Sample-wise central limit theorem with deterministic centering for nonsingular random dynamical systems

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1. Introduction

Let $(M, \mathcal{B}, m)$ be a probability space. Consider a family $\{\tau_s\}_{s \in S}$ of m-nonsingular transformations on $(M, \mathcal{B}, m)$ indexed by a measurable space $(S, \mathcal{B}(S))$ such that the map $S \times M \ni (s, x) \mapsto \tau_s x \in M$ is $(\mathcal{B}(S) \times \mathcal{B})/\mathcal{B}$-measurable. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\sigma : \Omega \to \Omega$ a P-preserving transformation. Take an S-valued random variable ξ on $(\Omega, \mathcal{F}, P)$ and define an S-valued strictly stationary process $\{\xi_n\}_{n=0}^{\infty}$ by $\xi_n = \xi \circ \sigma^n$ ($n \geq 0$). The family $\mathcal{X} = \{X_n\}$ of random maps $X_n : M \to M$ is called *the random dynamical system given by* $(\{\tau_s\}_{s \in S}, \sigma, \xi)$ if the maps in $\mathcal{X}$ are defined by

$$X_0(\omega)x = x, \quad X_{n+1}(\omega)x = \tau_{\xi_n(\omega)} X_n(\omega)x \text{ for } (x, \omega) \in M \times \Omega, \quad (n \geq 0)$$

For a random dynamical system $\mathcal{X}$ given by $(\{\tau_s\}_{s \in S}, \sigma, \xi)$, we introduce its (*direct*) *product* as the random dynamical system given by $(\{\tau_s \times \tau_s\}_{s \in S}, \sigma, \xi)$ and we denote it by $\mathcal{X} \times \mathcal{X}$. Following Kakutani [8], we introduce the skew product transformations $T_1 : M \times \Omega \to M \times \Omega$ and $T_2 : M^2 \times \Omega \to M^2 \times \Omega$ corresponding to $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ by

$$T_1(x, \omega) = (X_1(\omega)x, \sigma\omega) \quad \text{for } (x, \omega) \in M \times \Omega$$

and

$$T_2(x, y, \omega) = (X_1(\omega)x, X_1(\omega)y, \sigma\omega) \quad \text{for } (x, y, \omega) \in M^2 \times \Omega,$$

respectively.

In this article we consider the case where there exists a unique $m \times P$-absolutely continuous probability measure Q_1 with density H_1 such that the measure-preserving dynamical system (T_1, Q_1) is exact, i.e. $\bigcap_{n=0}^{\infty} T_1^{-n}(\mathcal{B} \times \mathcal{F})$ is trivial with respect to Q_1 (see [6] Chapter 10 for details). To make things much simpler we assume that the transfer operator $\mathcal{L}_{T_1} = \mathcal{L}_{T_1, m \times P}$ of T_1 with respect to $m \times P$ is asymptotically stable. (Furthermore, we will need to assume the asymptotic stability of $\mathcal{L}_{T_2} = \mathcal{L}_{T_2, m^2 \times P}$ later). Then the m-absolutely continuous probability measure $\rho \cdot m$ on $\mathcal{B}$ with density $\rho(\cdot) = \int_{\Omega} H(\cdot, \omega) P(d\omega)$ can be regarded as the physical measure for P-almost every sample ω. In fact, for any observable f in $L^{\infty}(m)$ there exists $\Gamma \in \mathcal{F}$ with $P(\Gamma) = 1$ such that $\omega \in \Gamma$ yields that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_n(\omega)x) = \int_X f\rho \, dm \quad m\text{-a.e. } x. \text{ So it is not too much to say that 'almost}$$

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sure sample-wise or quenched strong law of large numbers' holds. Therefore it is only natural to consider the central limit theorem (abbr. CLT) for the sequence $\{f(X_n(\omega)x)\}_{n\geq 0}$ as the next problem.

For an observable f we put $S_n(T_1)f(x,\omega) = \sum_{k=0}^{n-1} f(T_1^n(x,\omega)) = \sum_{k=0}^{n-1} f(X_n(\omega)x)$ for $(x,\omega) \in M \times \Omega$. Our main concern is the sample-wise (quenched) central limit problem for the sequence $\{X_n(\omega)\}_{n\geq 0}$ of random variables on the space $(X, \mathcal{B}, m)$ with fixed $\omega \in \Omega$. More practically, we consider the problem allowing an exceptional P-null set, i.e. the study of the condition for that there exists $\Gamma \in \mathcal{F}$ with probability one such that for any $\omega \in \Gamma$, the random variable on the probability space $(M, \mathcal{B}, m)$ obtained by making an appropriate centering on $S_n(T_1)f(\cdot,\omega)$ converges in law to a normal distribution as $n \to \infty$.

From general theory of weakly dependent stationary sequence of random variables, the sample-averaged (annealed) CLT with deterministic centering holds true for a large class of observables f, i.e., $Z_n(T_1)f = (1/\sqrt{n})(S_n(T_1)f - n\int_M f\,\rho\,dm)$ converges in law to an appropriate normal distribution $N(0, v(f))$ with respect to $m \times P$ as $n \to \infty$ (see [5] and [7]). Moreover, we can show that such a sample-averaged CLT implies that for any $u \in C_0(\mathbb{R})$, the sequence $\int_M u(Z_n(T_1)f)\,dm$ in $L^1(P)$ converges to $\int_{\mathbb{R}} u(t)\,N(0, v(f))(dt)$ weakly in $L^1(P)$ as $n \to \infty$ (see [11] and cf. Theorem 3.1 below), where $C_0(\mathbb{R})$ denotes the totality of continuous functions on $\mathbb{R}$ with compact support. Therefore this fact may lead us to wishful thinking that the sample-wise CLT with the same deterministic centering also holds true. But the results in [12] (see also [9]) make us recognize that the noise dependent centering $S_n(T_1)f(\cdot,\omega) - \sum_{k=0}^{n-1}\int_M f(X_k(\omega)y)\,H(y,\omega)\,m(dy)$ seems natural and even proper in view of the martingale approximation for stationary processes. In fact, a counter-example given in Section 4.4 of [1] (see also Example 4.2 in Section 4) shows that the deterministic centering does not work in general, even in the case where the $\{\xi_n\}_{n\geq 0}$ is independent and identically distributed if we consider 'almost sure sample-wise CLT'. Therefore, it turns out that some of observations and assertions in [11] have difficulties and fail to hold.

The purpose of the article is to announce the author's recent results on the condition for that the sample-averaged CLT for $S_n(T_1)f/\sqrt{n}$ implies the a.s. sample-wise CLT with the same deterministic centering. To be more precise, the following results are stated in Section 3. First we give the conditions for the sample-averaged CLT under the asymptotic stability of the transfer operator for the skew-product transformation T_1. One may notice that Theorem 3.1 is a sort of corrected version of Proposition 3.2 in [11]. Next we discuss about the conditions which guarantee that the sequence of random variables

$\int_M u(S_n(T_1)f/\sqrt{n})\,dm$ in the above converges strongly to $\int_{\mathbb{R}} u\,dN(0,v(f))$ in $L^1(P)$ (Theorem 3.2). Finally, we shall announce an a.s.sample-wise CLT with deterministic centering under some additional conditions (Theorem 3.4). Section 4 is devoted to examples. On account of limited space we have to restrict ourselves just give the statements of theses results. So proofs and details will be given elsewhere.

2. Asymptotic stability of transfer operator

In order to state the present results, we need to introduce the notion of asymptotic stability of the transfer operator. Let $(M,\mathcal{B},m)$ be a probability space and τ an m-nonsingular transformation. Then *the transfer operator(the Perron-Frobenius operator)* $\mathcal{L}_\tau = \mathcal{L}_{\tau,m}$ of τ with respect to m is defined to be the operator on $L^1(m)$ satisfying

$$\int_M f \cdot \mathcal{L}_\tau g\,dm = \int_M (f \circ \tau)g\,dm \quad \text{for } f \in L^\infty(m) \text{ and } g \in L^1(m).$$

For the sake of convenience we put

$$\Delta(\tau,g,n;h) = \mathcal{L}_\tau^n g - \left(\int_M g\,dm\right)h \quad \text{for } g,\,h \in L^1(m).$$

The transfer operator $\mathcal{L}_\tau$ is said to be *asymptotically stable with respect to* $h \in L^1(m)$ if for each $g \in L^1(m)$ $\|\Delta(\tau,g,n;h)\|_{1,m} \to 0\,(n \to \infty)$ holds. Note that such an h in the condition turns out to be a unique τ-invariant density and the measure-theoretic dynamical system (τ,hm) is exact i.e. $\bigcap_{n=0}^\infty \tau^{-n}\mathcal{B}$ is trivial with respect to the m-absolutely continuous τ-invariant measure hm with density h.

Consider the random dynamical system $\mathcal{X}$ given by $(\{\tau_s\}_{s \in S}, \sigma, \xi)$ and its product $\mathcal{X} \times \mathcal{X}$. Since T_1 and T_2 are $m \times P$-nonsingular and $m^2 \times P$-nonsingular, respectively, we can consider the transfer operators $\mathcal{L}_{T_1} = \mathcal{L}_{T_1,m \times P}$ on $L^1(m \times P)$ and $\mathcal{L}_{T_2} = \mathcal{L}_{T_2,m^2 \times P}$ on $L^1(m^2 \times P)$, respectively. It is not hard to see that if $\mathcal{L}_{T_2}$ is asymptotically stable with respect to $H_2 \in L^1(m^2 \times P)$, so is $\mathcal{L}_{T_1}$ with respect to $H_1 \in L^1(m \times P)$, where $H_1(x,\omega) = \int_M H_2(x,y,\omega)\,m(dy)\quad ((x,\omega) \in M \times \Omega)$.

3. Results

First we state that the sample-averaged CLT is equivalent to a sort of weak L^1-version of sample-wise CLT if $\mathcal{L}_{T_1}$ is asymptotically stable. Precisely, we have the following.

THEOREM 3.1. *Consider the random dynamical system given by* $(\{\tau_s\}_{s \in S}, \sigma, \xi)$ *such that the transfer operator* $\mathcal{L}_{T_1}$ *on* $L^1(m \times P)$ *is asymptotically stable. Let* $N(0,v)$ *denote the normal distribution with mean* 0 *and variance* $v \geq 0$, *where* $N(0,0)$ *is regarded as the point mass* δ_0 *at* $0 \in \mathbb{R}$. *Let* f *be a real-valued element in* $L^\infty(m)$ *satisfying the deterministic centering condition* $\int_m f\rho\,dm = 0$, *where* ρ *is defined as* $\rho(\cdot) = \int_\Omega H_1(\cdot,\omega)P(d\omega)$ *by using*

the unique invariant density H_1 for T_1 with respect to $m \times P$. Then the following are equivalent.

(1) $S_n(T_1)f/\sqrt{n}$ converges in law to $N(0, v)$ with respect to some $m \times P$-absolutely continuous probability measure.

(2) $S_n(T_1)f/\sqrt{n}$ converges in law to $N(0, v)$ with respect to any $m \times P$-absolutely continuous probability measure.

(3) There exists a probability density $g \in L^1(m)$ such that for any bounded continuous function u on $\mathbb{R}$, $\int_M u(S_n(T_1)f(x, \cdot)/\sqrt{n})g(x) \, dm$ converges weakly to $\int_{\mathbb{R}} u(t) \, N(0, v)(dt)$ in $L^1(P)$ as $n \to \infty$.

(4) For any bounded continuous function u on $\mathbb{R}$ and any probability density $g \in L^1(m)$, $\int_M u(S_n(T_1)f(x, \cdot)/\sqrt{n})g(x) \, dm$ converges to $\int_{\mathbb{R}} u(t) \, N(0, v)(dt)$ weakly in $L^1(P)$ as $n \to \infty$.

(5) There exists a probability density $g \in L^1(m)$ such that for any $t \in \mathbb{R}$, $\int_M e^{\sqrt{-1}t(S_n(T_1)f(x, \cdot)/\sqrt{n})}g(x) \, dm$ converges to $e^{-vt^2/2}$ weakly in $L^1(P)$ as $n \to \infty$.

(6) For any $t \in \mathbb{R}$ and any probability density $g \in L^1(m)$, $\int_M e^{\sqrt{-1}t(S_n(T_1)f(x, \cdot)/\sqrt{n})}g(x) \, dm$ converges to $e^{-vt^2/2}$ weakly in $L^1(P)$ as $n \to \infty$.

For $f \in L^1(m)$, we denote by F_f and $\tilde{f}$ the elements in $L^1(m^2)$ defined by $F_f(x, y) = f(x) - f(y)$ and $\tilde{f}(x, y) = f(x)f(y)$, respectively. For $\mathcal{B}$-measurable function f and $\mathcal{B}^2$-measurable function F, we abuse the notation $S_n f(x, \omega)$ and $S_n F(x, y, \omega)$ to denote $S_n(T_1)f(x, \omega)$ and $S_n(T_2)F(x, y, \omega)$, respectively if there is no fear of confusion. Note that if $f \in L^\infty(m)$ satisfies the deterministic centering condition $\int_M f\rho \, dm = 0$ with respect to T, then $\tilde{f}$ also satisfies the deterministic centering condition $\int_{M^2} \tilde{f}\rho_2 \, dm^2 = 0$ with respect to T_2, where $\rho_2(x, y) = \int_\Omega H_2(x, y, \omega) \, P(d\omega)$. Indeed,

$$\int_{M^2} \tilde{f}\rho_2 \, dm^2 = \int_{M^2} \left((f(x) - f(y)) \int_\Omega H_2(x, y, \omega) \, dP \right) dm^2$$

$$= \int_{M^2 \times \Omega} (f(x) - f(y))H_2(x, y, \omega) \, d(m^2 \times P) = 0.$$

holds true since H_2 is symmetric.

From now on we impose the stronger condition that $\mathcal{L}_{T_2}$ is asymptotically stable on our system. Recall that this yields that $\mathcal{L}_{T_1}$ is asymptotically stable. For $\Phi \in L^1(m \times P)$,

$\Psi \in L^1(m^2 \times P)$, and nonnegative integer n, we put

$$
\begin{aligned}
\Delta(T_1, \Phi, n) &= \mathcal{L}_{T_1}^n \Phi - \int_{M \times \Omega} \Phi \, d(m \times P) \cdot H_1, \text{ and} \\
\Delta(T_2, \Psi, n) &= \mathcal{L}_{T_2}^n \Psi - \int_{M^2 \times \Omega} \Psi \, d(m^2 \times P) \cdot H_2.
\end{aligned}
$$
(3.1)

Next for a real-valued element $f \in L^\infty(m)$, we consider autocorrelations

$$
\begin{aligned}
C(T_1, f, n) &= \int_{M \times \Omega} (f \circ T_1^n) f H_1 \, d(m \times P) - \left(\int_{M \times \Omega} f H_1 \, d(m \times \Omega) \right)^2 \text{ and} \\
C(T_2, F_f, n) &= \int_{M^2 \times \Omega} (F_f \circ T_2^n) f H_2 \, d(m^2 \times P) - \left(\int_{M^2 \times \Omega} F_f H_2 \, d(m^2 \times \Omega) \right)^2.
\end{aligned}
$$
(3.2)

Under the condition

$$
(3.3) \qquad \sum_{n=0}^\infty \|\Delta(T_1, f H_1, n)\|_{1, m \times P} < \infty \quad \text{and} \quad \sum_{n=0}^\infty \|\Delta(T_2, F_f H_2, n)\|_{1, m^2 \times P} < \infty,
$$

the limit variances are given by the following absolutely convergent series.

$$
(3.4) \quad v(f) = C(T_1, f, 0) + 2 \sum_{n=1}^\infty C(T_1, f, n) \text{ and } v(F_f) = C(T_2, F_f, 0) + 2 \sum_{n=1}^\infty C(T_2, F_f, n).
$$

Now we are in a position to state the next result.

THEOREM 3.2. *Assume that the transfer operator of the slew product transformation T_2 with respect to $m^2 \times P$ is asymptotically stable with respect to $H_2 \in L^1(m^2 \times P)$. Let f be a real-valued element in $L^\infty(m)$ satisfying the deterministic centering condition $\int_M f \rho \, dm = 0$. Furthermore, we assume that f and F_f satisfy the conditions in (3.3). Then the following are equivalent.*

(1) *There exists a probability density $g \in L^1(m)$ such that $S_n F_f / \sqrt{n}$ converges in law to $N(0, 2v(f))$ with respect to $m^2 \times P$-absolutely continuous probability measure with density $\tilde{g}$.*

(2) *For any probability density $g \in L^1(m)$, $S_n F_f / \sqrt{n}$ converges in law to $N(0, 2v(f))$ with respect to $m^2 \times P$-absolutely continuous probability measure with density $\tilde{g}$.*

(3) *There exists a probability density $g \in L^1(m)$ such that for any bounded continuous function u on $\mathbb{R}$, $\int_M u(S_n / \sqrt{n}) g \, dm$ converges to $\int_\mathbb{R} u \, dN(0, v)$ strongly in $L^1(P)$ as $(n \to \infty)$.*

(4) *For any probability density $g \in L^1(m)$ and for any bounded continuous function u on $\mathbb{R}$, $\int_M u(S_n / \sqrt{n}) g \, dm$ converges to $\int_\mathbb{R} u \, dN(0, v)$ strongly in $L^1(P)$ as $(n \to \infty)$.*

(5) There exists a probability density $g \in L^1(m)$ such that for any $t \in \mathbb{R}$,
$\int_M e^{\sqrt{-1}t(S_n/\sqrt{n})} g \, dm$ converges to $e^{-vt^2/2}$ strongly in $L^1(P)$ as $(n \to \infty)$.

(6) For any probability density $g \in L^1(m)$ and for any $t \in \mathbb{R}$, $\int_M e^{\sqrt{-1}t(S_n/\sqrt{n})} g \, dm$ converges to $e^{-vt^2/2}$ strongly in $L^1(P)$ as $(n \to \infty)$.

(7) $\qquad v(F_f) = 2v(f).$

(8) $\int_{M^2 \times \Omega} \tilde{f} H_2 \, d(m^2 \times P) + 2 \sum_{n=1}^{\infty} \int_{M^2 \times \Omega} f(x) f(X_n(\omega)y) H_2 \, d(m^2 \times P) = 0.$

(9) $\lim_{n \to \infty} \frac{1}{n} \int_{M^2 \times \Omega} S_n f(x, \omega) S_n f(y, \omega) H_2 \, d(m^2 \times P) = 0.$

REMARK 3.3. In the assertion (9) in Theorem 3.2, the single point set $\{H_2\}$ can be replaced by the totality of probability densities in $L^1(m^2 \times P)$ for which $\|\Delta(T_2, F, n)\|_{1,m^2 \times P}$ decays sufficiently fast. For example, under the same conditions in Theorem 3.2, (9) is equivalent to the following.

(9)′ $\lim_{n \to \infty} \frac{1}{n} \int_{M^2 \Omega} S_n f(x, \omega) S_n f(y, \omega) F(x, y, \omega) \, d(m^2 \times P) = 0$ holds for any probability density $F \in L^1(m^2 \times P)$ satisfying $\lim_{n \to \infty} n^5 \|\Delta(T_2, F, n)\|_{1,m^2 \times P} = 0$.

We note that the condition '$\lim_{n \to \infty} n^5 \|\Delta(T_2, F, n)\|_{1,m^2 \times P} = 0$' is rather technical and it is not optimal.

Finally, we give the almost sure sample-wise CLT with deterministic centering. In [1], [2], and [3] a kind of quenched CLT(called almost sure sample-wise CLT in this article) with deterministic centering condition is obtained by using an averaged large deviation estimate. Here we give a slightly abstract result by using the estimate of fourth moment rather than the large deviation estimate. Precisely, we have the following.

THEOREM 3.4. Assume that the transfer operator of the slew product transformation T_2 with respect to $m^2 \times P$ is asymptotically stable. Let f be a real-valued element in $L^\infty(m)$ satisfying the deterministic centering condition $\int_M f \rho \, dm = 0$ such that f and F_f satisfy the conditions in (3.3) and $v(T_2, F_f) = 2v(T_1, f)$ holds. Let $g \in L^1(m)$ be a probability density. We assume that there exist constants $C > 0$, $\alpha > 0$, and $\beta > 0$ such that the following conditions are fulfilled.

(a) $\left| \int_{M \times \Omega} e^{\sqrt{-1}tS_n f/\sqrt{n}} H_1 \, d(m \times P) - e^{-vt^2/2} \right| \leq C \left(1 + |t|^\alpha\right) \frac{1}{n^\beta}.$ and

$\left| \int_{M^2 \times \Omega} e^{\sqrt{-1}tS_n F_f/\sqrt{n}} H_2 \, d(m^2 \times P) - e^{-vt^2} \right| \leq C \left(1 + |t|^\alpha\right) \frac{1}{n^\beta},$ where $v = v(f)$.

(b) $\displaystyle\sup_n \int_{M\times\Omega} \frac{(S_n f)^4}{n^2} H_1\, d(m\times P) \leq C.$

(c) $\displaystyle \|\Delta(T_1,g,n)\|_{1,m\times P} \leq \frac{C}{n^2}$ $\quad and \quad$ $\displaystyle \|\Delta(T_2,\tilde{g},n)\|_{1,m^2\times P} \leq \frac{C}{n^2}.$

Then for P-a.s.ω, $S_n f(\cdot,\omega)/\sqrt{n}$ converges in law to the normal distribution $N(0,v)$ with respect to the m-absolutely continuous probability measure gm, where $\tilde{g} \in L^1(m^2)$ is defined by $\tilde{g}(x,y) = g(x)g(y)$ for $(x,y) \in M^2$.

4. Examples

Throughout this section, $(M,\mathcal{B},m)$ is the unit interval with the usual Lebesgue space structure, $S = \{0,1\}$ with the discrete topology, and $(\Omega,\mathcal{F}) = (S^{\mathbb{Z}_+},\mathcal{B}(S^{\mathbb{Z}_+}))$, where Z_+ is the totality of nonnegative integers and $\mathcal{B}(S^{\mathbb{Z}_+})$ is the topological Borel field of $S^{\mathbb{Z}_+}$ endowed with the product topology of S. Each element $\omega \in \Omega$ can be expressed as $\omega = (\omega(0)\omega(1)\omega(2)\cdots)$. Thus the shift $\sigma : \Omega \to \Omega$ satisfies $(\sigma\omega)(k) = \omega(k)$ for $k \in \mathbb{Z}_+$. Choose p and q satisfying $0 < p, q < 1$ and $p+q = 1$ and define a locally constant function U so that

$$U(\omega) = \begin{cases} \log p & \text{if } \omega(0) = \omega(1), \\ \log q & \text{if } \omega(0) \neq \omega(1). \end{cases}$$

Let $P = P_U$ be the Gibbs measure corresponding to the potential U, i.e. the unique σ-invariant probability measure satisfying $\int_\Omega (f \circ \sigma) g\, dP = \int_\Omega f \mathcal{L}_U g\, dP$ for continuous functions f and g, where $\mathcal{L}_U$ is the transfer operator defined by $\mathcal{L}_U g(\omega) = e^{U(0\omega)}g(0\omega) + e^{U(1\omega)}g(1\omega)$ (see [4]). As an S-valued random variable $\xi : \Omega \to S$, we employ $\xi(\omega) = \omega(0)$. Clearly, P makes $\{\xi_n = \xi \circ \sigma^n, \ n \geq 0\}$ a symmetric stationary Markov chain.

Put $I(0) = [0,1/2]$ and $I(1) = [1/2,1]$. Let $\mathcal{T}$ be the totality of piecewise C^2 uniformly expanding maps $\tau : M \to M$ such that ess.inf$|d\tau/dx| > 1$ and $\tau|_{\text{int}I(j)} : \text{int}I(j) \to M$ is a C^2-diffeomorphism which can be extended to a C^2 function on $I(j)$ for $j = 0, 1$. It is well known that for each $\tau \in \mathcal{T}$ there exists a unique invariant density $h_\tau \in L^1(m)$ with Lipschitz continuous version and the transfer operator $\mathcal{L}_{\tau,m}$ is asymptotically stable with respect to h_τ. Choose $\tau_0, \tau_1 \in \mathcal{T}$ and consider a random dynamical system $\mathcal{X}$ given by $(\{\tau_0, \tau_1\}, \sigma, \xi)$. Write as $h_0 = h_{\tau_0}$ and $h_1 = h_{\tau_1}$ for simplicity. Note that the transfer operators of the skew product transformations T_1 and T_2 with respect to $m \times P$ and $m^2 \times P$ are given by

(4.1)
$$\mathcal{L}_{T_1}\Phi(x,\omega) = e^{U(0\cdot\omega)}\mathcal{L}_{\tau_0}(\Phi(\cdot,0\omega))(x) + e^{U(1\cdot\omega)}\mathcal{L}_{\tau_1}(\Phi(\cdot,1\omega))(x)$$

$$\mathcal{L}_{T_2}\Psi(x,y,\omega) = e^{U(0\cdot\omega)}\mathcal{L}_{\tau_0\times\tau_0}(\Psi(\cdot,\cdot,0\omega))(x,y) + e^{U(1\cdot\omega)}\mathcal{L}_{\tau_1\times\tau_1}(\Psi(\cdot,\cdot,1\omega))(x,y)$$

for $\Phi \in L^1(m \times P)$ and $\Psi \in L^1(m^2 \times P)$, where $\mathcal{L}_{\tau_j} = \mathcal{L}_{\tau_j,m}$ is the transfer operator of τ_j with respect to the Lebesgue measure m for $j = 0, 1$. We have the following.

PROPOSITION 4.1. *Let $(M, \mathcal{B}, m)$, $(\Omega, \mathcal{F}, P, \sigma)$, and $\{\tau_0, \tau_1\} \subset \mathcal{T}$ be as above. Consider the random dynamical system $\mathcal{X}$ given by $(\{\tau_0, \tau_1\}, \sigma, \xi)$ and its direct product $\mathcal{X} \times \mathcal{X}$. Then there exist a probability densities $H_1 \in L^1(m \times P)$ and $H_2 \in L^1(m^2 \times P)$ such that the transfer operators $\mathcal{L}_{T_1}$ and $\mathcal{L}_{T_2}$ are asymptotically stable with respect to H_1 and H_2, respectively. Moreover, H_1 and H_2 have continuous versions and for any Lipschitz continuous real-valued function f on M, the autocorrelations given by (3.2) decay exponentially fast as $n \to \infty$.*

Note that if $p = q = 1/2$, then $\{\xi_n\}_{n=0}^\infty$ is independent and densities H_1 and H_2 have deterministic versions by virtue of the result in [10]. On the contrary, if $h_0 \neq h_1$ we can show by using the formula (4.1) that $p \neq q$ implies that H_1 and H_2 can not have deterministic versions.

First we give an example for which the deterministic centering does not work.

EXAMPLE 4.2. (cf. [1]) Choose τ_0 and τ_1 so that

$$
\tau_0 x = \begin{cases} 2x, & \text{if } x \in I(0) \\ 2x - 1, & \text{otherwise} \end{cases}, \quad \tau_1 x = \begin{cases} x^2 + (3/2)x, & \text{if } x \in I(0) \\ x^2 + (1/2)x - 1/2, & \text{otherwise} \end{cases}.
$$

Clearly $h_0 = 1$. By $\mathcal{L}_{\tau_1} h_1 = h_1$, it is not hard to see that $h_1(0) = 2h_1(1/2)$ and $h_1(1) = (2/3)h_1(1/2)$. Therefore $h_0 \neq h_1$. Consider the observable $f = h_0 - h_1$. Then we have $\int_M f h_0 \, dm - \int_M f h_1 \, dm = \int_M (h_0 - h_1)^2 \, dm > 0$. Thus the symmetricity of the Markov chain $\{\xi_n\}_{n=0}^\infty$ enable us to prove $v(F_f) < 2v(f)$ in the same way as in [1]. Note that we also show that $v(f) > 0$.

It is obvious that if $h_0 = h_1$ the deterministic centering trivially works. Therefore we are interested in a nontrivial example for which the deterministic centering works.

EXAMPLE 4.3. Let $\iota : M \to M$ be the involution $\iota x = 1 - x$. Choose $\tau_j \in \mathcal{T}$ such that $\iota \circ \tau_j = \tau_j \circ \iota$ holds for $j = 0, 1$. Then it is not hard to see that the invariant densities H_1 and H_2 satisfy $H_1(\iota x, \omega) = H_1(x, \omega)$ and $H_2(\iota x, y, \omega) = H_2(x, \iota y, \omega) = H_2(\iota x, \iota y, \omega) = H_2(x, y, \omega)$. Therefore if f satisfies $f \circ \iota = -f$, we have $\int_M f(x) H_2(x, y, \omega) \, m(dx) = 0$. This yields the condition (8) in Theorem 3.2. To be more concrete choose τ_0 and τ_1 so that

$$
\tau_0 x = \begin{cases} 2x, & \text{if } x \in I(0) \\ 2x - 1, & \text{otherwise} \end{cases}, \quad \tau_1 x = \begin{cases} x^2 + (3/2)x, & \text{if } x \in I(0) \\ -x^2 + (7/2)x - 3/2, & \text{otherwise} \end{cases}.
$$

Consider the observable $f(x) = \cos(k\pi x)$ with k odd. Obviously $f \circ \iota = -f$. Moreover, we can show that Theorem 3.4 is applicable to f with $v(f) > 0$.

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