A forward-backward distribution dependent SDE: a drift-less backward case

Takahiro Tsuchiya

School of Computer Science and Engineering, the University of Aizu.

Abstract

We show that a local existence and uniqueness condition implies the global solution on *drift-less* one-dimensional forward and high dimensional backward stochastic differential equations with Lipschitz coefficients.

1 Introduction

A global solution of Forward Backward Stochastic Differential Equations (FBSDEs for short) has a rich mathematical structure and there exists a lot of various contributions, we refer an excellent book [6]. For an application to Neural Ordinary Differential Equations [1], it plays an important role to develop to Neural Stochastic Differential Equations via a stochastic flow approach [5]. However, the solvability have been not disclosed for the following fundamental system,

$$\begin{cases} X(r) = X(t) + \int_t^r \sigma\left(s, X(s), Y(s), Z(s)\right) \mathrm{d}W(s) \\ Y(r) = \varphi\left(X(T)\right) - \int_r^T Z(s) \mathrm{d}W(s), \quad r \in [t, T]. \end{cases} \tag{1}$$

Thanks to Delarue in [2]: If the diffusion coefficient satisfies a non-degenerated condition and it is independent of Z, the smoothness of the coefficients is sufficient to obtain the global unique solution. For Z-dependent diffusion coefficients, the smoothness does not imply the well-definedness of the solution. To be more precise, the local solution exists if the Lipschitz continuous coefficient and terminal function satisfy

$$L_{\varphi,x} L_{\sigma,z} < 1, \tag{2}$$

where $L_{\varphi,x}$ and $L_{\sigma,z}$ are defined by the infimum of the collection of the Lipschitz constants. This is a local property close to the maturity T. To extend the solution, we need to estimate the Lipschitz continuity of a so called decoupling field,

$$\sup_{t\in[0,T]} L_{u(t,\cdot),x} < L_{\varphi,x}, \tag{3}$$

which is formalized by Fromm's contradiction method [3, Lemma 2.5.12]. Recently, a unified approach have been proposed in [6]. Their approach relies on the well-posedness of a one dimensional ordinary differential equation. As it is strong tool to solve one-dimensional linear FBSDE, it is not clear the following degenerate case; for an instance $\sigma(s, x, y, z) = \sin z$ and $\varphi(x) = (x/2) + \cos x$ in (1). Fromm et. al. solved a specific FBSDE that is related to a stochastic utility maximization problem in [4]. They considered that the terminal function is uniformly bounded to tame a quadratic structure with respect to Z. Moreover, we note that the assumption allows us to tame the singular term of so called characteristic BSDEs.

In this paper, we consider a different approach. We work on a framework such that the forward process takes one dimensional value but the backward process may be high dimensional system:

Theorem 1. *We suppose that σ and φ are Lipschitz continuous satisfying* (2). *Then, we have* (3) *holds. Moreover, we obtain that there exists the unique global solution of* (1).

The paper is organized as follows. Section 2 is to prepare notion and assumptions. The main result is provided in Section 3. In Section 4, we show the key estimate using an argument of stochastic flow.

2 Preliminaries

Let W be a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $\{\mathscr{F}_t\}_{t\geq 0}$ is an argument of natural filtration of W which satisfies usual condition. $\mathbb{R}^{m\times d}$ is identified with the space of ream matrices with m rows and d columns. If $z \in \mathbb{R}^{m\times d}$, we have $|z|^2 = \operatorname{trace}(zz^*)$ where $|\cdot|$ stands for the Frobenius norm.

For any real $l \in \mathbb{N}$ and $T > 0$, $\mathscr{S}\left(\mathbb{R}^l\right)$, denotes the set of $\mathbb{R}^l$-valued, adapted and cadlag process $\{X(t)\}_{t\in[0,T]}$ such that $\|X\| = \mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)|^2\right]^{\frac{1}{2}} < +\infty$. A collection $\mathscr{H}^2\left(\mathbb{R}^{m\times d}\right)$ denotes the set of (equivalent classes of) predictable processes $\{Z(t)\}_{t\in[0,T]}$ with values in $\mathbb{R}^{m\times d}$ such that $\|Z\| = \mathbb{E}\left[\left(\int_0^T |Z_r|^2\,\mathrm{d}r\right)\right]^{\frac{1}{2}} < +\infty$. We write a Banach space $\mathscr{S}^2(\mathbb{R}^l)\times\mathscr{S}^2(\mathbb{R}^m)\times\mathscr{H}^2\left(\mathbb{R}^{m\times d}\right) \equiv \mathscr{S}^2\times\mathscr{S}^2\times\mathscr{H}^2$ if there is no risk to confuse. For $(X,Y,Z)\in\mathscr{S}^2\times\mathscr{S}^2\times\mathscr{H}^2$, we note that

$$(X,Y,Z,\mathrm{W}) : [0,T]\times\Omega \longrightarrow \mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^d.$$

Assumption 1. *We say that the functions σ and φ satisfy* (A1) *if*

(A1.1) $\forall t\in[0,T]$, $\forall(x,y,z)\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$, *the functions $(x,y,z)\mapsto\sigma(t,x,y,z)$ and $x\mapsto\varphi(x)$ are infinitely differentiable with uniformly bounded derivatives.*

(A1.2) *There exists a constant Λ such that* $\forall t\in[0,T]$, $\forall(x,y,z)\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$,

$$|\sigma(t,x,y,z)| \leq \Lambda(1+|x|+|y|+|z|), \quad |\varphi(x)| \leq \Lambda(1+|x|).$$

Assumption 2. *We say that the function σ and φ satisfy* (A2) *if it holds*

(A2.1) *The forward SDEs takes a one dimensional process: $l = 1$.*

(A2.2) $L_{\varphi,x}L_{\sigma,z} < 1$,

where we set

$$\begin{aligned} L_{\varphi,x} &\triangleq \inf\left\{L>0 : \forall x_i\in\mathbb{R}^l\ (i=1,2),\quad |\varphi(x_1)-\varphi(x_2)|\leq L|x_1-x_2|\right\} \\ L_{\sigma,z} &\triangleq \inf\left\{L>0 : \forall t\in[0,T],\ \forall(x_i,y_i,z_i)\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\ (i=1,2),\right. \\ &\qquad \left.|\sigma(t,x_1,y_1,z_1)-\sigma(t,x_2,y_2,z_2)|\leq L(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|)\right\}. \end{aligned}$$

If $\forall t\in[0,T]$, $\forall(x,y,z)\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$, the function $x\mapsto u(t,x)$ is differentiable, we denote

$$\nabla_x u(s,x) = \left\{\frac{\partial u_i}{\partial x_j}(s,x) : j=1,\ldots,l,\ i=1,\ldots,m\right\}\in\mathbb{R}^{l\times m}.$$

Similarly, when σ satisfies the condition (A1.1), we write

$$\nabla_z\sigma(s,x,y,z) = \left\{\frac{\partial\sigma_{ij}}{\partial z_{pq}}(s,x,y,z) : i,p=1,\ldots,m,\ j,q=1,\ldots,d\right\}\in\mathbb{R}^{m\times d}\otimes\mathbb{R}^{m\times d}.$$

3 Drift-less coupled FBSDEs

It is known that it does not follow existence and uniqueness of the solution of FBSDEs only from smoothness like (A1). In spite of constructing a local solution, we need an additional condition such that (A2). The purpose of this section is to show that the local solution's condition (A2) becomes the condition to construct the global solution of FBSDEs when it is drift-less type (1).

3.1 An iterated scheme via implicit function theorem

In order to prove Theorem 1, we shall introduce the following iteration scheme.

Lemma 2. *Suppose that* (A1) *and* (A2) *hold. Denote*

$$X_0^{t,\xi}(t) = \xi, \quad (t,\xi) \in [0,T] \times \mathbb{R}^l.$$

Then, for all $n \in \mathbb{N}$, there exists a pair of smooth functions (u_{n-1}, v_{n-1}) and a unique weak solution $X_n \in \mathscr{S}^2$ such that

$$\begin{aligned} &u_{n-1}(s,x) \triangleq \mathbb{E}[\varphi\left(X_{n-1}^{s,x}(T)\right)], \quad (s,x) \in [0,T] \times \mathbb{R}^l, \\ &\left(\nabla_x u_{n-1}\right)(s,x)\sigma(s, u_{n-1}(s,x), v_{n-1}(s,x)) = v_{n-1}(s,x), \quad (s,x) \in [0,T] \times \mathbb{R}^l, \\ &X_n(r) = x + \int_t^r \sigma\left(s, X_n(s), u_{n-1}(s, X_n(s)), v_{n-1}(s, X_n(s))\right), \quad r \in [t,T]. \end{aligned}$$

Proof. When $n = 1$, we have and denote

$$\begin{aligned} &u_0(s,x) \triangleq \mathbb{E}[\varphi\left(X_0^{s,x}(T)\right)] = \varphi(x), \quad (s,x) \in [0,T] \times \mathbb{R}^l, \\ &F_0(s,x,z) \triangleq z - \nabla_x u_0(s,x)\sigma\left(s,x,u_0(s,x),z\right), \quad (s,x,z) \in [0,T] \times \mathbb{R}^l \times \mathbb{R}^{m\times d}. \end{aligned}$$

For all $(s_0, x_0) \in [0,T] \times \mathbb{R}^l$, we consider the map from $(|\cdot|_1, \mathbb{R}^{m\times d})$ to $(|\cdot|_2, \mathbb{R}^{m\times d})$ such that

$$G_0 : z \longmapsto \nabla_x u_0(s_0,x_0)\sigma\left(s_0, x_0, u_0(s_0,x_0), z\right).$$

As we have

$$|G_0(z_1) - G_0(z_2)|_2 \le L_{\varphi,x} L_{\sigma,z} |z_1 - z_2|_1, \quad z_1, z_2 \in \mathbb{R}^{m\times d}.$$

It shows that G_0 is contraction from (A2). Therefore, we have $F_0(s_0,x_0,z_0) = 0$ for some $z_0 \in \mathbb{R}^{m\times d}$. Again by the assumption (A2), applying the implicit function theorem, we obtain a unique smooth function v_0 on a neighborhood $B_{(s_0,x_0)}$ such that

$$\nabla_x u_0(s,x)\sigma(s, u_0(s,x), z) = z, \quad z = v_0(s,x), \quad (s,x) \in B_{(s_0,x_0)} \subset [0,T] \times \mathbb{R}^l.$$

As we have $L_{u_0,x} L_{\sigma,z} = L_{\varphi,x} L_{\sigma,z}$, the construction of v_0 is independent of the selection (s_0,x_0). Thus, v_0 can be extended to $[0,T] \times \mathbb{R}^l$. As we have $\sigma(s, u_0(s,x), v_0(s,x))$ is Lipschitz continuous, we have a unique solution in $\mathscr{S}^2$ such that

$$X_1(r) = x + \int_t^r \sigma\left(s, X_1(s), u_0(s, X_1(s), X_n), v_0(s, X_1(s))\right) \mathrm{dW}(s).$$

Now, let us assume that there exists desired $(u_{n-1}(s,x), v_{n-1}(s,x))$ and $X_n \in \mathscr{S}^2$. It defines u_n. For all $(s_0,x_0) \in [0,T] \times \mathbb{R}^l$, we consider the map from $(|\cdot|_1, \mathbb{R}^{m\times d})$ to $(|\cdot|_2, \mathbb{R}^{m\times d})$ such that

$$G_n : z \longmapsto \nabla_x u_n(s_0,x_0)\sigma\left(s_0, x_0, u_n(s_0,x_0), z\right)$$

and we have

$$|G_n(z_1) - G_n(z_2)|_2 \le L_{u_n,x} L_{\sigma,z} |z_1 - z_2|_1, \quad z_1, z_2 \in \mathbb{R}^{m\times d}.$$

It follows from Lemma 8 and (A2) that it is contraction. Thus, we obtain v_n via the implicit function theorem. The continuity of the coefficients $\sigma(s, u_n(s,x), v_n(s,x))$ implies that there exists a weak unique solution in $\mathscr{S}^2$ such that

$$X_{n+1}(r) = x + \int_t^r \sigma\left(s, X_{n+1}(s), u_0(s, X_{n+1}(s)), v_0(s, X_{n+1}(s))\right) \mathrm{dW}(s).$$

For the weak existence and uniqueness via the continuity, see [9]. □

Remark 3 (Loss of derivatives)**.** To relax the smooth condition (A1.1), we may face a problem, so called *loss of derivatives*;

$$u_0 \in C^k \Rightarrow v_0 \in C^{k-1} \Rightarrow u_n \in C^{k-n}, \quad n \leq k.$$

As it may overcome using an argument of Nash-Moser theorem, we do not go this direction in this paper.

For any $t \leq T$, let us define

$$Y_n(r) \triangleq u_{n-1}(r, X_n(r)), \quad Z_n(r) \triangleq v_{n-1}(r, X_n(r)), \quad r \in [t, T].$$

and we write $(\overline{X}_n, \overline{Y}_n, \overline{Z}_n) = (X_{n+1} - X_n, Y_{n+1} - Y_n, Z_{n+1} - Z_n)$ for $n \in \mathbb{N}$.

Lemma 4. *Suppose that* (A1) *and* (A2) *hold. There exists a constant $\delta > 0$ determined by σ, φ and l such that*

$$\forall T > 0, \quad (X_n, Y_n, Z_n) \xrightarrow[n\to\infty]{\mathscr{S}^2\times\mathscr{S}^2\times\mathscr{H}^2[T-\delta,T]} (X, Y, Z),$$

where (X, Y, Z) is a local solution to (1) *on $[T - \delta, T]$.*

Proof. Applying Lemma 8, we have for all $r \in [t, T]$,

$$\mathbb{E}\left[\left|\overline{Y}_n(r)\right|^2 \mathrm{d}s\right] \leq L_{\varphi,x}\mathbb{E}\left[\left|\overline{X}_n(r)\right|^2\right],$$

and

$$\mathbb{E}\left[\int_r^T \left|\overline{Z}_n(s)\right|^2 \mathrm{d}s\right] \leq \mathbb{E}\left[\left|\overline{Y}_n(T)\right|^2\right] \leq L_{\varphi,x}\mathbb{E}\left[\left|\overline{X}_n(T)\right|^2\right].$$

Since we have,

$$\overline{X}_n(r) = \int_t^r \sigma\big(s, X_{n+1}(s), Y_{n+1}(s), Z_{n+1}(s)\big) - \sigma\big(s, X_n(s), Y_n(s), Z_n(s)\big)\,\mathrm{dW}(s)$$

we obtain a constant $C > 0$ such that

$$\mathbb{E}\left[\left|\overline{X}_n(r)\right|^2\right] \leq C\int_t^r \mathbb{E}\left[\left|\overline{X}_n(s)\right|^2\right] + \mathbb{E}\left[\left|\overline{X}_n(T)\right|^2\right] \mathrm{d}s.$$

It follows from the Gronwall inequality that

$$\mathbb{E}\left[\left|\overline{X}_n(r)\right|^2\right] \leq Ce^{CT}\int_t^r \mathbb{E}\left[\left|\overline{X}_n(T)\right|^2\right] \mathrm{d}s.$$

Therefore, taking $\delta \in (0, 1)$ such that $Ce^{CT}\delta \leq \frac{1}{2}$ and $T - t < \delta$, we obtain that $\overline{X}_n(t) = 0$ for $T - t < \delta$ and

$$\mathbb{E}\left[\left|\overline{X}_n(T)\right|^2\right] \leq \frac{1}{2}\mathbb{E}\left[\left|\overline{X}_n(T)\right|^2\right].$$

Therefore, by the Burkholder-Davis-Gundy inequality shows that $X_n \to X$ in $\mathscr{S}^2$ on the local interval $[T - \delta, T]$. It induces the convergence $(Y_n, Z_n) \to (Y, Z)$ in $\mathscr{S}^2 \times \mathscr{H}^2$. Moreover, from the continuous of the coefficients, this is the desired solution. □

Remark 5. Generally, the above constant δ is depends on time t or T. On the drift-less case, this can be selected by the small length and it is independent of the time.

Lemma 6 (Stability problem)**.** *Suppose that for any $m \in \mathbb{N}$, σ_m and φ_m satisfy* (A1) *and* (A2) *and the Lipschitz constants is uniformly bounded,*

$$\sup_{m\in\mathbb{N}} (L_{\varphi_m,x} + L_{\sigma_m,x} + L_{\sigma_m,y} + L_{\sigma_m,z}) < \infty.$$

Let $(X_m, Y_m, Z_m) \in \mathscr{S}^2 \times \mathscr{S}^2 \times \mathscr{H}^2$ be the corresponding solution with initial condition $X_m(0) = \eta$ for an integrable random variable η. Then, there exists $C > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq s\leq T} |X_{m+1}(s) - X_m(s)|^2\right] + \mathbb{E}\left[\sup_{0\leq s\leq T} |Y_{m+1}(s) - Y_m(s)|^2\right] + \mathbb{E}\left[\int_0^T |Z_{m+1}(s) - Z_m(s)|^2\,\mathrm{d}s\right]$$
$$\leq C\mathbb{E}\left[|\varphi_{m+1} - \varphi_m|^2 (X_{m+1}(T)) + \int_0^T |\sigma_{m+1} - \sigma_m|^2 (s, X_{m+1}(s), Y_{m+1}(s), Z_{m+1}(s))\,\mathrm{d}s\right]$$

Proof. Note that for all $m \in \mathbb{N}$ it holds that

$$|\varphi_{m+1}(X_{m+1}(T)) - \varphi_m(X_m(T))| \leq L_{\varphi_m,x} |X_{m+1}(T) - X_{m+1}(T)| + |\varphi_{m+1} - \varphi_m| (X_m(T)).$$

It follows from the same argument of Lemma 4. □

Proof of Theorem 1. For a given Lipschitz continuous coefficients, it can be uniformly approximated by the smooth function satisfying (A1). Thus, it follow from Lemma 6 and the completeness of the Banach space $\mathscr{S}^2 \times \mathscr{S}^2 \times \mathscr{H}^2$ that it is sufficient to prove the existence and uniqueness when σ and φ satisfy (A1) and (A2).

Now, let us consider the following FBSDEs,

$$\begin{cases} X(r) & = x + \displaystyle\int_t^r \sigma\left(s, X(s), Y(s), Z(s)\right) \mathrm{dW}(s), \\ Y(r) & = u\left(T-\delta, X(T-\delta)\right) - \displaystyle\int_r^{T-\delta} Z(s)\, \mathrm{dW}(s), \quad r \in [t, T-\delta]. \end{cases}$$

It follows from Lemma 4 that (X, Y, Z) can be constructed on $[T-2\delta, T-\delta]$. By the uniqueness of the forward SDEs, (X, Y, Z) can be constructed uniquely on $[T-2\delta, T]$. Applying the same argument, we obtain that X can be constructed by the patched forward processes on the whole interval,

$$X(r) = X(0) + \int_0^r \sigma\left(s, X(s), u(s, X(s)), v(s, X(s))\right) \mathrm{dW}(s), \quad r \in [0, T].$$

We note that u_n converges uniformly to a function denoted by u and it satisfies

$$Y(t) = u(t, X(t)), \quad t \in [0, T].$$

It shows $Y \in \mathscr{S}^2$. Finally, it concludes that (X, Y, Z) is a desired result. □

3.2 Applications

A unified approach in [6] showed the equivalent condition, $L_{\varphi,x} L_{\sigma,z} \neq 1$ to get the global solution to the following one dimensional FBSDE,

$$\begin{cases} X(r) & = x + \displaystyle\int_0^r L_{\sigma,z} Z(s)\, \mathrm{dW}(s) \\ Y(r) & = L_{\varphi,x} X(T) - \displaystyle\int_r^T Z(s)\, \mathrm{dW}(s), \quad r \in [0, T]. \end{cases}$$

The condition can be extended to dimensional FBSDEs under a non-degenerate condition.

Assumption 3. *We say σ and φ satisfies Assumption* (A3) *if there exist positive constants $l_{\varphi,x}$ and $l_{\sigma,z}$ such that*

(A3.1) $l_{\varphi,x} l_{\sigma,z} > 1$ *and* $m = 1$

(A3.2) $\forall (s, x, y) \in [t, T] \times \mathbb{R}^l \times \mathbb{R}^m$ *and* $\forall\ z_1, z_2 \in \mathbb{R}^{m \times d}$,

$$\begin{aligned} l_{\varphi,x}|x_1 - x_2| &\leq |\varphi(x_1) - \varphi(x_2)|, \quad x_1, x_2 \in \mathbb{R}^l, \\ l_{\sigma,z}|z_1 - z_2| &\leq |\sigma(s, x, y, z_1) - \sigma(s, x, y, z_2)|. \end{aligned}$$

Corollary 7. *Suppose that the assumptions* (A1) *and* (A3) *are in force. Then, it admits a unique existence of the solution to* (2).

Proof. It is sufficient to prove the existence and uniqueness of (1) when $l_{\varphi,x} l_{\sigma,z} > 1$. The lower Lipschitz condition implies that the functions $x \mapsto \varphi(x)$ and $z \mapsto \sigma(s, x, y, z)$ for any fixed (s, x, y) are bijection maps, see [8, Section 4]. Thus, consider the inverse functions and we have

$$\begin{aligned} &|\varphi^{-1}(\tilde{x}_1) - \varphi^{-1}(\tilde{x}_2)| \leq l_{\varphi,x}^{-1} |\tilde{x}_1 - \tilde{x}_2|, \\ &|\sigma^{-1}(s, x, y, \tilde{z}_1) - \sigma^{-1}(s, x, y, \tilde{z}_2)| \leq l_{\sigma,z}^{-1} |\tilde{z}_1 - \tilde{z}_2|, \\ &\quad \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^l,\ (s, x, y) \in [t, T] \times \mathbb{R}^m,\ \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^{m \times d}. \end{aligned}$$

Then, we obtain $L_{\varphi^{-1},x} L_{\sigma_\varphi^{-1},z} \leq l_{\varphi,x}^{-1} l_{\sigma_\varphi,z}^{-1} < 1$. Thus, let us consider the following FBSDE,

$$\begin{cases} Y(r) &= Y(t) + \int_t^r \sigma^{-1}\left(s, X(s), Y(s), \tilde{Z}(s)\right) \mathrm{d}W(s), \\ X(r) &= \varphi^{-1}(Y(T)) - \int_r^T \tilde{Z}(s) \mathrm{d}W(s). \end{cases}$$

It follows from Theorem 1 that the uniqueness and existence of the solution. Moreover, we have

$$Z(s) = \sigma^{-1}\left(s, X(s), Y(s), \tilde{Z}(s)\right),$$

$$\tilde{Z}(s) = \sigma\left(s, X(s), Y(s), Z(s)\right), \quad s \in [t, T].$$

This implies the existence and uniqueness of the desired FBSDE (1). □

4 Appendix

Roughly, on drift-less and smooth coefficient SDEs, the spatial derivative of the decoupling field is given by the stochastic flow. Moreover, it is a non-negative exponential martingale $\{\nabla_x X^{t,x}(r)\}_{r\in[t,T]}$ such that

$$\nabla_x u(t,x) = \mathbb{E}[\nabla_x \varphi\left(X^{t,x}(T)\right) \cdot \nabla_x X^{t,x}(T)].$$

Formally, we show the following lemma.

Lemma 8 (Key lemma)**.** *Suppose that σ and φ satisfy* (A1) *and $\sigma(x,y,z) = \sigma(s,x)$ and X takes one dimensional value; $l = 1$. Let $X^{t,x} = \{X^{t,x}(r)\}_{r\in[t,T]}$ be a solution to the equation,*

$$X(r) = x + \int_t^r \sigma\left(s, X(s)\right) \mathrm{dW}(s), \quad t \leq r \leq T.$$

Then, for all $0 \leq t \leq r$ and $x, h \in \mathbb{R}^l$,

$$\left|\mathbb{E}\left[\varphi\left(X^{t,x+h}(r)\right) - \varphi\left(X^{t,x}(r)\right)\right]\right| \leq L_{\varphi,x}\left|h\right|.$$

In particular, denoting $u(t,x) = \mathbb{E}[\varphi(X^{t,x}(T))]$, it shows that

$$\sup_{t\in[0,T]} L_{u(t,\cdot),x} < L_{\varphi,x}.$$

Proof. From the mean-value theorem that it holds that for all $t \leq r \leq T$,

$$\varphi\left(X^{t,x+h}(r)\right) - \varphi\left(X^{t,x}(r)\right) = H(r)\left(X^{t,x+h}(r) - X^{t,x}(r)\right),$$

where $H(s)(\omega)$ is a linear map from $\mathbb{R}^l$ to $\mathbb{R}^m$ such that

$$H(r)U = \int_0^1 \nabla_x \varphi\left((1-\theta)X^{t,x+h}(r) + \theta X^{t,x}(r)\right) U \mathrm{d}\theta, \quad U \in \mathbb{R}^l.$$

Thus, we have

$$\left|H(r)\left(X^{t,x+h}(r) - X^{t,x}(r)\right)\right| \leq L_{\varphi,x}\left|\left(X^{t,x+h}(r) - X^{t,x}(r)\right)\right|, \quad r \in [t,T]$$

Applying Jensen's inequality, we have

$$\left|\mathbb{E}\left[\varphi\left(X^{t,x+h}(r)\right) - \varphi\left(X^{t,x}(r)\right)\right]\right| \leq L_{\varphi,x}\mathbb{E}\left[\left|X^{t,x+h}(r) - X^{t,x}(r)\right|\right].$$

Again, it follows from the mean-value theorem that it holds that for all $t \leq r \leq T$ and

$$\begin{aligned} X^{t,x+h}(r) - X^{t,x}(r) &= h + \sum_{k=1}^d \int_t^r \sigma_k\left(s, X^{t,x+h}(s)\right) - \sigma_k\left(s, X^{t,x}(s)\right) \mathrm{d}W_k(s) \\ &= h + \sum_{k=1}^d \int_t^r G_k(s)\left(X^{t,x+h}(s) - X^{t,x}(s)\right) \mathrm{d}W_k(s) \\ &= h \exp\left[-\frac{1}{2}\int_t^r \langle G(s), G(s)\rangle_{\mathbb{R}^d} \,\mathrm{d}s + \sum_{k=1}^d \int_t^r G_k(s) \mathrm{d}W_k(s)\right], \end{aligned}$$

where $G_k(s)$ is a functions,

$$G_k(s) = \int_0^1 (\nabla_x \sigma_k) \left(s, (1-\theta) X^{t,x+h}(s) + \theta X^{t,x}(s)\right) \mathrm{d}\theta.$$

Note that the exponential term is a scalar under $l = 1$, cf. [7, Lemma 3.2.3]. Thus, we obtain that

$$\begin{aligned} &|X^{t,x+h}(r) - X^{t,x}(r)| \\ &= |h| \exp\left[-\frac{1}{2}\int_t^r \langle G(s), G(s)\rangle_{\mathbb{R}^d} \,\mathrm{d}s + \sum_{k=1}^d \int_t^r G_k(s)\mathrm{d}W_k(s)\right] \end{aligned}$$

As we have $|G(s)| \leq L_{\sigma,x}$, the exponential local martingale is non-negative martingale. Then, we obtain that

$$\mathbb{E}\left[\left|X^{t,x+h}(r) - X^{t,h}(r)\right|\right] = |h|.$$

In short, we obtain

$$\left|\mathbb{E}\left[\varphi\left(X^{t,x+h}(r)\right) - \varphi\left(X^{t,x}(r)\right)\right]\right| \leq L_{\varphi,x} |h|.$$

□

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