Non-i.i.d. random holomorphic dynamical systems

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Abstract

We consider non-i.i.d. random holomorphic dynamical systems whose choice of maps depends on Markovian rules. We show that generically, such a system is stable on average or chaotic with full Julia set. This generalizes a result for i.i.d. random dynamical systems of rational maps. This is a joint work with Hiroki Sumi (Kyoto University).

1 Introduction

We consider dynamical systems of rational maps on the Riemann sphere. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with the spherical distance d. Let Rat be the space of all rational maps of degree two or more from $\widehat{\mathbb{C}}$ to itself with metric $\kappa(f, g) := \sup_{z \in \widehat{\mathbb{C}}} d(f(z), g(z))$.

We are in particular interested in non-i.i.d. random dynamical systems, whose choices of rational maps satisfy "Markov" rules. It is defined as follows, see [5].

Definition 1.1. Let $m \in \mathbb{N}$. Suppose that m^2 regular Borel measures $(\tau_{ij})_{i,j=1,\dots,m}$ on Rat satisfy $\sum_{j=1}^{m} \tau_{ij}(\mathrm{Rat}) = 1$ for all $i = 1, \dots, m$. We call τ a *Markov random dynamical system* (MRDS for short).

For a MRDS $\tau = (\tau_{ij})_{i,j=1,\dots,m}$, we define a Markov chain on $\widehat{\mathbb{C}} \times \{1, \dots, m\}$ whose transition probability from a point (z, i) to a Borel set of the form $B \times \{j\}$ is defined by $\tau_{ij}(\{f \in \mathrm{Rat} \colon f(z) \in B\})$. This Markov chain is the main object of this article.

We realized "Markov" rules as the chain on the extended space $\widehat{\mathbb{C}} \times \{1, \dots, m\}$. This enables us to analyze (random) dynamics more systematically. Also, Markov chain on $\widehat{\mathbb{C}} \times \{1, \dots, m\}$ describes the following random dynamical systems on $\widehat{\mathbb{C}}$.

Let $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ be a MRDS. Fix an initial point $z_0 \in \widehat{\mathbb{C}}$, and choose $i_0 = 1, \dots, m$ (with some probability if you like). For each point $(z_n, i_n) \in \widehat{\mathbb{C}} \times \{1, \dots, m\}$, define the next point $(z_{n+1}, i_{n+1}) \in \widehat{\mathbb{C}} \times \{1, \dots, m\}$ randomly with transition probability induced by τ for each $n \in \mathbb{N}_0$. As a consequence, it defines the random orbit $\{z_n\}_{n \in \mathbb{N}_0}$ of the form $z_n = f_n \circ \cdots \circ f_2 \circ f_1(z_0)$, where f_n are chosen with respect to the probability measure $\tau_{i_n i_{n+1}} / \tau_{i_n i_{n+1}}(\mathrm{Rat})$.

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One of our motivation is to generalize the theory of i.i.d. random dynamical systems. MRDS defined above include the setting of i.i.d. RDS in the sense that if $m = 1$, then $\tau = (\tau_{11})$ induces i.i.d. RDS on $\widehat{\mathbb{C}}$.

We now mention some results about i.i.d. RDS. The first study of i.i.d. RDS of the holomorphic maps was due to Fornaess and Sibony [1]. Sumi investigated more general setting and found many new phenomena which cannot occur in the deterministic dynamics [2, 3, 4]. These are called randomness-induced phenomena or noise-induced phenomena, which are one of the main topics of the study of RDS.

Our goal of this article is to extend the generic dichotomy from i.i.d. RDS to Markov RDS. On the i.i.d. case, it was shown in [3]. We consider the space of all MRDS, and divide it into two types regarding the dynamical properties. One is stable on average and the other is chaotic on the whole space.

2 Definition of mean stability locus and chaos locus

We consider MRDSs which are irreducible and compactly generated. In this section, we define these concepts and basic notation, utilizing the notion of directed graphs.

Definition 2.1. Let $\tau = (\tau_{ij})_{i,j=1,\ldots,m}$ be a MRDS. We define the directed graph (V, E) in the following way. We define the vertex set as $V := \{1, 2, \ldots, m\}$ and the edge set as

$$E := \{(i, j) \in V \times V : \ \tau_{ij}(\mathrm{Rat}) > 0\}.$$

Define $i : E \to V$ (resp. $t : E \to V$) as the projection to the first (resp. second) coordinate and we call $i(e)$ (resp. $t(e)$) the initial (resp. terminal) vertex of $e \in E$. Also, for each $e = (i, j)$, we denote $\tau_e = \tau_{ij}$.

Definition 2.2. Let MRDS be the set of all Markov random dynamical systems τ which satisfy the following two condition.

- The directed graph (V, E) of τ is strongly connected.
- For each $e \in E$, the support $\operatorname{supp} \tau_e$ is compact in Rat.

We endow the space MRDS with the following topology. A sequence $\{\tau^n\}_{n\in\mathbb{N}}$ converges to τ as $n \to \infty$ if

- the directed graphs of τ^n coincide that of τ for sufficiently large $n \in \mathbb{N}$,
- τ_e^n converges to τ_e with weak*-topology for each $e \in E$ and
- $\operatorname{supp} \tau_e^n$ converges to $\operatorname{supp} \tau_e$ for each $e \in E$ with respect to the Hausdorff metric.

In the following, let $\tau \in$ MRDS and let (V, E) be the directed graph of τ. We define the Fatou sets and the Julia sets as follows.

Definition 2.3. (i) A word $e = (e_1, e_2, \ldots, e_N) \in E^N$ with length $N \in \mathbb{N}$ is said to be *admissible* if $t(e_n) = i(e_{n+1})$ for all $n = 1, 2, \ldots, N-1$. For this word e, we call $i(e_1)$ (resp. $t(e_N)$) the initial (resp. terminal) vertex of e and we denote it by $i(e)$ (resp. $t(e)$).

(ii) For each $i, j \in V$, we set

$$H_i^j(S_\tau) := \{f_N \circ \cdots \circ f_2 \circ f_1 \colon N \in \mathbb{N}, e = (e_1, e_2, \ldots, e_N) \in E^N \text{ is admissible}, \\ f_n \in \operatorname{supp} \tau_{e_n} (\forall n = 1, \ldots, N), i = i(e), t(e) = j\}.$$

(iii) For each $i \in V$, we denote by $F_i(S_\tau)$ the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood U in $\widehat{\mathbb{C}}$ such that the family $\cup_{j \in V} H_i^j(S_\tau)$ is equicontinuous on U. $F_i(S_\tau)$ is called the *Fatou set* of τ at the vertex i and the complement $J_i(S_\tau) := \widehat{\mathbb{C}} \setminus F_i(S_\tau)$ is called the *Julia set* of τ at the vertex i.

The Julia sets are the set of all initial points where dynamics sensitively depends. Since the system contains rational maps of degree two or more, the Julia sets are not empty. Thus, chaotic part always exists in this sense.

Compared to the dynamics of iteration of single map, the Fatou sets are not completely invariant. However, these are forward invariant in the following sense.

Definition 2.4. We consider subsets $L_i \subset \widehat{\mathbb{C}}$ indexed by $i \in V$, and regard it as the family $(L_i)_{i \in V}$. We say $(L_i)_{i \in V}$ is forward invariant if $f(L_{i(e)}) \subset L_{t(e)}$ for each $e \in E$ and $f \in \operatorname{supp} \tau_e$.

Definition 2.5. Let $\mathcal{F}$ be the set of all forward invariant set $(L_i)_{i \in V}$ whose element L_i is non-empty and compact in $\widehat{\mathbb{C}}$. We say that $(L_i)_{i \in V}$ is minimal if $(L_i)_{i \in V}$ is a minimal element of $\mathcal{F}$ with respect to the order $\subset$. Here, $(L_i)_{i \in V} \subset (K_i)_{i \in V}$ if $L_i \subset K_i$ for all $i \in V$.

Definition 2.6. Define $\mathcal{C}$ as the set of all $\tau \in$ MRDS such that $(\widehat{\mathbb{C}})_{i \in V}$ is minimal and $J_i(S_\tau) = \widehat{\mathbb{C}}$ for all $i \in V$.

Note that the Julia set $J_i(S_\tau)$ is equal to the closure of all repelling fixed points of all $h \in H_i^i(S_\tau)$. Thus, for each $\tau \in \mathcal{C}$, fixed points are dense in the minimal set $\widehat{\mathbb{C}}$. It should be noted that the meaning of terminology is slightly different from that of deterministic dynamics.

We defined the chaos locus $\mathcal{C}$. We next define the stablility locus $\mathcal{A}$.

Definition 2.7. Define $\mathcal{A}$ as the set of all $\tau \in$ MRDS such that the following holds. There exist $N \in \mathbb{N}$ and non-empty open sets U_i and W_i for each $i \in V$ such that

(I) $U_i \subset \overline{U_i} \subset W_i \subset \overline{W_i} \subset F_i(S)$ for each $i \in V$.

(II) For each $z \in \widehat{\mathbb{C}}$ and $i \in V$, there exist $j \in V$ and $h \in H_i^j(S)$ such that $h(z) \in W_j$.

(III) For each admissible word $e = (e_1, \ldots, e_N)$ with length N and each $f_n \in \operatorname{supp} \tau_{e_n}$ $(n = 1, \ldots, N)$, we have $\overline{f_N \circ \cdots \circ f_1(W_{i(e)})} \subset U_{t(e)}$.

Each $\tau \in \mathcal{A}$ is called mean stable. A mean stable system has marvelous contracting property on average. More precisely, the following holds.

Proposition 2.8. Let $\tau \in \mathcal{A}$. Then there exists $\alpha < 0$ such that for each $z \in \widehat{\mathbb{C}}$, the random Lyapunov exponent satisfies $\limsup_{n \to \infty} n^{-1} \log \|D(f_n \circ \cdots \circ f_2 \circ f_1)(z)\| \leq \alpha$ for almost every admissible sequence. Here, Dg denotes the differential of a map g and $\| \cdot \|$ is the norm induced by spherical metric.

Remark that the Lyapunov exponent is positive (for almost every initial point with respect to the Lyubich measure) in deterministic dynamics of iteration of a single map.

Definition 2.9. Let $(L_i)_{i\in V}$ be a minimal set of MRDS τ. We say that $(L_i)_{i\in V}$ is attracting if there exist $N \in \mathbb{N}$ and open sets U_i and W_i for each $i \in V$ such that

(i) $L_i \subset U_i \subset \overline{U_i} \subset W_i \subset \overline{W_i} \subset F_i(S)$ for each $i \in V$.

(ii) For each admissible word $e = (e_1, \dots, e_N)$ with length N and each $f_n \in \operatorname{supp} \tau_{e_n}$ $(n = 1, \dots, N)$, we have $f_N \circ \cdots \circ f_1(W_{i(e)}) \subset U_{t(e)}$.

Proposition 2.10. Let $\tau \in$ MRDS. Then τ is mean stable if and only if every minimal set is attracting.

3 Main Result

We present our main theorem of this article. In the previous section, we defined two loci $\mathcal{A}$ and $\mathcal{C}$. These two are completely opposite; the former is stable and the latter is chaotic. The main result states that most systems are either of these.

Theorem 3.1. The disjoint union $\mathcal{A} \cup \mathcal{C}$ is dense in the space MRDS. Also, $\mathcal{A}$ and $\mathcal{C}$ are non-empty, and $\mathcal{A}$ is open.

In the following, we restrict our system to polynomial maps. Note that every polynomial map has an (super)attracting fixed point at infinity, and hence the Fatou set contains a neighborhood of point at infinity. Let Poly be the space of all polynomial maps of degree two or more.

Corollary 3.2. Define PMRDS as the set of all τ whose supports consist of polynomials, in the sense that $\operatorname{supp} \tau_e \subset$ Poly for each $e \in E$. Then $\mathcal{A} \cap$ PMRDS is dense in PMRDS and $\mathcal{C} \cap$ PMRDS is empty.

The key of proof is the classification of minimal sets. Every minimal set of a Markov random dynamical system satisfies one of the following three condition; minimal set intersects with the Julia set, intersects with some rotation domain of a map or is attracting.

References

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