Invariant measure of a random map with a rare entrance of the neighborhood of an indifferent fixed point

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1 Introduction and examples

We consider a family of transformations and study a random dynamical system such that one transformation is randomly selected from the family and then applied on each iteration. We study random dynamical systems with an indifferent fixes point. Focusing the "entrance" (inverse image) of the neighborhood of an indifferent fixed point of a random map, we study and estimate the absolutely continuous invariant measures of the random dynamical system. There are some types of entrances. In this article we mainly study a rare entrance which is only appeared in random dynamical systems.

First, for a deterministic one-dimensional map, we make clear what is an indifferent fixed point and we consider what is an entrance of the neighborhood of an indifferent fixed point.

Let $T : [0,1] \to [0,1]$ be a piecewise C^1 map. A fixed point p ($T(p) = p$) is called an indifferent fixed point of T if $|T'(p)| = 1$.

To clarify the notion of an entrance of the neighborhood of an indifferent fixed point, we are going to consider the following example:

Example 1. Define $\tau : [0,1] \to [0,1]$ by

$$\tau(x) = \begin{cases} x + 2^{d-1}x^d, & [0,1/2), \\ 2x - 1, & [1/2,1], \end{cases}$$

where $d > 1$ is a constant.

In this example, 0 is an indifferent fixed point of τ. It is well-known that τ has an invariant measure μ which is absolutely continuous with respect to the Lebesgue measure. Further, it is also well-known that

$$\mu([0,1]) < \infty \quad \text{if and only if} \quad d < 2.$$

In particular, if $d \geq 2$, $\mu([0,\varepsilon)) = \infty$ and $\mu([\varepsilon,1]) < \infty$ for any small $\varepsilon > 0$. For example, see [I1, I2, and T]. $[0,\varepsilon)$ is a neighborhood of the indifferent fixed point 0. If $x \in [1/2,(1+\varepsilon)/2)$, then $\tau(x) \in [0,\varepsilon)$. So, $[1/2,(1+\varepsilon)/2)$ is the entrance of $[0,\varepsilon)$ for this map τ.

The entrance of $[0,\varepsilon)$ is vanished, if we modify τ on $[1/2,1]$ as in the following example.

Example 2. Define $\tau : [0,1] \to [0,1]$ by

$$\tau(x) = \begin{cases} x + 2^{d-1}x^d, & [0,1/2), \\ 3 \cdot 2^{-1}(x - 2^{-1}) + 4^{-1}, & [1/2,1], \end{cases}$$

where $d > 1$ is a constant.

This τ also has an invariant measure μ which is absolutely continuous with respect to the Lebesgue measure. However, this measure is not an infinite measure, even if $d \geq 2$. Since $\tau^n(x) \in [1/4,1]$ for large n and for a.e.-x, the neighborhood $[0,\varepsilon)$ of the indifferent fixed point 0 has no entrance.

If we consider random maps, we can find an intermediate case between Examples 1 and 2. We are going to consider the following example.

Example 3. Let $W = [0,1/4]$. For $t \in W$, define $\tau_t : [0,1] \to [0,1]$ by

$$\tau_t(x) = \begin{cases} x + 2^{d-1}x^d, & [0,1/2), \\ 2(1-t)(x - 2^{-1}) + t, & [1/2,1], \end{cases}$$

where $d > 1$ is a constant. We consider the random map such that t is randomly selected from W according to the probability density function $p(t,x) = 4$. (The precise definition of random maps will be given later.)

We note that τ_0 is same to the map in Example 1 and that $\tau_{1/4}$ is same to the map in Example 2. We think that $[0, \varepsilon)$ seems to have a rare entrance or a stochastically small entrance.

We will study absolutely continuous invariant measures of random maps such as Example 3.

2 Position dependent random maps

In this section we define position dependent random maps and define invariant measures for random maps. (There is a detailed explanation in [I3]. See also [Ba-G].) Our definition of position dependent random maps is more general than it of position independent random maps as well as deterministic maps.

Let $(W, \mathcal{B}, \nu)$ be a σ-finite measure space. We use W as a parameter space. Let $(X, \mathcal{A}, m)$ be a σ-finite measure space. We use X as a state space. Let $\tau_t : X \to X$ $(t \in W)$ be a nonsingular transformation, which means that $m(\tau_t^{-1}A) = 0$ if $m(A) = 0$ for any $A \in \mathcal{A}$. Assume that $\tau_t(x)$ is a measurable function of $t \in W$ and $x \in X$.

Let $p : W \times X \to [0, \infty)$ be a measurable function which is a probability density function of $t \in W$ for each $x \in X$, that is, $\int_W p(t,x)\nu(dt) = 1$ for each $x \in X$.

The random map $T = \{\tau_t; p(t,x) : t \in W\}$ is defined as a Markov process with the following transition function:

$$\mathbf{P}(x, D) := \int_W p(t,x) 1_D(\tau_t(x))\nu(dt) \quad \text{for any} \quad D \in \mathcal{A},$$

where 1_D is the indicator function for D. The transition function $\mathbf{P}$ induces an operator $\mathbf{P}_*$ on measures on X defined by

$$\mathbf{P}_*\mu(D) := \int_X \mathbf{P}(x, D)\mu(dx) = \int_X \int_W p(t,x)\ 1_D(\tau_t(x))\ \nu(dt)\ \mu(dx)$$

for any measure μ on X and any $D \in \mathcal{A}$.

If $\mathbf{P}_*\mu = \mu$, μ is called an *invariant measure* for the random map $T = \{\tau_t; p(t,x) : t \in W\}$. This definition of invariant measures for random maps is an extension of it for deterministic maps.

3 Setting for this topic and results

In the setting of the previous section, let $X = [0,1]$ and let m be the Lebesgue measure.

We are going to consider random maps which are more general than the random map in Example 3.

Let a and b be constants with $1 < a \le b$. Let $\delta > 0$ be a constant. Put $W_1 = [a,b]$, $W_2 = [0,\delta]$, and $W = W_1 \times W_2$. Let ν be the Lebesgue measure on W.

For $(d,t) \in W$, define $\tau_{d,t} : [0,1] \to [0,1]$ by

$$\tau_{d,t}(x) = \begin{cases} x + m_d x^d, & [0,1/2), \\ q_t(x) + t, & [1/2,1], \end{cases}$$

where $m_d > 0$ is a constant depended on d and $q_t(x)$ is a measurable function of $t \in W_2$ and $x \in [1/2,1]$. Further, $q_t(x)$ satisfies the following conditions:

(i) $0 \le q_t(x) + t \le 1$ for $t \in W_1$ and $x \in [1/2,1]$.

(ii) There exists two constants L_1 and L_2 such that $L_1(x - \frac{1}{2}) \le q_t(x) \le L_2(x - \frac{1}{2})$.

(iii) $q_t(x)$ is a C^1 function of $x \in [1/2,1]$.

(iv) $q_t'(x) > 1$ for $x \in (1/2,1)$.

(v) There exists a constant M such that $\bigvee_{[0,1]} \dfrac{p(d,t,\cdot)}{\tau'_{d,t}} < M$ for a.s. $(d,t) \in W$.

The following theorem follows from Theorem 6.1 in [I4].

Theorem 1. *Let the random map $T = \{\tau_{d,t}, p(d,t,x);\ (d,t) \in W\}$ satisfy the above conditions. Then, $T = \{\tau_{d,t}, p(d,t,x);\ (d,t) \in W\}$ has an absolutely continuous σ-finite invariant measure μ. Further, for any small $\varepsilon > 0$, $\mu((\varepsilon,1]) < \infty$.*

In [I4], we have shown the existence of an absolutely continuous σ-finite invariant measure for a random map which satisfies more general conditions.

Using Theorem 4.3 in [I4] and the idea of the proofs of Theorems 6.2 and 6.3 in [I4], we can show the following theorem.

Theorem 2. *Let the random map $T = \{\tau_{d,t}, p(d,t,x);\ (d,t) \in W\}$ satisfy the above conditions and let μ be a measure as in Theorem 1. Let $p_0 > 0$ be a constant and assume that $p(d,t,x) > p_0$. Then, $\mu([0,1]) < \infty$ if and only if $a < 3$.*

In Example 3, d is a fixed constant. So, we can consider that Example 3 is the case $a = b = d$ in the above setting. Thus, the random map T of Example 3 satisfies the assumption of Theorems 1 and 2. Hence, T has an absolutely continuous σ-finite invariant measure μ. Further, $\mu([0,1]) < \infty$ if and only if $d < 3$.

Remark 1. Assume that $W_2 = \{0\}$ that is, $\delta = 0$. Then, $\mu([0,1]) < \infty$ if and only if $a < 2$. This is a consequence of Theorems 6.2 and 6.3 in [I4].

Remark 2. Let $\kappa > 0$ be a constant. Replace $\tau_{d,t}(x)$ with the following:

$$\tau_{d,t}(x) = \begin{cases} x + m_d x^d, & [0, 1/2), \\ q_t(x) + t^\kappa, & [1/2, 1], \end{cases}$$

that is, in the setting, replace $q_t(x) + t$ with $q_t(x) + t^\kappa$. Then, the random map $T = \{\tau_{d,t}, p(d,t,x);\ (d,t) \in W\}$ has an absolutely continuous σ-finite invariant measure μ. Further, $\mu([0,1]) < \infty$ if and only if $a < 2 + \kappa^{-1}$.

More general and deeper results related to this subject will be discussed in [I5].

References

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