From normal to anomalous diffusion in simple dynamical systems

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I. INTRODUCTION

We briefly review a string of work that explores diffusion in simple dynamical systems [1–3]. For this purpose we consider dynamics generated by piecewise linear one-dimensional maps. These systems have the advantage that they are amenable to easy analytical calculations. We start with a simple example of a map that exhibits diffusive dynamics for which we exactly calculate the parameter-dependent diffusion coefficient. We do so by using a method that boils down to solving a functional recursion relation yielding fractal generalised Takagi functions. As a result we obtain a diffusion coefficient that is a highly non-trivial function under parameter variation. Its complexity can be explained with respect to prameter sensitivity of the underlying microscopic chaotic dynamics [2].

While our first example is a dynamical system where diffusion originates from microscopic deterministic chaos, our second example defines a random dynamical system generating diffusion. For obtaining the latter we mix randomly in time normal diffusive with trivial localising dynamics. As a result we obtain a random dynamical system that exhibits a non-trivial transition under variation of the probability of mixing both types of underlying dynamics. Most notably, there is a critical parameter value at which there emerges so-called subdiffusion reflecting the intermittent motion that we generate [3].

We conclude with a brief summary in which we relate the results obtained in this review to other strings of literature, where similar results have been obtained for physically more relevant dynamical systems.

II. CHAOTIC DIFFUSION IN ONE-DIMENSIONAL DETERMINISTIC MAPS

To set the scene let us recall a random walk in one dimension, where steps of length s to the left and to the right are drawn as independent and identically distributed random variables with probability $p(\pm s) = 1/2$ [4]. Since the single steps are uncorrelated this yields a Markov process. Karl Pearson was the first one to pose the question about how to assess such a random walk in terms of diffusive spreading [5]. For this purpose we define the diffusion coefficient as

$$D := \lim_{n\to\infty} \frac{1}{2n} < (x_n - x_0)^2 > \qquad (1)$$

with discrete time step $n \in \mathbb{N}$ and an ensemble average over the initial probability distribution function $\varrho(x)$ of positions $x = x_0$, $x \in \mathbb{R}$ at time $n = 0$, $< \ldots >:= \int dx\ \varrho(x) \ldots$. The quantity in the numerator above is called the mean square displacement (MSD) of a process, which gives information about the spreading of points with time n. For the random walk defined above one trivially obtains $D = s^2/2$ [6].

This setting motivates diffusion in terms of a simple stochastic process. Starting from the 1980's, however, theoretical physicists became interested in modeling diffusion on the basis of deterministic chaos [7–9]. The key idea is that in this case the single steps at discrete

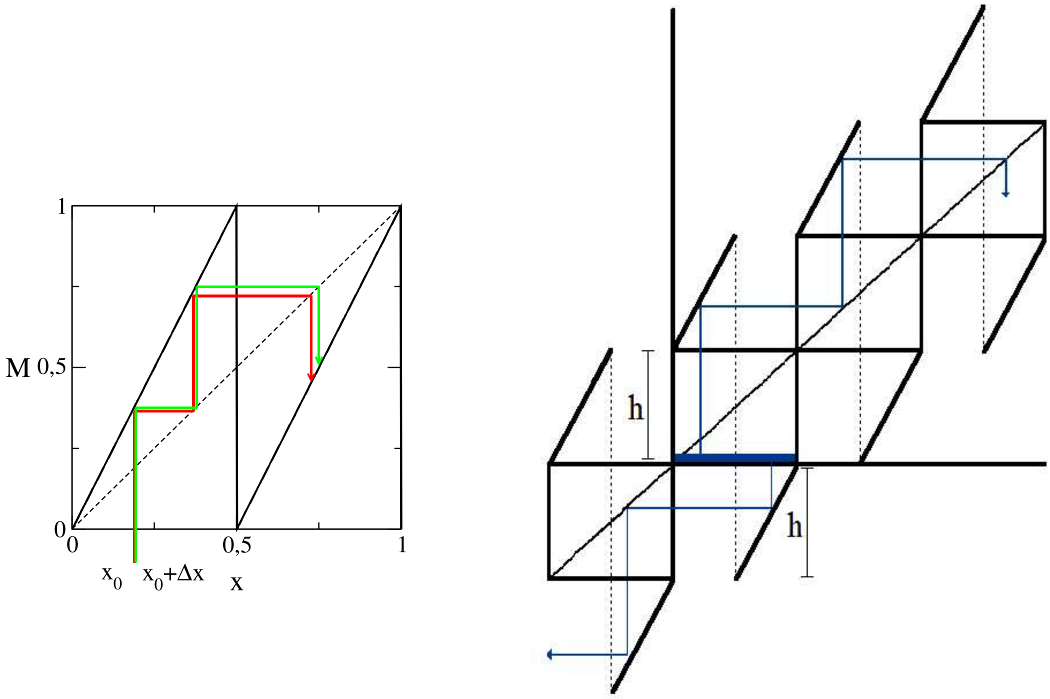

FIG. 1: Chaotic diffusion in a simple piecewise linear one-dimensional map. Left: Exponential separation of nearby trajectories in the Bernoulli shift Eqs.(2),(3). Right: A slightly generalised version of Bernoulli shift dynamics yielding a deterministic random walk on the line.

times n are not generated by stochastic coin tossing as above but instead by using a one-dimensional deterministic map such as, e.g., the famous Bernoulli shift, see Fig. 1(left) [10],

$$M(x) = 2x \bmod 1 \tag{2}$$

governing the equation of motion

$$x_{n+1} = M(x_n) \; . \tag{3}$$

If one applies an infinitesimally small perturbation $\Delta x_0 := \tilde{x}_0 - x_0 \ll 1$ to a given initial position x_0 and generates respective orbits according to Eq.(3), it is straightforward to see that this deterministic dynamical system exhibits an exponential dynamical instability,

$$\Delta x_n = 2\Delta x_{n-1} = 2^n \Delta x_0 = e^{n \ln 2} \Delta x_0 \; . \tag{4}$$

Here $\lambda := \ln 2$ defines the Lyapunov exponent of the dynamical system. If this quantity is positive one speaks of a dynamical system being chaotic [11]. However, as the Bernoulli shift above acts on a compact set it cannot generate a MSD that keeps growing linearly in the long time limit, as is required for the limit in Eq.(1) to yield a non-trivial result. In order to obtain a dynamical system that displays long-time diffusion on an unbounded set we start from a slightly modified version of the Bernoulli shift,

$$M_h(x) := \begin{cases} 2x + h & 0 \le x < \frac{1}{2} \\ 2x - 1 - h & \frac{1}{2} \le x < 1 \end{cases} \tag{5}$$

with symmetric shift $h \ge 0$ as a control parameter [2, 12]. This map $M_h(x)$ is continued on the whole real line by a lift of degree one, $M_h(x+1) = M_h(x) + 1$. The dynamical system $x_{n+1} = M_h(x_n)$ now yields a deterministic walk on the line, similar to the random walk mentioned at the very beginning but here defined on the basis of deterministic chaos [7–9]. There is a rich literature on studying the diffusive dynamics resulting from this kind of models [1].

An interesting question is to calculate the parameter-dependent diffusion coefficient $D(h)$ of this and related models. By now many analytical and numerical methods are available to solve this problem (see Refs. [1, 2, 13] and further references therein). Here we outline only one of them, which we may call Takagi function method, as at some point it involves fractal functions of the type of the famous Takagi function [14].

The starting point is the definition of the diffusion coefficient Eq.(1), where the MSD therein $< (x_n - x_0)^2 >$ we now define with respect to the invariant density $\varrho_h(x)$ of the map $m_h(x) := M_h(x) \mod 1$ on the unit interval, $< \ldots > := \int_0^1 dx\, \varrho_h(x) \ldots$. It is easy to see that for the map $m_h(x)$ we have $\forall_h\ \varrho_h(x) = 1$. By defining integer jumps $j_k := \lfloor x_{k+1} \rfloor - \lfloor x_k \rfloor$ at discrete time k one can show that $D(h)$ in Eq.(1) can be re-written via telescopic summation in terms of the Taylor-Green-Kubo formula [13, 15, 16]

$$D(h) = \frac{1}{2} \left\langle j_0^2 \right\rangle + \sum_{k=1}^{\infty} \left\langle j_0 j_k \right\rangle \ . \tag{6}$$

The function $\langle j_0 j_k \rangle$ therein is called the velocity autocorrelation function of the process. This formula displays a convenient structure: If the latter function was zero, as in case of a Markov process without any temporal memory, the first term in this formula yields the random walk solution $j_0^2/2$. Deviations from this random walk result are obtained if the correlation function decay is non-zero, meaning the process exhibits higher-order dynamical correlations.

For calculating $D(h)$ it thus remains to evaluate

$$\left\langle j_0 \sum_{k=0}^{\infty} j_k \right\rangle = \int_0^1 dx\, j_0 \sum_{k=0}^{\infty} j_k(x) \ . \tag{7}$$

Note that $J_h^n(x) := \sum_{k=0}^{n} j_k(x)$ defines a kind of scattering function, which tells us exactly how far a point that starts at a given initial position x moves after n time steps (see Ref. [16] for details). For calculational purposes it is convenient to employ the corresponding cumulative function

$$T_h^n(x) := \int_0^x dy \sum_{k=0}^{n} j_k(y) \ , \tag{8}$$

which after a short calculation [2, 13] yields the de Rham-type recursion relation [17]

$$T_h^n(x) = t(x) + \frac{1}{2}\, T_h^{n-1}(m_h(x)) \tag{9}$$

with $dt(x)/dx := j_0(x)$. This equation can be solved to [2, 13]

$$T_h^n(x) = \sum_{k=0}^{n} \frac{1}{2^k} t(m_h(x)) \,. \tag{10}$$

For $0 \leq h$ and $T_h(x) := \lim_{n\to\infty} T_h^n(x)$, combining Eqs.(1),(6),(8),(10) leads to

$$D(h) = \frac{\lceil h \rceil^2}{2} + \left(\frac{1-\hat{h}}{2}\right)(1 - 2\lceil h \rceil) + T_h\left(\hat{h}\right) \tag{11}$$

with $\hat{h} := h \mod 1 \ (h \notin \mathbb{N}_0), \hat{h} := 1 \ (h \in \mathbb{N}_0)$ [2]. Figure 2(left) displays numerical results for Eq.(11). On large scales we recover the simple random walk diffusion coefficient $D(h) = h^2/2\,(h \gg 1)$ as a lower bound of the exact $D(h)$. Figure 2(right) depicts a blow-up of the initial parameter region $0 < h < 0.5$. On small scales and on certain parameter intervals $D(h)$ is a function exhibiting a lot of fine structure. One can indeed show that in terms of Eq.(11) $D(h)$ is in this regime a fractal function of the control parameter h, which reflects a topological instability of the dynamics under parameter variation. This can be understood in terms of specific periodic orbits as outlined in the insets of Fig. 2(right). They show two examples of periodic orbits at specific parameter values h, a ballistic orbit (left inset) and an eventually localised one (right inset). The different symbols correspond to parameter values h generating respective classes of periodic orbits in the dynamics of the map. While these orbits are of Lebesgue measure zero in position space there is an environment of orbits exhibiting a similar type of dynamics for long transient times. It is intuitively clear that diffusion will be enhanced if the dynamics is dominated by the former type of dynamics while it is suppressed by the latter, which exactly is seen in $D(h)$ of Fig. 2(right). This provides a microscopic explanation for the fractality of the diffusion coefficient, i.e., it is due to long-time correlations in the microscopic scattering process of the dynamics [1, 13, 16, 18, 19].

Apart from fractal parameter regions at $[k, k+1/2]$, $k \in \mathbb{N}_0$ there are parameter regions at $[k+1/2, k+1]$, $k \in \mathbb{N}_0$ where $D(h)$ simply exhibits plateaus. This is also somewhat counter-intuitive, as naively one would expect $D(h)$ to monotonically increase with h, like the random walk approximation in Fig. 2(left). The explanation of these plateaus is provided by Fig. 3. One can see that at respective parameter values the phase space splits into two disjoint invariant sets (colored in blue, respectively $[0, 1/4) \cup [3/4, 1)$, and colored in red, respectively $[1/4, 3/4)$ in Fig. 3(right), periodically continued in (left)), which by action of the map are getting mapped onto themselves. While Fig. 3(left) displays the map $M_h(x)$, this splitting is more clearly seen in Fig. 3(right) by using the map $m_h(x)$. For a uniform initial density and $0.5 < h < 1$ the diffusion coefficient of the blue part is calculated to $D_b(h) = 1 - h$ while the diffusion coefficient of the red part is $D_r(h) = h - 1/2$. Combining both results yields the total diffusion coefficient

$$D(h) = D_b(h) + D_r(h) = \frac{1}{2} \,, \tag{12}$$

which explains the first plateau. The other ones are understood analogously.

In summary, we have demonstrated that simple deterministic random walks on the line can generate surprisingly non-trivial diffusive dynamics under variation of a control param-

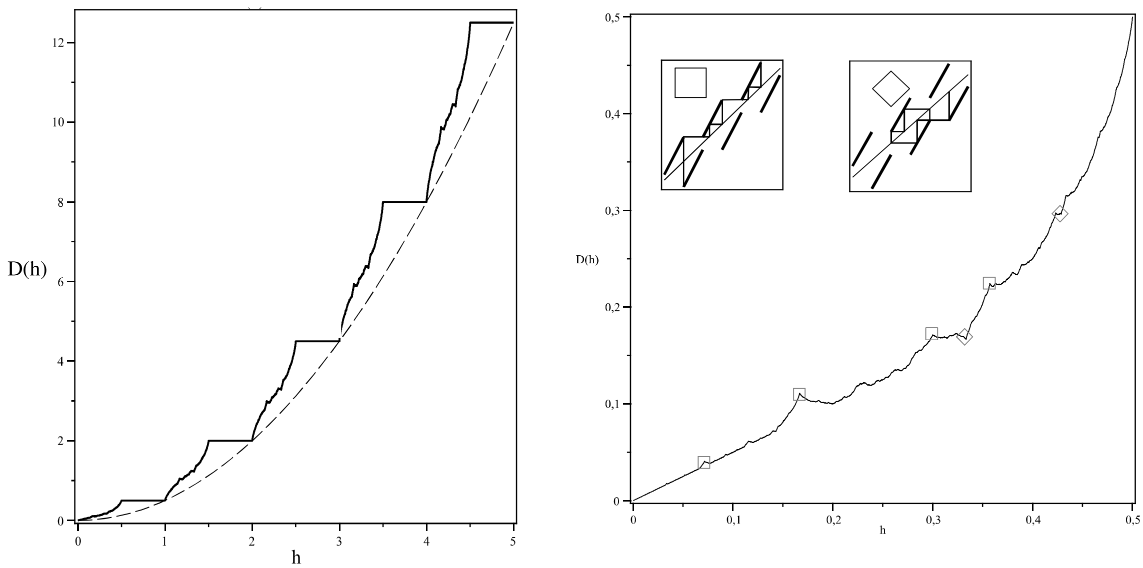

FIG. 2: Parameter-dependent diffusion coefficient for the deterministic random walk displayed in Fig. 1(right). Left: The diffusion coefficient $D(h)$ defined by Eq.(1) for the lifted map Eq.(5). Plotted is the analytical results Eq.(11) in comparison to a simple random walk approximation (dashed line). Right: Blow-up of the initial region of the left figure. The symbols therein refer to two different types of periodic orbits, ballistic (squares) as well as eventually localised ones (diamonds). These two different classes of periodic orbits occur at specific parameter values h explaining the fratcal structure of the corresponding diffusion coefficient $D(h)$.

eter. This is due to the deterministic chaos underlying the diffusion process, which can generate long-time dynamical correlations that are very sensitive with respect to parameter variation. In the next section we will explore yet another surprising feature of random walks on the line generated by simple maps.

III. A SIMPLE DIFFUSIVE RANDOM DYNAMICAL SYSTEM

We now introduce a simple scheme, based on the idea of random dynamical systems, which generates diffusive dynamics that is non-trivial in yet a fundamentally different way than discussed before. For this purpose let us start with Fig. 4(left), which outlines the basic concept. This figure displays three different time series for the position x_t of a point, or particle, at discrete time t. In the upper left the time series is generated by a deterministic dynamical system D yielding normal diffusion, where the MSD in Fig. 1(left) grows linearly with time in the long time limit. In the upper right we have a deterministic dynamical system L for which all particles eventually localize in space in the long time limit, meaning the MSD is eventually approaching a constant value for long times. One may now combine these two very different types of diffusive dynamics by randomly switching between the type of dynamical system that generates the position x_t at the next time step t. One may do so by tossing a coin whose two sides appear with probability p, respectively $1 - p$. This recipe yields a new random dynamical system R that mixes these two types of dynamics in time. As a result one may naively expect a kind of intermittent motion as displayed by the third

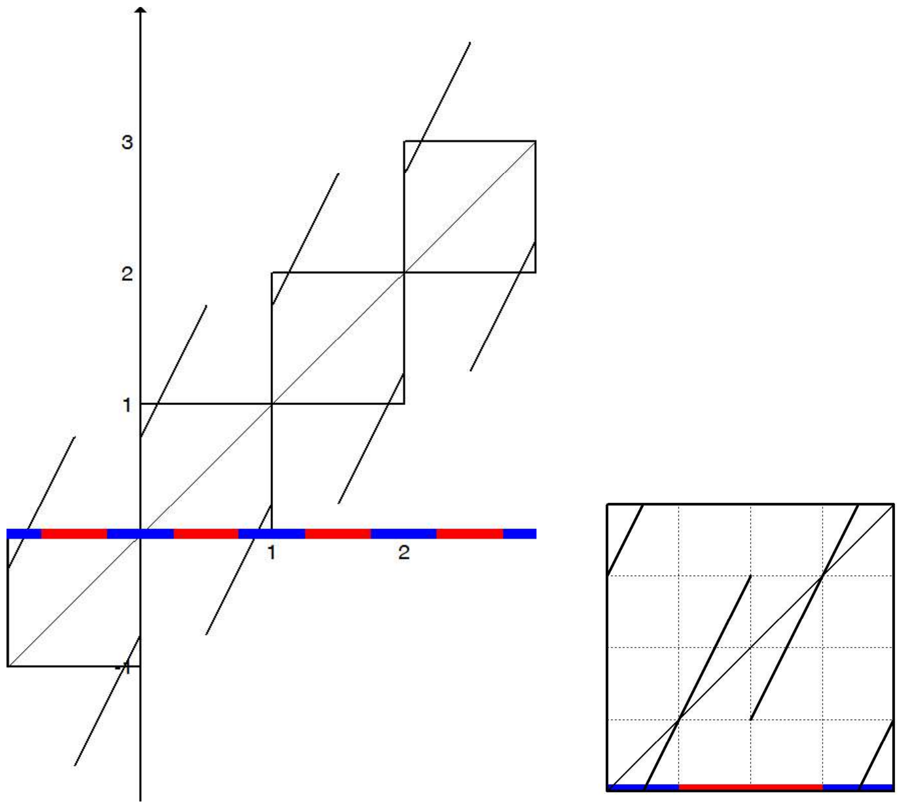

FIG. 3: Explanation of the plateau regions of the parameter-dependent diffusion coefficient $D(h)$ in Fig. 2. Left: The diffusive map $M_h(x)$ Eq.(5), see Fig. 1(right), for a parameter h in a plateau region. Here the system becomes non-ergodic. That is, the phase space splits into two disjoint invariant sets displayed by the blue and red intervals that are mapped onto themselves. Right: The same map mod 1 constrained onto the unit interval, called $m_h(x)$ in the text, for which the non-ergodicity can be identified graphically more clearly.

time series at the bottom of this figure.

That this expectation indeed holds true is demonstrated in Fig. 1(right). As an example we chose a simple piecewise linear one-dimensional map similar to the one discussed before defined by the equation of motion $x_{t+1} = M_a(x_t)$, where

$$M_a(x) = \begin{cases} ax & , \quad 0 \le x < \frac{1}{2} \\ ax + 1 - a & , \quad \frac{1}{2} \le x < 1 \end{cases} , \; a > 0 \, , \tag{13}$$

cf. the inset in Fig. 4(right). For $a > 2$ this model exhibits normal diffusion with a Lyapunov exponent $\lambda(a) = \ln a$ [1, 13, 18, 19]. The sample trajectory in the upper left of Fig. 4(left) was obtained from $D = M_4(x)$, where the dynamics is chaotic, $\lambda(4) = \ln 4 > 0$. The trajectory in the upper right of Fig. 4(left) corresponds to $L = M_{1/2}(x)$, where the dynamics is non-chaotic, $\lambda(1/2) = -\ln 2 < 0$. Here all particles contract onto stable fixed points at integer positions $x \in \mathbb{Z}$. For defining the random map R we now let the slope a be an independent and identically distributed, multiplicative random variable: At any time step t we choose for our map $R = M_a(x)$ with probability $p \in [0, 1]$ the slope $a = 1/2$ while with probability $1 - p$ we pick $a = 4$ [3]. Random maps of this type are also called iterated function systems [20]. They have been studied by both mathematicians and physicists in

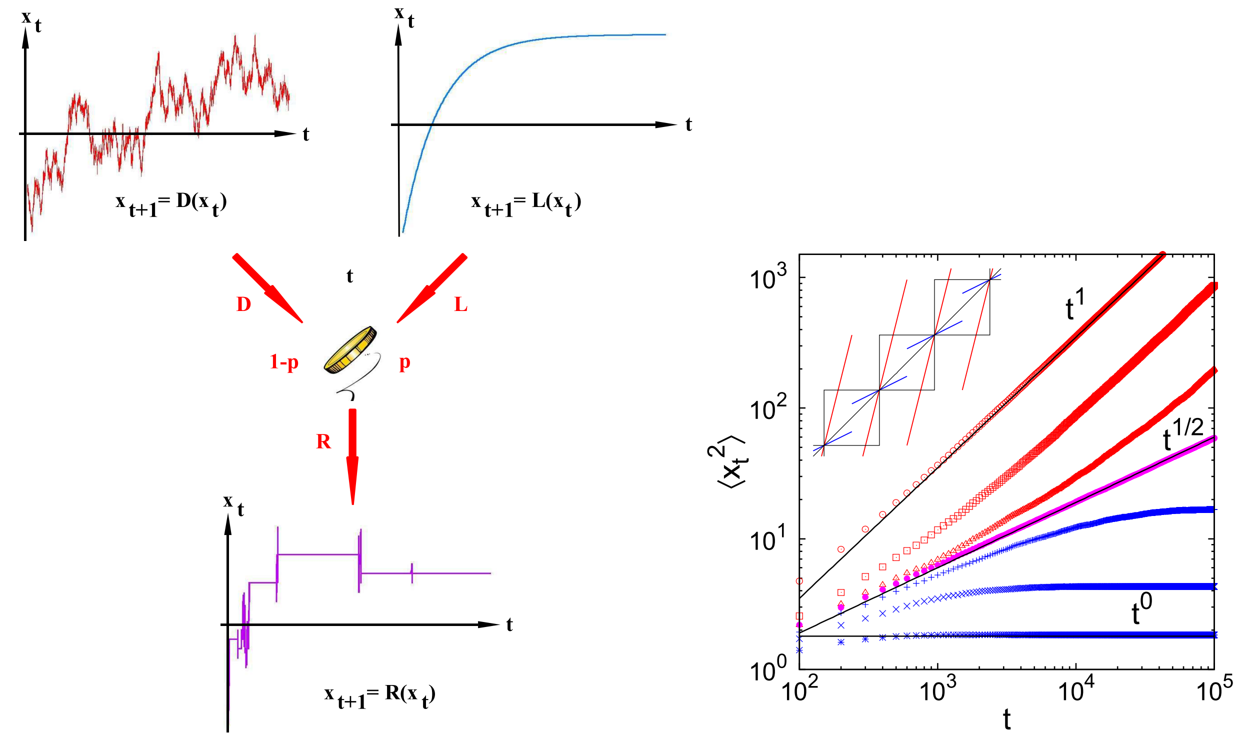

FIG. 4: A simple diffusive random dynamical system. Left: Sketch of a basic recipe to generate a random dynamical system by mixing normal diffusive (top left) and localising (top right) dynamics randomly in time. The result is intermittent motion (bottom). Right: Under variation of the probability p of mixing these two different types of dynamics the resulting random dynamical system R displays a transition from normal diffusion to localisation characterised by subdiffusive dynamics. Shown is the mean square displacement $< x_t^2 >$, cf. Eq.(1), as a function of discrete time t. The inset displays the specific random dynamical system generating these results, which consists of a random combination of two piecewise linear maps with different parameter values, see Eq.(13).

view of their measure-theoretic [21] and statistical physical properties [22].

It is easy to show that the Lyapunov exponent $\lambda(p)$ of the random map R is zero at probability $p_c = 2/3$. Since $\lambda(p) > 0$ for $p < p_c$ the map R should generate normal diffusion in this regime while $p > p_c$ with $\lambda(p) < 0$ should lead to localization for long times. The numerical results for the MSD of R under variation of p depicted in Fig. 4(right) confirm that this is indeed the case; for details of the simulations we refer to Ref. [3]. However, passing through p_c the dynamics displays a subtle transition: Right at p_c we obtain so-called anomalous diffusion, where the MSD grows nonlinearly in time [23]. More precisely here we encounter subdiffusion, $\langle x^2(t)\rangle \sim t^{1/2}$, which around p_c survives for long transient times.

We conclude that this simple random dynamical system can generate a novel, very different type of diffusive dynamic compared to the first one. This dynamic is characterised by interesting properties like ageing, weak ergodicity breaking, breaking of self-averaging and infinite invariant densities [3].

IV. SUMMARY

We have introduced to the problem of chaotic diffusion generated by deterministic dynamical systems. As the simplest examples possible we chose piecewise linear one-dimensional maps defined on the whole real line. For these dynamical systems the parameter-dependent diffusion coefficient can be calculated exactly analytically. We outlined a straightforward method of how to do so, which is based on evaluating fractal generalised Takagi functions. Surprisingly, the resulting diffusion coefficient exhibits fractal structures under parameter variation, which can be explained with respect to non-trivial dynamical correlations on microscopic scales.

For certain classes of such maps the Hausdorff and the box counting dimensions of the corresponding diffusion coefficient curves under parameter variation have been calculated rigorously mathematically. Interestingly, these curves belong to a very special type of fractals for which both dimensions are exactly equal to one. The fractal structure emerges from logarithmic corrections in continuity properties that display an intricate dependence on parameter variation [24].

These fractal diffusion coefficients are not an artefact specific to one-dimensional maps. A line of work demonstrated that they also appear in Hamiltonian particle billiards [1] and even for particles moving in soft potential landscapes [25]. The latter system relates to electronic transport in artificial graphene that can be studied experimentally. Yet another string of work investigated anomalous diffusion in intermittent maps of Pomeau-Manneville type, which also turned out to display fractal parameter dependencies of suitably generalised anomalous diffusion coefficients [1],

In this brief review we only discussed a simple type of random dynamical system as a second example generating non-trivial diffusion. We showed that mixing normal diffusive with localised dynamics randomly in time yields a novel type of intermittent dynamics characterised by the emergence of subdiffusion. To which extent the recipe that we proposed for obtaining this kind of random diffusive dynamical system can generate other types of anomalous diffusion remains to be explored.

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