}{\lambda_{PS}} m_s. \tag{3.12}$$

Although m_s is linear, it is said that A_t is exponentially decayed, so it can be seen that the ask price is also exponentially decayed. Figure 4 illustrates the average price movement that satisfies system equation (3.6). The justification for the exponential movement of the entire LOB over time is explained in the following two ways. Firstly, patient sellers form the equilibrium by referring their relative position with other sellers rather than their prices. Nextly, even though the upper and the lower bounds change, that is, if the absolute prices change they will not cancel their orders when the price is relatively reaching the equilibrium.

4 Conclusion

In this paper, we have considered an one-sided LOB based on the framework of [6] and [14] and developed an equilibrium price formation model under time-homogeneous arrival rate of the patient sellers and the impatient buyers. Then we have showed the market impact function have convex form. Furthermore, as a sufficient condition to satisfy the price model with linear impact and exponential decay in [11], it was shown that only the patient sellers submit their orders and the upper and lower boundaries are both exponentially decayed in [14] model. The intervals between the upper and the lower bound are fixed, so both boundaries are exponentially decayed. This means that the lower bound, the potential best ask price, will be updated by revealing new information, and the belief of the patient sellers who have submitted their orders on the LOB will be updated accordingly. At the same time, it also means that the patient sellers who have overpriced too much will cancel their order and leave the book. We can explain more realistic by using a model that considers the time-inhomogeneous arrival rate, which is our future research.

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